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Reconstruction of density and wave velocity from reflection and transmission data

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Abstract: Consider an inverse problem of reconstructing the coefficient in a linear wave equation on an inhomogeneous slab with density $\rho(z)$ and wave velocity $c(z)$. The inversion input information is the reflection and transmission data corresponding to a point source. By applying the characteristic theory for hyperbolic equations, we establish an integral system from which $\rho(z)$ and $c(z)$ can be recovered simultaneously. In contrast to some known results, our inverse approach is carried out for depth variable, rather than for travel-time variable. Therefore inversion results in this paper are more appropriate for the physical interpretation of a medium slab.

Key words: inverse problem; wave equation; characteristic theory; integral equations

The problem of reconstructing the physical parameters of a layered-inhomogeneous medium from measurable data on the medium surface has received much attention. Let z be the Cartesian coordinate measuring medium depth and t the time. Then the 1-D hyperbolic system governing the wave motion in isotropic elastic layered-inhomogeneous medium with density $\rho(z)$ and lamé constants $\lambda(z), \mu(z)$ is[1]

$$\rho w_t + p_z = 0, \quad p_t + (\lambda + 2\mu) w_z = 0 \tag{1}$$

where $w(z, t)$ is the particle velocity and $p(z, t)$ the pressure. The characteristic wave speed associated with Eq. (1) is $c(z) = \sqrt{[\lambda(z) + 2\mu(z)]/\rho(z)}$.

By introducing the transformation $x = \int_0^z \frac{\mathrm{d}y}{c(y)}$, which maps depth variable z to travel-time variable x, Eq. (1) becomes

$$\zeta w_t + p_x = 0, \quad p_t + \zeta w_x = 0 \tag{2}$$

where $\zeta(x) := \rho(z) c(z)$ is the characteristic impedance.

For inverse problems based on Eq. (2) or other analogous equations, there are some works related to recovering single parameters from reflection data[1,2]. Indeed, $\zeta(x)$ can be determined uniquely from reflection data $w(0, t) = f(t)$. Research on transmission inverse problems has been very active[3-5]. However, it is impossible to reconstruct $\rho(z)$ and $c(z)$ simultaneously only from reflection data $f(t)$ [1] or transmission data. In order to reconstruct multiple media profiles clearly, more information should be added to inverse problems. In this paper, we use both the reflection data and the transmission data[6,7] to recover multiple parameters of the medium.

The aim of this paper is to reconstruct $\rho(z)$ and $c(z)$ simultaneously. There are some papers on reconstructing coefficients of the first derivative term in the equation[5,8]. Our work is concerned with the recovery of the coefficient of $\partial_z^2 w$, which has an influence on the characteristics of hyperbolic equations. By combining the characteristic propagation theory with singularity analysis, we establish an integral system from which both $\rho(z)$ and $c(z)$ can be recovered simultaneously.

1 Establishment of Inverse Problem Model

It is easy to know from Eq. (1) that

$$\frac{\partial^2 w}{\partial t^2} - c^2(z) \frac{\partial^2 w}{\partial z^2} - \frac{1}{\rho} \frac{\partial(\rho c^2)}{\partial z} \frac{\partial w}{\partial z} = 0 \tag{3}$$

For this equation in a slab media $z \in [0, l]$, we add the following initial-boundary conditions:

$$w_z(0, t) = w_z(l, t) = 0, \quad w(z, 0) = 0, \quad w_t(z, 0) = \delta(z)$$

to constitute a direct problem. In order to identify the unknowns $\rho(z)$ and $c(z)$, the following reflection and transmission data are taken as our inversion input data.

$$w(0, t) = f(t), \quad w(l, t) = g(t)$$

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Define

$$B(z):=-\frac{1}{\rho(z)}\frac{\partial(\rho c^2)}{\partial z},\quad w(z,t):=F(z)u(z,t) \tag{4}$$

with $F(z)$ to be determined. Substituting Eq. (4) into Eq. (3) and letting the coefficient of $\partial_z u$ be zero yields

$$B(z)F(z)-2c^2(z)F'(z)=0,\quad \frac{\partial^2 u}{\partial t^2}-c^2(z)\frac{\partial^2 u}{\partial z^2}+b(z)u=0 \tag{5}$$

where

$$b(z)=\frac{1}{4}\frac{B^2}{c^2}-\frac{1}{2}c^2\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{B}{c^2}\right) \tag{6}$$

If we can determine $(c(z), b(z))$ in Eq. (5), then $\rho(z)$ and $c(z)$ can be recovered uniquely from Eq. (4) and Eq. (6). Under the transform Eq. (4), we are led to the following inverse problem:

$$\frac{\partial^2 u}{\partial t^2}-c^2(z)\frac{\partial^2 u}{\partial z^2}+b(z)u=0 \qquad 0<z<l,\ t>0 \tag{7}$$

$$u_z-hu\,\big|_{z=0}=u_z+Hu\,\big|_{z=l}=0 \qquad t>0 \tag{8}$$

$$u\,\big|_{t=0}=0,\quad u_t\,\big|_{t=0}=\delta(z) \tag{9}$$

$$u(0,t)=f(t),\quad u(l,t)=g(t) \qquad t>0 \tag{10}$$

where h, H are known constants related to $\rho(z)$, $c(z)$ at $z=0, l$, respectively. We want to recover $(c(z), b(z))$ for $0<z<l$ from $(f(t), g(t))$.

The inverse problem constituted by Eq. (7) to Eq. (10) is treated by the following steps. We firstly consider the singularity propagation for forward problem Eq. (7) to Eq. (9) in section 2. The correspondent results will be used in section 3 and section 4. Then we establish a relationship between $f(t)$ and $(c(z), b(z))$ in section 3, while the relation between $g(t)$ and $(c(z), b(z))$ is obtained in section 4. Finally we simplify our inversion system for some special cases in section 5, which coincides with some well-known results.

2 Singularity Propagation of Forward Problem

The propagation of a singularity in semi-infinite media $z>0$ is well-known[1,2]. But in Eq. (7) to Eq. (9), we consider a slab with finite depth $0<z<l$ (see Fig. 1). Define $q(z)=\int_0^z\frac{\mathrm{d}y}{c(y)}$. It is easy to know that the characteristics of Eq. (7) are described by

$$\overset{\frown}{OS_1}: t=q(z) \qquad 0<z<l$$

$$\overset{\frown}{S_1S_2}: t=2q(l)-q(z) \qquad 0<z<l$$

$$\overset{\frown}{S_2S_3}: t=2q(l)+q(z) \qquad 0<z<l$$

and so on. Now we consider $u(x,t)$ in the domain

$$D_1=\{(z,t): 0<t<2q(l)+q(z), 0<z<l\}$$

Since u_t is of impulse singularity along characteristics, $u(z,t)$ has jump discontinuities between two sides of $\overset{\frown}{OS_1}$ and $\overset{\frown}{S_1S_2}$. The following approach to the determination of these jump discontinuities is standard[9] from the theory of characteristics of hyperbolic equations.

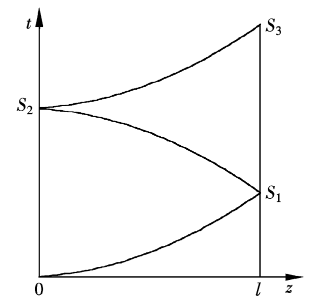

Fig. 1 Singularity propagation

Lemma 1 The jump discontinuities of $u(z,t)$ between two sides of $\overset{\frown}{OS_1}$ and $\overset{\frown}{S_1S_2}$ satisfy

$$\Big[u(z,t)\Big]_{t=2q(l)-q(z)-}^{t=2q(l)-q(z)+}=\Big[u(z,t)\Big]_{t=q(z)-}^{t=q(z)+}=\frac{1}{c(0)}\sqrt{\frac{c(z)}{c(0)}} \qquad 0<z<l$$

Proof Let $u(x,t)$ have the following decomposition: $u(z,t)=A^-(z)H(t-q(z))+A^+(z)H(t-2q(l)+q(z))+U(z,t)$, for $0<t<2q(l)+q(z)$, $0<z<l$. By the smoothness of $U(z,t)$ along its characteristics, we can determine $A_\pm(z)$ and then complete the proof of lemma 1.

3 Relation between $f(t)$ and $(c(z), b(z))$

Since the forward problem Eq. (7) to Eq. (9) is symmetric with respect to t, we extend its solution $u(z,t)$ by defining $u(z,t)=-u(z,-t)$ for $t\leqslant 0$ and $0<z<l$. In this case the reflection data $f(t)$ should also be extended in an odd way. Introduce two domains (see Fig. 2):

$$D_2=\{(z,t): |t|<2q(l)-q(z), 0<z<l\},\quad D_3=\{(z,t): |t|<q(t), 0<z<l\}$$

According to the singularity analysis in section 2, we define the following Cauchy problem:

$$\begin{cases} \dfrac{\partial^2 \bar{u}}{\partial t^2} - c^2(z)\dfrac{\partial^2 \bar{u}}{\partial z^2} + b(z)\bar{u} = 0 & z > 0, t \in \mathbf{R} \\ \bar{u}_z - h\bar{u} \mid_{z=0} = 0 & t \in \mathbf{R} \\ \bar{u}(0, t) = f(t) & t \in \mathbf{R} \end{cases}$$

Then it follows for $u(z, t)$ to satisfy Eq. (7) to Eq. (10) that

$$u(z, t) = \bar{u}(z, t) \qquad (z, t) \in D_2 \tag{11}$$

$$u(z, t) \equiv 0 \qquad (z, t) \in D_3 \tag{12}$$

Introduce the following problem:

$$\left.\begin{array}{ll} \dfrac{\partial^2 G}{\partial t^2} - c^2(z)\dfrac{\partial^2 G}{\partial z^2} + b(z)G = 0 & z > 0, t \in \mathbf{R} \\ G \mid_{z=0} = \delta(t), G_z \mid_{z=0} = h\delta(t) & t \in \mathbf{R} \end{array}\right\} \tag{13}$$

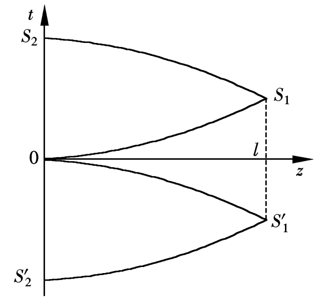

Fig. 2 Extension of solution

It is obvious that

$$G(z, t) \equiv 0 \qquad |t| > q(z), z > 0 \tag{14}$$

Similarly to the treatment in section 2, let

$$G(z, t) = a(z)[\delta(t - q(z)) + \delta(t + q(z))] + \tilde{G}(z, t) \qquad z > 0, t \in \mathbf{R} \tag{15}$$

where $a(z)$ is a function to be determined such that $\tilde{G}(z, t)$ has only jump discontinuities at characteristics $|t| = q(z)$ for $z > 0$. Inserting Eq. (15) into the first relation in Eq. (13) and taking the coefficient of $\delta'(t + q) - \delta'(t - q)$ as zero yield

$$2a'(z)q'(z) + a(z)q''(z) = 0 \qquad z > 0 \tag{16}$$

On the other hand, taking $z = 0$ in Eq. (15) and letting $\tilde{G}(0, t) = 0$ say $a(0) = 1/2$. Hence we get

$$a(z) = \frac{1}{2}\sqrt{\frac{c(z)}{c(0)}} \qquad 0 \leqslant z \leqslant l \tag{17}$$

Now it is easy to verify that $\bar{u}(z, t)$ can be expressed as

$$\bar{u}(z, t) = a(z)\Big[f(t - q(z)) + f(t + q(z))\Big] + \int_{-q(z)}^{q(z)} f(t - \tau)\tilde{G}(z, \tau)\,\mathrm{d}\tau \tag{18}$$

for $z > 0, t \in \mathbf{R}$. Especially, Eq. (18) becomes

$$0 = a(z)\Big[f(t - q(z)) + f(t + q(z))\Big] + \int_{-q(z)}^{q(z)} f(t - \tau)\tilde{G}(z, \tau)\,\mathrm{d}\tau \tag{19}$$

for $(z, t) \in D_3$. From lemma 1 and Eq. (12), we know $u(0, 0+) = 1/c(0)$. Then we get from Eq. (19) and the odd extension of $f(t)$ that $f(0+) - f(0-) = 2/c(0)$. By differentiating Eq. (19) with respect to t, we get

$$-a(z)[f'(t - q(z)) + f'(t + q(z))] = \frac{2}{c(0)}\tilde{G}(z, t) + \int_0^{q(z)} \Big[f'(t - \tau) + f'(t + \tau)\Big]\tilde{G}(z, \tau)\,\mathrm{d}\tau \tag{20}$$

for $0 < t < q(z)$. Since this equation gives the relation between $f(t)$ and $\bar{G}(z, t)$, now we establish the relation between $\tilde{G}(z, t)$ and $(c(z), b(z))$. From Eq. (13) to Eq. (17), we know that $\tilde{G}(z, t)$ satisfies

$$\left.\begin{array}{l} \dfrac{\partial^2 \tilde{G}}{\partial z^2} - \dfrac{1}{c^2(z)}\dfrac{\partial^2 \tilde{G}}{\partial t^2} - \dfrac{b(z)}{c^2(z)}\tilde{G} + \Big[a''(z) - \dfrac{a(z)b(z)}{c^2(z)}\Big][\delta(t - q(z)) + \delta(t + q(z))] = 0 \\ \tilde{G}\mid_{z=0} = 0, \tilde{G}_z \mid_{z=0} = [h - 2a'(0)]\delta(t) \end{array}\right\} \tag{21}$$

for $(z, t) \in \mathbf{R}^+ \times \mathbf{R}$.

We first give the following lemma describing the impulse strength of $\tilde{G}_t$ on characteristic $\widehat{OS_1}$.

Lemma 2 $\tilde{G}(z, t)$ has the following decomposition: $\tilde{G}_t(z, t) = E(z)\delta(t - q(z)) + \bar{G}_t(z, t)$ along $\widehat{OS_1}$, where $\bar{G}_t$ is of jump discontinuities on $t = q(z)$ and

$$E(z) = \sqrt{\frac{c(z)}{c(0)}}\left\{\int_0^z \sqrt{\frac{c(0)}{c(z)}}\,\frac{c^2(z)a''(z) - b(z)a(z)}{2c(z)}\mathrm{d}z + [2a'(0) - h]\,c(0)\right\}$$

Proof Assume $\tilde{G}(z, t) = E(z)H(t - q(z)) + \bar{G}(z, t)$, $t > 0$. By inserting this decomposition into Eq. (21), we get a boundary problem with respect to $\bar{G}(z, t)$. Letting the coefficient of $\delta(t - q(z))$ in the corresponding equation be zero and $\bar{G}_z(0, t)$ not contain the impulse singularity, we get an ordinary equation of the first order with respect to $E(z)$, which yields the expression for $E(z)$.

Now we consider the integral form of Eq. (21) by integrating along characteristics (see Fig. 3). First, draw characteristic line $\widehat{P_1P_2}$: $\tau_1 - t = q(\zeta_1) - q(z)$ through $P_1(z, t) \in D_3$, which intersects $\tau_1 = -q(\zeta_1)$ at point P_2. For $P_3(\zeta, \tau) \in \widehat{P_1P_2}$, draw another characteristic

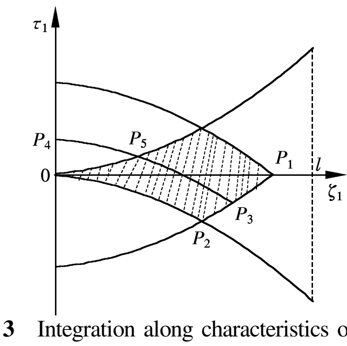

Fig. 3 Integration along characteristics on D_3

line $\widehat{P_3P_4}: \tau_1 - \tau = q(\zeta) - q(\zeta_1)$.

It is easy to know $\widehat{P_3P_4}$ intersects $\tau_1 = q(\zeta_1)$ at point $P_5(\zeta_1, \tau_1) = \left(q^{-1}\left(\frac{\tau + q(\zeta)}{2}\right), \frac{\tau + q(\zeta)}{2}\right)$. Inserting the decomposition in lemma 2 into Eq. (21) and integrating along $\widehat{P_3P_4}$ demonstrates

$$\frac{\partial\widetilde{G}(\zeta,\tau)}{\partial\zeta} + \frac{1}{c(\zeta)}\frac{\partial\widetilde{G}(\zeta,\tau)}{\partial\tau} + \int_0^\zeta \frac{c'(\zeta_1)}{c^2(\zeta_1)}\frac{\partial\widetilde{G}(\zeta_1,\tau_1)}{\partial\tau_1}\Big|_{\widehat{P_3P_4}} d\zeta_1 - \int_0^\zeta \frac{b(\zeta_1)}{c^2(\zeta_1)}\widetilde{G}(\zeta_1,\tau_1)\Big|_{\widehat{P_3P_4}} d\zeta_1 + \int_0^\zeta\left[a''(\zeta_1) - \frac{a(\zeta_1)b(\zeta_1)}{c^2(\zeta_1)}\right]\delta(\tau_1 - q(\zeta_1))\Big|_{\widehat{P_3P_4}} d\zeta_1 = 0 \tag{22}$$

for $P_3 \in D_3$. The first integral term can be computed by lemma 2. Since $\partial_t \widetilde{G}(z,t) \equiv 0$ for $t > q(z)$ and $\partial_t \widetilde{G}(z,t)$ has only jump discontinuities on $t = q(z)$, Eq. (22) takes the form

$$\frac{\partial\widetilde{G}(\zeta,\tau)}{\partial\zeta} + \frac{1}{c(\zeta)}\frac{\partial\widetilde{G}(\zeta,\tau)}{\partial\tau} + \int_0^\zeta F_1(\zeta_1)\delta(\tau_1 - q(\zeta_1))\Big|_{\widehat{P_3P_4}} d\zeta_1 + \int_{P_5(\zeta_1)}^\zeta F_2(\zeta_1,\tau_1)\Big|_{\widehat{P_3P_5}} d\zeta_1 = 0 \tag{23}$$

where

$$P_5(\zeta_1) = q^{-1}\left[\frac{\tau + q(\zeta)}{2}\right],\quad F_1(\zeta_1) = \frac{c'(\zeta_1)}{c^2(\zeta_1)}E(\zeta_1) + a''(\zeta_1) - \frac{a(\zeta_1)b(\zeta_1)}{c^2(\zeta_1)},\quad F_2(\zeta_1,\tau_1) = \frac{c'(\zeta_1)}{c^2(\zeta_1)}\frac{\partial\widetilde{G}(\zeta_1,\tau_1)}{\partial\tau_1} - \frac{b(\zeta_1)}{c^2(\zeta_1)}\widetilde{G}(\zeta_1,\tau_1) \tag{24}$$

Now integrating Eq. (23) again along $\widehat{P_2P_1}$ gets (see Fig. 3)

$$\widetilde{G}(z,t) - \widetilde{G}(P_2+) + \frac{1}{2}\int_{q^{-1}\left(\frac{\tau+q(\zeta)}{2}\right)}^z F_1(\zeta_2(\zeta))c(\zeta_2(\zeta))d\zeta + \iint_{D(z,t)} F_2(\zeta_1,\tau_1)d\zeta_1 d\tau_1 = 0$$

where

$$\zeta_2(\zeta) = q^{-1}\left[q(\zeta) + \frac{\tau - q(z)}{2}\right],\qquad \widetilde{G}(P_2+) = \widetilde{G}\left(q^{-1}\left(\frac{Q(z) - t}{2}\right), \frac{t - q(z)}{2} +\right)$$

and $D(z,t)$ is the shaded region in Fig. 3. By letting $t \to q(z)$, the above equality becomes

$$\widetilde{G}(z, q(z)) + \frac{1}{2}\int_0^z F_1(\zeta)c(\zeta)d\zeta = 0$$

from Eq. (21), which implies

$$\frac{c'(z)}{c(z)}E(z) + c(z)a''(z) - \frac{a(z)b(z)}{c(z)} = -2\frac{d}{dz}\widetilde{G}(z, q(z))$$

due to Eq. (24). This relation can be reduced to $E'(z) = -\frac{d}{dz}\widetilde{G}(z,q(z))$ from lemma 2, which tells us

$$\frac{b(z)a(z) - c^2(z)a''(z)}{c(z)} + (\ln c(z))'[\widetilde{G}(z,q(z)) - (2a'(0) - h)c(0)] = 2\frac{d}{dz}\widetilde{G}(z,q(z)) \tag{25}$$

again in terms of lemma 2, where $\widetilde{G}(z,q(z))$ means $\widetilde{G}(z,q(z)-)$. By combining Eq. (20) and Eq. (25) together, we get the relation between reflection data $f(t)$ and $(c(z), b(z))$. If $c(z) \equiv 1$, then we can determine $b(z)$ from Eq. (25). In this case, reflection data is enough for recovering $b(z)$. In our case, both $c(z)$ and $b(z)$ are unknown, hence we must use transmission data $g(t)$ to establish another relation.

4 Relation between $g(t)$ and $(c(z), b(z))$

It is easy to know $g(t) \equiv 0$ for $0 < t < q(l)$. Hence we consider Eq. (7) on the domain $D_4 = \{(z,t): 2q(l) - q(z) < t < 2q(l) + q(z), 0 < z < l\}$, see Fig. 4. As mentioned above, $u(z,t)$ has jump discontinuities between two sides of $\widehat{S_1S_2}$. Assume two characteristics $\frac{d\tau}{d\zeta} = \pm q'(\zeta)$ through $M_0(z,t) \in D_4$ intersect $\widehat{S_1S_2}$ and $\widehat{S_1S_3}$ at points M_1 and M_2, respectively. The coordinates of these points are $M_1\left(q^{-1}\left(q(l) + \frac{q(z) - t}{2}\right), q(l) + \frac{t - q(z)}{2}\right)$, $M_2(l, t + q(z) - q(l))$ and $M_3(l, \tau + q(\zeta) - q(l))$. Similarly to the treatment in getting Eq. (22), integrating Eq. (7) along $\widehat{M_4M_3}$ yields

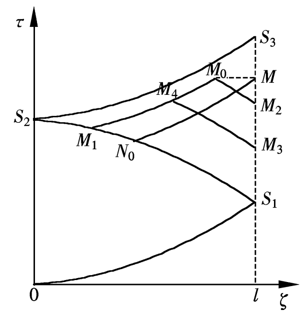

Fig. 4 Integration along characteristics on D_4

$$\frac{\partial u(\zeta,\tau)}{\partial\zeta} + \frac{1}{c(\zeta)}\frac{\partial u(\zeta,\tau)}{\partial\tau} - \left[\frac{1}{c(l)}g'(\tau + q(\zeta) - q(l)) - Hg(\tau + q(\zeta) - q(l))\right] + \int_l^\zeta F_3(\zeta_1, \tau - q(\zeta_1) + q(\zeta))d\zeta_1 = 0$$

where

$$F_3(\zeta_1,\tau_1) = \frac{c'(\zeta_1)}{c^2(\zeta_1)}\frac{\partial u(\zeta_1,\tau_1)}{\partial\zeta_1} - \frac{b(\zeta_1)}{c^2(\zeta_1)}u(\zeta_1,\tau_1) \tag{26}$$

Integrating the above relation again along $\widehat{M_1M_0}$ generates

$$u(z,t)-u(M_1)-\int_{\frac{q(z)+t}{2}}^{t}\left[\frac{1}{c(l)}g'(\tau+q(\zeta)-q(l))-Hg(\tau+q(\zeta)-q(l))\right]_{\widehat{M_1M_0}}\mathrm{d}\tau+\int_{\frac{q(z)+t}{2}}^{t}\mathrm{d}\tau\left[\int_{l}^{\zeta}F_3(\zeta_1,\tau-q(\zeta_1)+q(\zeta))\mathrm{d}\zeta_1\right]_{\widehat{M_1M_0}}=0 \quad (27)$$

where $u(M_1)$ is evaluated on the upper-side of $\widehat{S_1S_2}$, while the second integral term is in fact the integration of function $F_3(\zeta,\tau)$ on the domain (see Fig. 4).

$$Q(z,t)=\left\{(\zeta,\tau):2q(l)-q(\zeta)<\tau<t-|q(z)-q(\zeta)|,\ q^{-1}\left(q(l)+\frac{q(z)-t}{2}\right)<\zeta<l\right\}$$

The simplification of Eq. (27) results in

$$u(z,t)-u(M_1)+\frac{1}{2c(l)}\left[g(2q(z)-q(l))-g(t+q(z)-q(l))\right]+\frac{1}{2}H\int_{q(z)+t}^{2t}g(\tau-t+q(z)-q(l))\mathrm{d}\tau+\iint_{Q(z,t)}F_3(\zeta,\tau)\mathrm{d}\zeta\mathrm{d}\tau=0 \quad (28)$$

for $(z,t)\in D_4$. By substituting Eq. (26) into Eq. (28), we have

$$u(z,t)-u(M_1)+\frac{1}{2c(l)}\left[g(2q(z)-q(l))-g(t+q(z)-q(l))\right]+\frac{1}{2}H\int_{q(z)+t}^{2t}g(\tau-t+q(z)-q(l))\mathrm{d}\tau+\int_{q^{-1}\left(q(l)+\frac{q(z)-t}{2}\right)}^{l}\frac{c'(\zeta_1)}{c^2(\zeta_1)}\left[u(\zeta_1,t-|q(\zeta_1)-q(z)|)-u(\zeta_1,2q(l)-q(\zeta_1))\right]\mathrm{d}\zeta_1-\iint_{Q(z,t)}\frac{b(\zeta_1)}{c^2(\zeta_1)}u(\zeta_1,\tau_1)\mathrm{d}\zeta_1\mathrm{d}\tau_1=0 \quad (29)$$

which establishes the relation between $g(t)$ and $(u(\zeta_1,\ \tau_1),b(\zeta_1),c(\zeta_1))$ on domain D_4.

On the other hand, assume that characteristic line $\frac{\mathrm{d}\tau}{\mathrm{d}\zeta}=q'(\zeta)$ through $M(l,t)\in\widehat{S_1S_3}$ intersects $\widehat{S_1S_2}$ at point $N_0\left(q^{-1}\left(\frac{3q(l)-t}{2}\right),\frac{q(l)+t}{2}\right)$. Integrating Eq. (7) along $\widehat{MN_0}$ tells us

$$Hg(t)+\frac{1}{c(l)}g'(t)+\left[\frac{\partial u}{\partial\zeta}-\frac{1}{c(\zeta)}\frac{\partial u}{\partial\tau}\right]_{N_0}+\int_{q^{-1}\left(\frac{3q(l)-t}{2}\right)}^{l}F_3(\zeta,t+q(\zeta)-q(l))\mathrm{d}\zeta=0 \quad (30)$$

by virtue of Eq. (8) and Eq. (10), where $[\cdot]_{N_0}$ also is evaluated on the upper-side of $\widehat{S_1S_2}$.

Let $q^{-1}\left(\frac{3q(l)-t}{2}\right)=z$ in Eq. (30) for $q(l)<t<3q(l)$, then the coordinate of N_0 is $N_0(z,2q(l)-q(z))$ and it holds that

$$\frac{1}{c(l)}g'(3q(l)-2q(z))+Hg(3q(l)-2q(z))+\frac{\mathrm{d}}{\mathrm{d}z}u(z,2q(l)-q(z)+)+\int_{z}^{l}F_3(\zeta,2q(l)-q(z)+q(\zeta))\mathrm{d}\zeta=0 \qquad 0<z<l \quad (31)$$

since

$$\frac{\mathrm{d}}{\mathrm{d}z}u(z,2q(l)-q(z)+)=\left[\frac{\partial u}{\partial\zeta}+\frac{\partial u}{\partial\tau}\left(-\frac{1}{c(\zeta)}\right)\right]_{N_0}$$

Eq. (29) and Eq. (31) contain $u(M)$ evaluating on the upper-side of $\widehat{S_1S_2}$. For any point $P(z,2q(l)-q(z))\in\widehat{S_1S_2}$, according to lemma 1 we get

$$u(z,2q(l)-q(z)+)=u(z,2q(l)-q(z)-)+\frac{1}{c(0)}\sqrt{\frac{c(z)}{c(0)}} \qquad 0<z<l$$

Furthermore, it can be replaced by

$$u(z,2q(l)-q(z)+)=\frac{1}{c(0)}\sqrt{\frac{c(z)}{c(0)}}+a(z)[f(2q(l)-2q(z))+f(2q(l))]+\int_{-q(z)}^{q(z)}f(2q(l)-q(z)-\tau)\tilde{G}(z,\tau)\mathrm{d}\tau \quad (32)$$

from Eq. (11) and Eq. (18) for $0<z<l$.

By combining Eq. (20), Eq. (25), Eq. (29), Eq. (31) and Eq. (32) together, we get a set of equations. These equations contain the following unknown functions ($a(z)$ can be expressed by $c(z)$): ① $(c(z),b(z))$ for $0<z<l$; ② $\tilde{G}(z,t)$ for $|t|<q(z)$, $0<z<l$; ③ $u(z,t)$ for $(z,t)\in D_4$; ④ $u(z,2q(l)-q(z)+)$ for $0<z<l$. Here Eq. (20), Eq. (25), Eq. (29), Eq. (31) and Eq. (32) constitute a closed system from which $c(z)$ and $b(z)$ can be reconstructed simultaneously by numerical implementation.

5 Discussions

This paper is concerned with the inverse problem of reconstructing $\rho(z)$ and $c(z)$ simultaneously. Both reflection and transmission data are required for our inversion approach. However, if one of these two parameters is prescribed in advance, our inversion can be greatly simplified and coincides with some known inversion methods. In the case of only one unknown parameter, either reflection data or transmission data is enough to determine $\rho(z)$ or $c(z)$ uniquely. Starting from inverse problem Eq. (7) to Eq. (10) directly, we give two examples.

Case 1 $c(z)$ is prescribed and we take reflection data $f(t)$ as inversion input. This inversion problem, determining $b(z)$ from $f(t)$, has been discussed thoroughly. Since $c(z)$ is known, we can convert wave velocity to 1 for travel-time variable x. Therefore, without loss of generality, we assume $c(z)\equiv 1$ directly. Eq. (7) to Eq. (10) lead to

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial z^2}+b(z)u=0 \qquad 0<z<l,\ t>0 \tag{33}$$

$$u_z-hu\mid_{z=0}=u_z+Hu\mid_{z=l}=0 \qquad t>0 \tag{34}$$

$$u\mid_{t=0}=0,\quad u_t\mid_{t=0}=\delta(z) \tag{35}$$

$$u(0,t)=f(t) \qquad t>0 \tag{36}$$

In this case, Eq. (20) and Eq. (25) in section 3 can be simplified as

$$\begin{cases}\dfrac{1}{2}[f'(t-z)+f'(t+z)]+2\tilde{G}(z,t)+\displaystyle\int_0^z[f'(t-\tau)+f'(t+\tau)]\tilde{G}(z,\tau)\mathrm{d}\tau=0\\ b(z)=4\dfrac{\mathrm{d}}{\mathrm{d}z}\tilde{G}(z,z)\end{cases}$$

For known reflection data $f(t)$, we can solve $\tilde{G}(z,t)$ for $0<t<z<l$ from the first equation. Then $b(z)$ can be determined. The boundary condition at $z=l$, however, has always no influence on $b(z)$. This fact corresponds to the following well-known conclusion: reflection data $f(t)$ for $0<t<2l$ is enough to determine $b(z)$ for $0<z<l$.

Case 2 $c(z)$ is known, the inversion input is transmission data $g(t)$. We also assume $c(z)\equiv 1$. In this case, Eq. (36) is replaced by

$$u(l,t)=g(t) \qquad l<t<3l \tag{37}$$

we want to recover $b(z)$ in terms of Eq. (33) to Eq. (35) and Eq. (37). A similar problem is considered in Ref. [3]. From Ref. [3], we know $b(z)$ can be recovered from transmission data $g(t)$.

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利用透射和反射数据反演声波传播的介质密度和波速

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摘要: 研究点源作用下线性波动方程多个系数的反演问题, 其中介质密度 $\rho(z)$ 和波速 $c(z)$ 为待求量. 通过波动方程的特征理论, 利用点源产生的反射数据和透射数据, 建立了同时反演密度$\rho(z)$和波速 $c(z)$的封闭积分系统. 与已有的结果相比, 本文求解反问题是直接在深度变量 z 而非传输时间变量 x 下进行的, 因此更有助于结果的物理解释.

关键词: 反问题; 波动方程; 特征理论; 积分方程

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