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$L(d,1)$-labeling of regular tilings

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Abstract: $L(d,1)$-labeling is a kind of graph coloring problem from frequency assignment in radio networks, in which adjacent nodes must receive colors that are at least d apart while nodes at distance two from each other must receive different colors. We focus on $L(d,1)$-labeling of regular tilings for $d \geqslant 3$ since the cases $d=0$, 1 or 2 have been researched by Calamoneri and Petreschi. For all three kinds of regular tilings, we give their $L(d,1)$-labeling numbers for any integer $d \geqslant 3$. Therefore, combined with the results given by Calamoneri and Petreschi, the $L(d,1)$-labeling numbers of regular tilings for any nonnegative integer d may be determined completely.

Key words: regular tiling; frequency assignment problem; vertex labeling; $L(d,1)$-labeling; $L(2,1)$-labeling

$L(d,1)$-labeling of a graph G is a generation of $L(2,1)$-labeling. The $L(2,1)$-labeling, proposed by Griggs and Yeh[1], arose from a variation of the frequency assignment problem introduced by Hale[2]. Suppose that a number of transmitters or stations are given, in order to reduce the interference, "close" transmitters must receive different frequencies and "very close" transmitters must receive frequencies that are at least two frequencies apart. We can construct an interference graph for the frequency assignment problem so that each vertex represents a transmitter and two vertices are adjacent on G if the corresponding transmitters are "very close". Two vertices are of distance two on G if the corresponding transmitters are "close".

Thus, for a given graph G, an $L(2,1)$-labeling of G is defined as a function f: $V(G) \to \{0, 1, 2, 3, \ldots\}$ such that $|f(u)-f(v)| \geqslant 2$ if $uv \in E(G)$; and $|f(u)-f(v)| \geqslant 1$ if $d(u,v)=2$. The $L(2,1)$-labeling number $\lambda(G) = \min_f \max\{f(v): v \in V(G)\}$. $L(2,1)$-labelings of graphs have been extensively studied in the past decade[3 - 11].

Chang, et al.[3] generalized the $L(2,1)$-labeling to the $L(d,1)$-labeling. An $L(d,1)$-labeling of a given graph G, where d is a nonnegative integer, is a nonnegative integral function f on $V(G)$ satisfying the following condition:

$$|f(u)-f(v)| \geqslant \begin{cases} d & \text{if } d(u,v)=1 \\ 1 & \text{if } d(u,v)=2 \end{cases}$$

The $L(d,1)$-labeling number of G, denoted by $\lambda_d(G)$, is the smallest integer number k such that there exists an $L(d,1)$-labeling f: $V(G) \to \{0, 1, 2, \ldots, k\}$. We call the $L(d,1)$-labeling f of G optimal if $\max_{v \in V(G)} f(v) = \lambda_d(G)$.

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Chang, et al.[3] gave the upper bound of $L(d,1)$-labeling number for general graph G. They also obtained different lower and upper bounds of $\lambda_d(G)$ for some families of graphs including trees and chordal graphs.

The tiling problem consists in covering the plane with copies of the same polygons. If we tile the plane regularly, that is, use the same regular polygons to tile the plane, only hexagons, squares and triangles may be used. So, regular tilings considered here only have three kinds, including the hexagonal tiling, the squared tiling and the triangular tiling. We call the graph formed by tiling a plane a tiling graph. Let Δ be the degree of a regular tiling graph. Then $\Delta=3$ if the tiling is hexagonal tiling; $\Delta=4$ if the tiling is squared tiling and; $\Delta=6$ if the tiling is triangular tiling.

Calamoneri and Petreschi[4] have proved that, for $d=0$, 1 or 2 and for any regular tiling graph G with degree Δ, $\lambda_d(G) = \Delta + 2^d - 2$. In what follows, G_1, G_2 and G_3 denote the hexagonal tiling graph, the squared tiling graph and the triangular tiling graph, respectively. Then $\lambda_d(G_1) = 2^d+1$, $\lambda_d(G_2) = 2^d+2$, $\lambda_d(G_3) = 2^d+4$ for $d=0,1,2$. Note that, for any nonnegative integer d, $\lambda_d(G_1) \leqslant \lambda_d(G_2) \leqslant \lambda_d(G_3)$ since G_1 is a subgraph of G_2 and G_2 is a subgraph of G_3.

In this paper, we investigate the $L(d,1)$-labelings ($d \geqslant 3$) of three regular tiling graphs G_1, G_2 and G_3, and obtain the exact values of the $L(d,1)$-labeling numbers of them as follows:

$$\lambda_d(G_1) = d+4 \quad \text{if } d \geqslant 3$$

$$\lambda_d(G_2) = \begin{cases} 8 & \text{if } d=3 \\ d+6 & \text{if } d \geqslant 4 \end{cases}$$

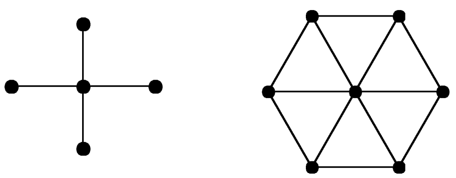
$$\lambda_d(G_3) = \begin{cases} 11 & \text{if } d=3 \\ 2d+6 & \text{if } d \geqslant 4 \end{cases}$$

Thus, together with the known results $\lambda_d(G_i) = \Delta + 2^d - 2$, $d=0,1,2$ and $i=1,2,3$, all $L(d,1)$-labeling numbers of three kinds of tiling graphs here for any nonnegative integer d are determined completely.

1 Main Results

Based on the fact that regular tiling graphs have infinite vertices, it is impossible to draw them thoroughly. In this paper, all shown regular tiling graphs are only a part of them.

The star $K_{1,4}$ and wheel W_6 are two graphs as shown in Fig. 1. These two graphs will be used in the proofs of the following theorems.



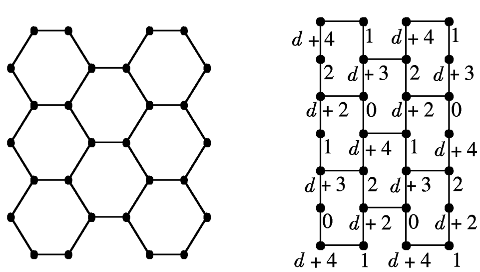
(a) (b)

Fig. 1 Graphs of star $k_{1,4}$ and wheel W_6. (a) $K_{1,4}$; (b) W_6

Suppose G is a graph with alternate vertices in row and column. An $(m \times n)$-periodic labeling of graph G means a periodic coloring of G with row period m and column period n. That is, for $i = 1, 2, \ldots, s - m$ and $j = 1, 2, \ldots, t - n$, vertices in the $(i + m)$-th row of graph G have the same labels with corresponding vertices in the i-th row of graph G; vertices in the $(j + n)$-th column have the same labels with corresponding vertices in the j-th column. Where s, t are the numbers of row and column of G, respectively. Call f an $(m \times n)$-periodic $L(d, 1)$-labeling of graph G if f is an $(m \times n)$-periodic labeling and also an $L(d, 1)$-labeling of graph G.

The method how we get the $L(d, 1)$-labeling numbers of regular tiling graphs is to find optimal periodic labelings of regular tiling graphs or their isomorphic graphs at first, and then get the $L(d, 1)$-labeling numbers by proving that the periodic labelings are optimal.

First of all, we investigate $L(d, 1)$-labeling of hexagonal tiling graph G_1 for $d \geqslant 3$. G_1 is isomorphic to a subgraph G_1' of squared tiling graph G_2. Fig. 2 shows such a subgraph G_1' obtained by deleting a family of edges on G_2. We use G_1' to find an optimal $L(d, 1)$-labeling and obtain $L(d, 1)$-labeling number of G_1 stated in theorem 1.



(a) (b)

Fig. 2 Graphs of G_1 and G_1'. (a) G_1; (b) G_1' and its (6×2)-periodic labeling

Theorem 1 $\lambda_d(G_1) = d + 4$, if $d \geqslant 3$.

Proof Since G_1 is isomorphic to G_1', their $L(d, 1)$-labeling numbers are equal. Fig. 2 gives graph G_1 and a (6×2)-periodic labeling of G_1' with the maximum label $d + 4$. It is straightforward to check that it is an $L(d, 1)$-labeling of G_1'. Thus $\lambda_d(G_1) = \lambda_d(G_1') \leqslant d + 4$ for $d \geqslant 3$.

Next we prove that $\lambda_d(G_1) \geqslant d + 4$ for $d \geqslant 3$. Notice that, for any $L(d, 1)$-labeling of graph G_1, if some number in $\{3, 4, 5, \ldots, d, d + 1\}$ is used to label a vertex u, then there must be an adjacent vertex of u with label at least $d + 4$. Suppose that $\lambda_d(G_1) \leqslant d + 3$, then there are only 5 numbers 0, 1, 2, $d + 2$, $d + 3$ available to label G_1 for any optimal $L(d, 1)$-labeling. However, this is impossible since G_1 contains a cycle C_6. A contradiction. So $\lambda_d(G_1) \geqslant d + 4$.

Hence, the (6×2)-periodic labeling in Fig. 2(b) is optimal and $\lambda_d(G_1) = d + 4$ for $d \geqslant 3$. This completes the proof of the theorem.

Now, we consider $L(d, 1)$-labeling of squared tiling graph G_2. Note that G_2 is isomorphic to graph $P_\infty \times P_\infty$, the product of two paths with infinite length.

Theorem 2 $\lambda_d(G_2) = \begin{cases} 8 & \text{if } d = 3 \\ d + 6 & \text{if } d \geqslant 4 \end{cases}$

Proof

Case 1 $d = 3$

For $d = 3$, there is a (3×9)-periodic $L(3, 1)$-labeling of G_2 with label no more than 8. See Fig. 3. So $\lambda_3(G_2) \leqslant 8$.

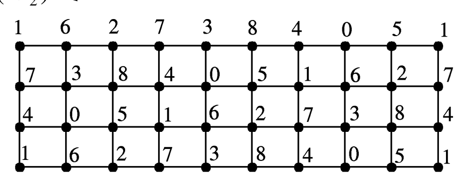


Fig. 3 A (3×9)-periodic labeling of G_2

We next prove that $\lambda_3(G_2) \geqslant 8$. Suppose that $\lambda_3(G_2) \leqslant 7$, then numbers 2, 3, 4, 5 cannot be used by any optimal $L(3, 1)$-labeling of G_2. Indeed, any vertex of G_2 with any label of 2, 3, 4, 5 has an adjacent vertex with more than 8 labels since G_2 is a regular graph of degree 4. So there are only four numbers 0, 1, 6, 7 available to form an optimal $L(3, 1)$-labeling of G_2. But G_2 contains star subgraph $K_{1,4}$, and any four numbers cannot form an $L(3, 1)$-labeling of $K_{1,4}$. Hence 0, 1, 6, 7 cannot form an $L(3, 1)$-labeling of G_2, which contradicts the assumption. So $\lambda_3(G_2) \geqslant 8$. Thus, $\lambda_3(G_2) = 8$.

Case 2 $d \geqslant 4$

For $d \geqslant 4$, there is a (4×4)-periodic $L(d, 1)$-labeling of G_2 with label no more than $d + 6$. See Fig. 4. So $\lambda_d(G_2) \leqslant d + 6$.

Further, we claim that $\lambda_d(G_2) \geqslant d + 6$. Suppose, by contraries, that $\lambda_d(G_2) \leqslant d + 5$. Then any numbers in $\{3, 4, 5, \ldots, d, d + 1, d + 2\}$ cannot be used by any

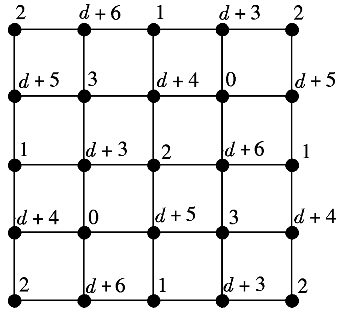
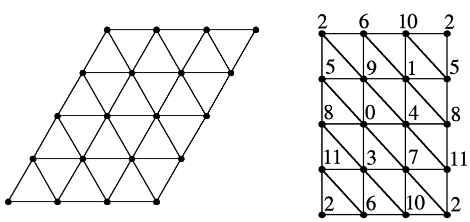


Fig. 4 A (4×4)-periodic labeling of G_2

optimal $L(d,1)$-labeling of G_2. Indeed, for any $L(d,1)$-labeling of G_2, if a vertex u of G_2 is labeled with some number in $\{3, 4, 5, \ldots, d, d+1, d+2\}$, then u has an adjacent vertex with label no less than $d+6$ since u is of degree 4 and $d \geqslant 4$. Thus, there only remain five numbers $0, 1, 2, d+3, d+4, d+5$ available to $L(d,1)$-label G_2 optimally. This is impossible since G_2 contains a subgraph $K_{1,4}$. A contradiction to the assumption. So $\lambda_d(G_2) \geqslant d+6$. Thus, $\lambda_d(G_2) = d+6$, which completes the proof.

Next we consider triangular tiling graph G_3. G_3 is isomorphic to the graph G'_3 obtained by adding a family of edges on the squared tiling graph G_2. See Fig. 5.



(a) (b)

Fig. 5 Graphs of G_3 and G'_3. (a) G_3; (b) G'_3 and its (4×3)-periodic labeling

Theorem 3 $\lambda_d(G_3) = \begin{cases} 11 & \text{if } d=3 \\ 2d+6 & \text{if } d \geqslant 4 \end{cases}$

Proof

Case 1 $d=3$

For $d=3$, there is a (4×3)-periodic $L(3,1)$-labeling of G'_3 with the maximum label 11, where G'_3 is an isomorphic graph of G_3. See Fig. 5. So $\lambda_3(G_3) = \lambda_3(G'_3) \leqslant 11$.

Now we prove that $\lambda_3(G_3) \geqslant 11$. Suppose $\lambda_3(G_3) \leqslant 10$. Then for any optimal labeling f of G_3, $\max\limits_{v \in G_3} f(v) \leqslant 10$. Thus, we claim that the numbers 2, 3, 4, 6, 7, 8 cannot be used to label any vertex of G_3 by f. Indeed, if some vertex of G_3 is labeled with the number 2, then its six adjacent vertices can only be labeled with the numbers 5, 6, 7, 8, 9, 10 since $d=3$. However, this is impossible since the six adjacent vertices of u induce a cycle and any two vertices of the six vertices are of distance two. Similarly, the numbers 3, 4, 6, 7, 8 cannot be used as vertex labels of G by f. So there only remain five numbers 0, 1, 5, 9, 10 available by f to label G_3. However, such an f does not exist since G_3 contains a subgraph W_6. Hence $\lambda_3(G_3) \geqslant 11$. And so for the case $d=3$, $\lambda_3(G_3) = 11$.

Case 2 $d=4$

For $d=4$, Fig. 6 shows a (3×3)-periodic $L(4,1)$-labeling of G'_3 with the maximum label 14. So $\lambda_4(G_3) = \lambda_4(G'_3) \leqslant 14$.

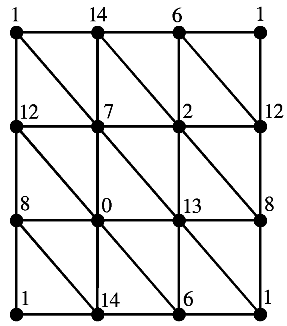


Fig. 6 A (3×3)-periodic labeling of G'_3

Next we prove that $\lambda_4(G_3) \geqslant 14$. Suppose $\lambda_4(G_3) \leqslant 13$, then for any optimal $L(4,1)$-labeling f of G_3, $\max\limits_{v \in G_3} f(v) \leqslant 13$. Thus, for any vertex u of G_3, $f(u) \neq 3, 4, 5, 8, 9, 10$. Indeed, if the center vertex of W_6 is labeled by f with number 3, then the cycle induced by its six adjacent vertices can only be labeled differently with the numbers in $\{7, 8, 9, 10, 11, 12, 13\}$ since $d=4$. But the numbers in $\{7, 8, 9, 10, 11, 12, 13\}$ are impossible to form an $L(4,1)$-labeling of the cycle. Hence $f(u) \neq 3$. Similarly, $f(u) \neq 4, 5, 8, 9, 10$. So there remain eight numbers 0, 1, 2, 6, 7, 11, 12, 13 available for f. Furthermore, among these eight numbers, for any vertex u of G_3, $f(u) \neq 0, 1, 2, 11, 12, 13$ for similar reasons. Thus, the labeling f does not exist. A contradiction to the assumption. Hence, $\lambda_4(G_3) \geqslant 14$. And so, $\lambda_4(G_3) = 14 = 2d+6$.

Case 3 $d \geqslant 5$

Fig. 7 is a (3×3)-periodic $L(d,1)$-labeling of G'_3 just with the maximum label $2d+6$. So $\lambda_d(G_3) = \lambda_d(G'_3) \leqslant 2d+6$.

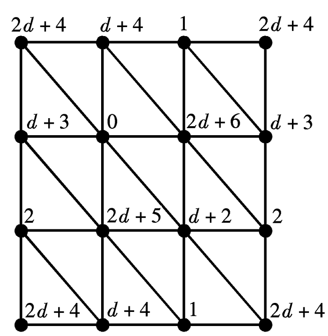


Fig. 7 A (3×3)-periodic labeling of G'_3

For any optimal $L(d,1)$-labeling f of G_3, the three vertices of any triangle in G_3 have labels differed by at least d. Thus, there exist vertex labels of G in

$[0, d-1]$, $[d, 2d-1]$ and $[2d, +\infty)$.

If $f(u) = d$ for some vertex u, then the six adjacent vertices of u have different labels in $\{0\} \cup \{2d, 2d+1, 2d+2, \dots\}$. Since the six adjacent vertices induce a cycle, there must be a vertex with label at least $3d+2$. But $3d+2 \geqslant 2d+7 > \lambda_d(G_3)$. A contradiction. Hence, for any vertex u of G_3, $f(u) \neq d$. Similarly, $f(u) \neq d+1, d+5, d+6, d+7, \dots, 2d-1$.

So $d+2, d+3, d+4$ are the only available numbers in $[d, 2d-1]$ for f. And $0, 1, 2, d+2, d+3, d+4, 2d+4, 2d+5, 2d+6$ are the only available numbers for f since G_3 contains a subgraph W_6 and $\lambda_d(G_3) \leqslant 2d+6$. Furthermore, if $f(u) = d+4$ for some vertex u of G_3, then u has an adjacent vertex with label no less than $2d+6$.

Assume that $\lambda_d(G_3) \leqslant 2d+5$, then the available numbers for f are only $0, 1, 2, d+2, d+3, 2d+4, 2d+5$. But the numbers in $\{0, 1, 2, d+2, d+3, 2d+4, 2d+5\}$ are impossible to form an optimal $L(d,1)$-labeling f of G_3. Contradict to the assumption. Hence, $\lambda_d(G_3) = 2d+6$ if $d \geqslant 5$.

From the above proof of three cases, we have

$$\lambda_d(G_3) = \begin{cases} 11 & \text{if } d = 3 \\ 2d+6 & \text{if } d \geqslant 4 \end{cases}$$

2 Concluding Remarks

Combined with the known results studied by Calamoneri and Petreschi for the cases $d = 0, 1$ or 2, we determine the $L(d,1)$-labeling numbers of three regular tilings for any nonnegative integer d. That is

$$\lambda_d(G_1) = \begin{cases} 2^d + 1 & \text{if } d = 0, 1 \text{ or } 2 \\ d+4 & \text{if } d \geqslant 3 \end{cases}$$

$$\lambda_d(G_2) = \begin{cases} 2^d + 2 & \text{if } d = 0, 1 \text{ or } 2 \\ 8 & \text{if } d = 3 \\ d+6 & \text{if } d \geqslant 4 \end{cases}$$

$$\lambda_d(G_3) = \begin{cases} 2^d + 4 & \text{if } d = 0, 1 \text{ or } 2 \\ 11 & \text{if } d = 3 \\ 2d+6 & \text{if } d \geqslant 4 \end{cases}$$

We point out that $L(j,k)$-labeling of regular tilings is the generalized problem of $L(d,1)$-labeling of regular tilings. It seems to be a little complicated. $L(j,k)$-labeling is a direction in which to further research labeling coloring of regular tilings.

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Regular tilings 的 $L(d,1)$-标号着色

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摘要: $L(d,1)$-标号着色是$L(2,1)$-标号着色的推广, 这一图的点着色问题来自于无线电波中的频道分配问题, 要求图中相邻顶点所着的颜色相差至少 d, 距离为 2 的顶点所着颜色必须不相同. 由于 $d = 0, 1, 2$ 时 regular tilings 的 $L(d,1)$-标号着色数已由 Calamoneri 和 Petreschi 给出, 本文研究 $d \geqslant 3$ 时所有 3 种 regular tilings 的 $L(d,1)$-标号着色, 给出它们的 $L(d,1)$-标号着色数. 结合 Calamoneri 和 Petreschi 的结果, 对所有非负整数 d, regular tilings 的 $L(d,1)$-标号着色数已完全确定.

关键词: regular tiling; 频道分配问题; 点着色; $L(d,1)$-标号着色; $L(2,1)$-标号着色

中图分类号: O157.5