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Homological properties of modules characterized by matrices

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Abstract: Some homological properties of R-modules were investigated by matrices over a ring R. Given two cardinal numbers α, β and an $\alpha \times \beta$ row-finite matrix $\boldsymbol{A}$, it was proved that $\mathrm{Ext}^1_R(R^{(\alpha)}/R^{(\beta)}\boldsymbol{A}, M) = 0$ if and only if $M_\alpha / \mathrm{r}_{M_\alpha}(R^{(\beta)}\boldsymbol{A}) \cong \mathrm{Hom}_R(R^{(\beta)}\boldsymbol{A}, M)$ if and only if $\mathrm{r}_{M_\beta}\mathrm{l}_{R^{(\beta)}}(\boldsymbol{A}) = \boldsymbol{A}M_\alpha$. Thus, the notion of (m, n)-injectivity was extended. Moreover, (α, β)-flatness was characterized via annihilators of matrices, factorizations of homomorphisms as well as homological groups so that (m, n)-flat modules, f-projective modules and n-projective modules were consolidated under the notion of (α, β)-flat modules. Furthermore, a characterization of left R-ML modules and some equivalent conditions for $R^{(\beta)}\boldsymbol{A}$ to be left R-ML were presented. Consequently, the notions of coherent rings, (m, n)-coherent rings and π-coherent rings were consolidated under that of (α, β)-coherent rings.

Key words: (α, β)-injective module; (α, β)-flat module; R-ML module; (α, β)-coherent ring

(m, n)-injective modules were discussed by Chen et al. in Ref. [1]. Given m and $n \in \mathbf{N}$ (the set of natural numbers), a right R-module M is called (m, n)-injective if every right R-homomorphism from an n-generated submodule of R^m to M extends to one from R^m to M. This definition unifies several definitions on generalizations of injectivity of modules, such as f-injective modules, P-injective modules and FP-injective modules. Many characterizations of (m, n)-injective modules were given and the notions of (m, n)-flat modules as well as (m, n)-coherent rings were introduced in Ref. [2].

Finite projectivity of modules was originally investigated by Simon[3] under the terminology of $\aleph_{-1}$-projectivity and was studied in Refs. [4, 5]. The concept of locally projective modules was introduced by Gruson and Raynaud[6].

It is well known that

$$\text{injective} \Rightarrow \begin{cases} \text{FP-injective} = (m, n)\text{-injective } (\forall m, n \in \mathbf{N}) \Rightarrow \text{P-injective} = (1, 1)\text{-injective} \\ \text{f-injective} = (1, n)\text{-injective } (\forall n \in \mathbf{N}) \Rightarrow \text{P-injective} = (1, 1)\text{-injective} \end{cases}$$

and

$$\text{projective} \Rightarrow \text{locally projective} \Rightarrow \text{finitely projective} \Rightarrow \text{flat} = (m, n)\text{-flat } (\forall m, n \in \mathbf{N})$$

But none of the above implications is invertible in general.

The purpose of the present discussion is to supersede m and $n \in \mathbf{N}$ by two (possibly infinite) cardinal numbers α and β when investigating some homological properties of modules so that some known results can be extended.

Throughout R is an associative ring with identity and all modules are unitary. α and β are two fixed cardinal numbers (unless specified otherwise). We write $M_R({}_RM)$ to indicate a right (left) R-module and $M_{(\alpha)}(M^{(\alpha)})$ to indicate the direct sum of α copies of $M_R({}_RM)$, while the direct product of α copies of $M_R({}_RM)$ is denoted as $M_\alpha(M^\alpha)$. Elements in $M_{(\alpha)}(M^{(\alpha)})$ are regarded as column (row) vectors and elements in $M_\alpha(M^\alpha)$ are regarded similarly. Thus, matrix product may freely be adopted. For example, given $a \in R^{(\beta)}$, $\beta \times \alpha$ row-finite matrix $\boldsymbol{A}$ over R, and $x \in M_\alpha$, we may define $a\boldsymbol{A} \in R^{(\alpha)}$ and $\boldsymbol{A}x \in M_\beta$ as usual. Hence $R^{(\beta)}\boldsymbol{A}$ may stand for the set $\{a\boldsymbol{A} \in R^{(\alpha)} \mid a \in R^{(\beta)}\}$. For a left R-module M, $\mathrm{r}_{M_\alpha}(R^{(\beta)}\boldsymbol{A})$ denotes the right annihilator of $R^{(\beta)}\boldsymbol{A}$ in M_α. Similarly, $\mathrm{l}_{R^{(\beta)}}(\boldsymbol{A})$ is the left annihilator of $\boldsymbol{A}$ in $R^{(\beta)}$.

Theorem 1 Let α and β be two cardinal numbers and $\boldsymbol{A}$ a $\beta \times \alpha$ row-finite matrix over R. The following are equivalent for a left R-module M:

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① Every left R-homomorphism from $R^{(\beta)}A \subseteq R^{(\alpha)}$ to M extends to one from $R^{(\alpha)}$ to M;

② $\mathrm{Ext}_R^1(R^{(\alpha)}/R^{(\beta)}A, M) = 0$;

③ Every short exact sequence $0 \to {}_RM \to {}_RL \to R^{(\alpha)}/R^{(\beta)}A \to 0$ splits;

④ If the sequence $0 \to {}_RK \to {}_RL \to R^{(\alpha)}/R^{(\beta)}A \to 0$ is exact then any R-homomorphism $f: {}_RK \to {}_RM$ extends to ${}_RL$;

⑤ $\varphi: \mathrm{Hom}_R(R^{(\alpha)}/R^{(\beta)}A, E) \to \mathrm{Hom}_R(R^{(\alpha)}/R^{(\beta)}A, E/M)$ derived from the canonical map $\pi: E \to E/M$ is an epimorphism, where E is the injective envelope of M;

⑥ $\sigma: M_\alpha/\mathrm{r}_{M_\alpha}(R^{(\beta)}A) \to \mathrm{Hom}_R(R^{(\beta)}A, M)$ derived from the canonical isomorphism $M_\alpha \cong \mathrm{Hom}_R(R^{(\alpha)}, M)$ is an isomorphism;

⑦ For any $f \in \mathrm{Hom}_R(R^{(\beta)}A, M)$, if (g, h) is the pushout of (f, i) in the following diagram (where i is the inclusion map)

$$\begin{array}{ccc} R^{(\beta)}A & \xrightarrow{i} & R^{(\alpha)} \\ \downarrow f & & \downarrow h \\ M & \xrightarrow{g} & L \end{array}$$

there exists a homomorphism $\varphi: L \to M$ such that $\varphi \circ g = 1_M$;

⑧ $\mathrm{r}_{M_\beta}\mathrm{l}_{R^{(\beta)}}(A) = AM_\alpha$.

Proof ① ⇔ ② ⇔ ③ ⇔ ④, ② ⇔ ⑤ and ③ ⇔ ⑦ are trivial.

① ⇔ ⑧. Note that $\mathrm{Hom}_R(R^{(\beta)}A, M) \cong \{x \in M_\beta \mid \mathrm{l}_{R^{(\beta)}}(A) \subseteq \mathrm{l}_{R^{(\beta)}}(x)\} = \mathrm{r}_{M_\beta}\mathrm{l}_{R^{(\beta)}}(A)$ and $\mathrm{Hom}_R(R^{(\alpha)}, M) \cong M_\alpha$ as abelian groups.

② ⇔ ⑥. Let $I = R^{(\beta)}A \subseteq R^{(\alpha)} = F$. Then the result follows by the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & \mathrm{r}_{M_\alpha}(I) & \longrightarrow & M_\alpha & \longrightarrow & M_\alpha/\mathrm{r}_{M_\alpha}(I) & \to 0 & & \\ & \cong\downarrow & & \cong\downarrow & & \sigma\downarrow & & & \\ 0 \to & \mathrm{Hom}_R(F/I, M) & \to & \mathrm{Hom}_R(F, M) & \to & \mathrm{Hom}_R(I, M) & \to & \mathrm{Ext}_R^1(F/I, M) & \to 0 \end{array}$$

Definition 1 Let α and β be two cardinal numbers. A left R-module M is called (α, β)-injective if it satisfies the equivalent conditions in theorem 1 for every $\beta \times \alpha$ row-finite matrix A over R.

Remark 1 ① It is easy to see that (α, β)-injectivity coincides with (m, n)-injectivity[1] in the case $\alpha = m$ and $\beta = n \in \mathbf{N}$. Thus some known results on (m, n)-injective (respectively, P-injective, n-injective, f-injective and FP-injective) modules in Refs. [1, 2] can be obtained as corollaries of theorem 1.

② From Baer's criterion of injectivity one can see that a left R-module is injective if and only if it satisfies the equivalent conditions in theorem 1 for every $|R| \times 1$ matrix A over R.

Proposition Let A be a $\beta \times \alpha$ row-finite matrix over R, $i: R^{(\beta)}A \to R^{(\alpha)}$ the inclusion map and $\pi: M \to M/K$ with ${}_RK \leqslant {}_RM$. Suppose that $\mathrm{Hom}_R(i, M)$ is an epimorphism, i. e., $\mathrm{r}_{M_\beta}\mathrm{l}_{R^{(\beta)}}(A) \subseteq AM_\alpha$. Then the following are equivalent:

① $\mathrm{Hom}_R(i, K)$ is an epimorphism, i. e., $\mathrm{r}_{K_\beta}\mathrm{l}_{R^{(\beta)}}(A) \subseteq AK_\alpha$;

② $\mathrm{Hom}_R(R^{(\alpha)}/R^{(\beta)}A, \pi)$ is an epimorphism;

③ $\{x \in M_\alpha \mid R^{(\beta)}Ax \subseteq K\} = K_\alpha + \mathrm{r}_{M_\alpha}(R^{(\beta)}A)$.

Proof ① ⇔② is straightforward.

① ⇒③. $\{x \in M_\alpha \mid R^{(\beta)}Ax \subseteq K\} \supseteq K_\alpha + \mathrm{r}_{M_\alpha}(R^{(\beta)}A)$ always holds. Suppose ① holds. Then, for any $x \in M_\alpha$ with $R^{(\beta)}Ax \subseteq K$, it is easy to see that $Ax \in \mathrm{r}_{K_\beta}\mathrm{l}_{R^{(\beta)}}(A)$ and hence $Ax = Ay$ for some $y \in K_\alpha$. Thus $x = y + (x - y) \in K_\alpha + \mathrm{r}_{M_\alpha}(R^{(\beta)}A)$.

③ ⇒①. For any $x \in \mathrm{r}_{K_\beta}\mathrm{l}_{R^{(\beta)}}(A)$, we have $x \in \mathrm{r}_{M_\beta}\mathrm{l}_{R^{(\beta)}}(A)$ and hence $x = Ay$ for some $y \in M_\alpha$. It follows that $R^{(\beta)}Ay \subseteq K$. Thus $y = y_1 + y_2$ with $y_1 \in K_\alpha$ and $y_2 \in \mathrm{r}_{M_\alpha}(R^{(\beta)}A)$. Therefore $x = Ay = Ay_1 + Ay_2 = Ay_1 \in AK_\alpha$.

Remark 2 One can deduce some equivalent conditions, under which a submodule of an (α, β)-injective (respectively, P-injective, n-injective, f-injective and FP-injective) module is (α, β)-injective (respectively, P-injective, n-injective, f-injective and FP-injective) from the above proposition.

Now let N be a right R-module and A a $\beta \times \alpha$ matrix over R. Suppose the i-th row of A is $a_i = (a_{ij})$ and $e_i \in$

$R^{(\beta)}$ with 1 in the i-th position and 0 elsewhere. Define f: $R^{(\beta)} \to R^{(\beta)}\boldsymbol{A}$ such that $f(e_i) = a_i$ (for all i), then f is an epimorphism of left R-modules with Ker $f = \mathrm{l}_{R^{(\beta)}}(\boldsymbol{A})$. Let τ: $\mathrm{l}_{R^{(\beta)}}(\boldsymbol{A}) \to R^{(\beta)}$ be the inclusion map. We have an exact sequence

$$N \otimes \mathrm{l}_{R^{(\beta)}}(\boldsymbol{A}) \xrightarrow{N\otimes\tau} N \otimes R^{(\beta)} \xrightarrow{N\otimes f} N \otimes R^{(\beta)}\boldsymbol{A} \to 0$$

Consequently, for each $x = (x_i) \in N^{(\beta)}$ with $\sum_i (x_i \otimes a_i) = 0$ in $N \otimes R^{(\beta)}\boldsymbol{A}$, it follows that $\sum_i (x_i \otimes e_i) \in \mathrm{Ker}(N \otimes f) = \mathrm{Im}(N \otimes \tau)$. Thus we have the following lemma.

Lemma For each $x = (x_i) \in N^{(\beta)}$ with $\sum_i (x_i \otimes a_i) = 0$ in $N \otimes R^{(\beta)}\boldsymbol{A}$, there is a positive integer k, $y \in N^k$ and $k \times \beta$ row-finite matrix $\boldsymbol{C}$ over R such that $\boldsymbol{CA} = \boldsymbol{0}$ and $x = y\boldsymbol{C}$.

For each $x \in N$, $a \in R^{(\beta)}$ and $b \in R_{(\alpha)}$, by usual multiplication of matrices, we have $xa\boldsymbol{A} \in N^{\alpha}$ and $xa\boldsymbol{A}b \in N$. Then, by the above lemma, there is a canonical map μ: $N \otimes R^{(\beta)}\boldsymbol{A} \to N^{\alpha}$ defined via $\mu(x \otimes a\boldsymbol{A}) = xa\boldsymbol{A}$ for each $x \in N$ and $a \in R^{(\beta)}$. Simultaneously, there is a canonical map ν: $N \otimes R^{(\beta)}\boldsymbol{A} \to \mathrm{Hom}_R(\boldsymbol{A}R_{(\alpha)}, N)$ such that $[\nu(x \otimes a\boldsymbol{A})](\boldsymbol{A}b) = xa\boldsymbol{A}b$ for all $x \in N$, $a \in R^{(\beta)}$ and $b \in R_{(\alpha)}$. It is easy to see that $\nu(x \otimes a\boldsymbol{A}) = 0$ if and only if $xa\boldsymbol{A} = 0$.

Theorem 2 Given two cardinal numbers α and β, a $\beta \times \alpha$ matrix $\boldsymbol{A}$ over R and a right R-module N_R. Let $S = \{f \in \mathrm{Hom}_R(R_\beta/\boldsymbol{A}R_{(\alpha)}, N_R) \mid \exists x \in N^{(\beta)}$ such that $f(a + \boldsymbol{A}R_{(\alpha)}) = xa$ for all $a \in R_\beta\}$. The following statements are equivalent:

① The canonical map μ: $N \otimes R^{(\beta)}\boldsymbol{A} \to N^{\alpha}$ is a monomorphism;

② The canonical map ν: $N \otimes R^{(\beta)}\boldsymbol{A} \to \mathrm{Hom}_R(\boldsymbol{A}R_{(\alpha)}, N)$ is a monomorphism;

③ For each $x \in \mathrm{l}_{N^{(\beta)}}(\boldsymbol{A})$, there is a positive integer k, $y \in N^k$ and $k \times \beta$ row-finite matrix $\boldsymbol{C}$ over R such that $\boldsymbol{CA} = \boldsymbol{0}$ and $x = y\boldsymbol{C}$;

④ For each $f \in S$, there is a finitely generated free (or projective) right R-module F such that $f = f_2 \circ f_1$ for some $f_1 \in \mathrm{Hom}_R(R_\beta/\boldsymbol{A}R_{(\alpha)}, F_R)$ and $f_2 \in \mathrm{Hom}_R(F_R, N_R)$;

⑤ For each $f \in S$, if φ: $L_R \to N_R$ is an epimorphism, there exists $g \in \mathrm{Hom}_R(R_\beta/\boldsymbol{A}R_{(\alpha)}, L_R)$ such that Img is contained in a finitely generated submodule of L and $f = \varphi \circ g$.

In the case that $\boldsymbol{A}$ is row-finite, the above conditions are equivalent to

⑥ $\mathrm{Ext}_R^1(R^{(\alpha)}/R^{(\beta)}\boldsymbol{A}, N^+) = 0$, where $N^+ = \mathrm{Hom}_Z(N, \mathbf{Q}/\mathbf{Z})$;

⑦ $\mathrm{Tor}_R^1(N, R^{(\alpha)}/R^{(\beta)}\boldsymbol{A}) = 0$.

Proof ① ⇔②⇔③ follows by the preceding lemma. ④⇔⑤ is easy.

③ ⇔④. Note that $S \cong \mathrm{l}_{N^{(\beta)}}(\boldsymbol{A})$ as abelian groups.

If $\boldsymbol{A}$ is row-finite, $R^{(\beta)}\boldsymbol{A} \subseteq R^{(\alpha)}$ and Im $\mu \subseteq N^{(\alpha)}$. Then ①⇔⑥ and ①⇔⑦ are clear.

Suppose that $\beta = n$ is a positive integer and α is a cardinal number. Given an $n \times \alpha$ row-finite matrix $\boldsymbol{A}$ over R and a right R-module N, we define φ: $N \otimes \mathrm{l}_{R^n}(\boldsymbol{A}) \to \mathrm{Hom}_R(R_n/\boldsymbol{A}R_{(\alpha)}, N)$ such that $[\varphi(x \otimes a)](b + \boldsymbol{A}R_{(\alpha)}) = xab$ for all $x \in N$, $a \in R^n$ and $b \in R_n$. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & N \otimes \mathrm{l}_{R^n}(\boldsymbol{A}) & \longrightarrow & N \otimes R^n & \longrightarrow & N \otimes R^n\boldsymbol{A} \to 0 \\ & & \downarrow\varphi & & \downarrow\cong & & \downarrow\nu \\ 0 & \to & \mathrm{Hom}_R(R_n/\boldsymbol{A}R_{(\alpha)}, N) & \to & \mathrm{Hom}_R(R_n, N) & \to & \mathrm{Hom}_R(\boldsymbol{A}R_{(\alpha)}, N) \end{array}$$

where $\mathrm{l}_{R^n}(\boldsymbol{A}) \cong \mathrm{Hom}_R(R_n/\boldsymbol{A}R_{(\alpha)}, R)$. Note that ν is a monomorphism if and only if φ is an epimorphism. So the equivalence of ① and ② in theorem 2 yields the following corollary.

Corollary 1 Given a positive integer n, a cardinal number α, an $n \times \alpha$ row-finite matrix $\boldsymbol{A}$ over R. Let $P = R_n/\boldsymbol{A}R_{(\alpha)}$ and $P^* = \mathrm{Hom}_R(P, R_R)$. The following statements are equivalent for a right R-module N_R:

① The canonical map μ: $N \otimes R^n\boldsymbol{A} \to N^{\alpha}$ is a monomorphism;

② The canonical map φ: $N \otimes \mathrm{l}_{R^n}(\boldsymbol{A}) \to \mathrm{Hom}_R(R_n/\boldsymbol{A}R_{(\alpha)}, N)$ is an epimorphism;

③ The canonical map σ: $N \otimes P^* \to \mathrm{Hom}_R(P, N_R)$ such that $[\sigma(x \otimes g)](z) = xg(z)$ ($\forall x \in N$, $g \in P^*$ and $z \in P$) is an epimorphism.

Definition 2 Let α and β be two cardinal numbers. A right R-module N is called (α, β)-flat if it satisfies the equivalent conditions ① to ⑤ in theorem 2 for every $\beta \times \alpha$ matrix $\boldsymbol{A}$ over R.

Remark 3 ① It is easy to see that (α, β)-flatness coincides with (m, n)-flatness[2] in the case $\alpha = m$ and $\beta = n \in \mathbf{N}$.

② For a positive integer n, it is easy to see that a right R-module N is n-projective[5] if and only if it satisfies the equivalent conditions in theorem 2 and corollary 1 for all cardinal numbers α and every $n\times\alpha$ matrix $\boldsymbol{A}$ over R. Furthermore, N is f-projective if and only if it is n-projective for all positive integers n.

③ As an immediate consequence of theorem 2 and corollary 1, we can obtain corresponding characterizations of, respectively, (m,n)-flat modules, flat modules, n-projective modules and f-projective modules.

Recall that a right R-module M is said to be R-Mittag Leffler (R-ML) in the case the canonical map $\mu_{M,I}: M\otimes R^I\to M^I$ defined via $\mu_{M,I}(x\otimes\{r_i\})=\{xr_i\}$ is a monomorphism for every set I[7]. Left R-ML modules can be defined similarly. It is well-known that M is R-ML if and only if, for every finitely generated submodule N of M, the inclusion map factors through a finitely presented module. Moreover, M is finitely presented if and only if M is finitely generated and R-ML. M is f-projective if and only if M is flat and R-ML.

Theorem 3 Let M be a right R-module and $\pi: F_R\to M_R$ an epimorphism with $\mathrm{Ker}\pi=K$, where F is free. Then the following are equivalent:

① M is R-ML;

② For any $n\in\mathbf{N}$ and any $x\in M^n$, there exists $y\in M^m$, $m\times k$ matrix $\boldsymbol{A}$ and $m\times n$ matrix $\boldsymbol{B}$ over R for some m, $k\in\mathbf{N}$ such that $x=y\boldsymbol{B}$, $y\boldsymbol{A}=\boldsymbol{0}$ and $\boldsymbol{B}\mathrm{r}_{R^n}(x)\subseteq\boldsymbol{A}R_k$.

Proof ①⇒②. For any $x\in M^n$, xR_n is a finitely generated submodule of M. Suppose the inclusion map $i: xR_n\to M$ factors through $R_m/\boldsymbol{A}R_k$, where $\boldsymbol{A}$ is an $m\times k$ matrix $\boldsymbol{A}$ over R, $m,k\in\mathbf{N}$. Say $i=\psi\circ\varphi$ with $\varphi\in\mathrm{Hom}_R(xR_n, R_m/\boldsymbol{A}R_k)$ and $\psi\in\mathrm{Hom}_R(R_m/\boldsymbol{A}R_k, M)$. Let $\pi_1: R_n\to xR_n$ and $\pi_2: R_m\to R_m/\boldsymbol{A}R_k$ be the natural epimorphisms. Then $\varphi\circ\pi_1=\pi_2\circ f$ for some homomorphism $f: R_n\to R_m$. Note that $f(a)=\boldsymbol{B}a$ for some $m\times n$ matrix $\boldsymbol{B}$ over R and for all $a\in R_n$. In addition, $(\psi\circ\pi_2)(b)=yb$ for some $y\in M^m$ and for all $b\in R_m$. It is easy to see that y, $\boldsymbol{A}$ and $\boldsymbol{B}$ are as desired.

② ⇒①. Let xR_n be a finitely generated submodule of M with $x\in M^n$. Then the inclusion map $i: xR_n\to M$ factors through $R_m/\boldsymbol{A}R_k$.

Theorem 4 Given two cardinal numbers α and β and a $\beta\times\alpha$ matrix $\boldsymbol{A}$ over R. The following statements are equivalent:

① $R^{(\beta)}\boldsymbol{A}$ is a left R-ML module;

② For any cardinal number γ, the canonical map $\mu_{R_\gamma}: R_\gamma\otimes_R R^{(\beta)}\boldsymbol{A}\to(R_\gamma)^\alpha$ is a monomorphism;

③ For any free right R-module F and any cardinal number γ, the canonical map $\mu_{F_\gamma}: F_\gamma\otimes_R R^{(\beta)}\boldsymbol{A}\to(F_\gamma)^\alpha$ is a monomorphism;

④ The canonical map $\mu_{M_i}: (\prod M_i)\otimes_R R^{(\beta)}\boldsymbol{A}\to(\prod M_i)^\alpha$ is a monomorphism whenever $\{M_i\}_{i\in I}$ is a set of right R-modules such that, for each M_i, the canonical map $\mu_i: M_i\otimes_R R^{(\beta)}\boldsymbol{A}\to(M_i)^\alpha$ is a monomorphism.

Proof ①⇔②, ③⇒② and ④⇒② are clear.

②⇒③. Let $F=R_{(I)}$ where I is a set. Note that F is a pure submodule of R_I by Ref. [4].

③⇒④. For each $i\in I$, there is a natural epimorphism $\pi_i: F_i\to M_i$ with $\mathrm{Ker}\pi_i=K_i$ (where F_i is a free right R-module). Then we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
(\prod M_i)\otimes R^{(\beta)}\boldsymbol{A} \arrow[r] \arrow[d] & (\prod F_i)\otimes R^{(\beta)}\boldsymbol{A} \arrow[r] \arrow[d] & (\prod M_i)\otimes R^{(\beta)}\boldsymbol{A} \arrow[r] \arrow[d, "\mu_{\Pi M_i}"] & 0 \\
0 \arrow[r] (\prod M_i)^\alpha \arrow[r] & (\prod F_i)^\alpha \arrow[r] & (\prod M_i)^\alpha \arrow[r] & 0
\end{tikzcd}$$

It is trivial to verify that $\mu_{\prod M_i}$ is a monomorphism.

Recall that R is said to be left (m,n)-coherent[2] in the case that every n generated submodule of ${}_RR^m$ is finitely presented. R is said to be left coherent[8] in the case that every finitely generated left ideal R is finitely presented. R is called a left π-coherent[9] ring if every finitely generated torsionless left R-module is finitely presented. Note that $R^{(\beta)}\boldsymbol{A}$ is a β generated (torsionless) submodule of R^α in theorem 4 and, in the case that β is finite, $R^{(\beta)}\boldsymbol{A}$ is R-ML if and only if it is finitely presented. Naturally, we call R an (α,β)-coherent ring if $R^{(\beta)}\boldsymbol{A}$ is a left R-ML module for each $\beta\times\alpha$ matrix $\boldsymbol{A}$ over R. Thus (α,β)-coherence coincides with (m,n)-coherence in the case $\alpha=m$ and $\beta=n\in\mathbf{N}$. Moreover, R is left coherent (π-coherent) if and only if it is left (α,β)-coherent for all $\alpha=m$ and $\beta=n\in\mathbf{N}$ (for all $\beta\in\mathbf{N}$ and all cardinal number α).

We complete this paper with the following corollary, which is an immediate consequence of theorem 4.

Corollary 2[2,8,9] The following are equivalent for a ring R:

① R is left (m, n)-coherent (respectively, coherent, π-coherent);
② Every direct product of R is (m, n)-flat (respectively, flat, f-projective) as a right R-module;
③ The class of (m, n)-flat (respectively, flat, f-projective) right R-modules is closed under direct product.

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用矩阵刻画模的同调性质

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摘要: 用环 R 上的矩阵研究了 R-模的一些同调性质. 对于任给的基数 α,β 以及 $\beta\times\alpha$ 行有限矩阵 $\boldsymbol{A}$, 证明了 $\mathrm{Ext}_R^1(R^{(\alpha)}/R^{(\beta)}\boldsymbol{A}, M)=0$ 当且仅当 $M_\alpha/\mathrm{r}_{M_\alpha}(R^{(\beta)}\boldsymbol{A})\cong \mathrm{Hom}_R(R^{(\beta)}\boldsymbol{A}, M)$ 当且仅当 $\mathrm{r}_{M_\beta}\mathrm{l}_{R^{(\beta)}}(\boldsymbol{A})=\boldsymbol{A}M_\alpha$, 进一步推广了$(m,n)$-内射性的概念, 并从矩阵的零化子, 同态的分解和同调群等角度给出(α,β)-平坦性的等价刻画, 从而使(m,n)-平坦模, f-投射模和 n-投射模统一到(α,β)-平坦模的概念之下. 此外还给出了左 R-ML 模的一个刻画和 $R^{(\beta)}\boldsymbol{A}$ 是左 R-ML 模的等价条件, 从而把凝聚环、(m,n)-凝聚环、π-凝聚环等概念统一到(α,β)-凝聚环的概念之下.

关键词: (α,β)-内射模; (α,β)-平坦模; R-ML 模; (α,β)-凝聚环

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