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Weak solution for a fourth-order nonlinear wave equation

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Abstract: The existence and the nonexistence, the uniqueness and the energy decay estimate of solution for the fourth-order nonlinear wave equation $u_{tt}+\alpha\Delta^2 u-b\Delta u_t-\beta\Delta u+u_t|u_t|^r+g(u)=0$ in $\Omega\times(0,\infty)$ are studied with the boundary condition $u=\frac{\partial u}{\partial \boldsymbol{v}}=0$ on $\partial\Omega$ and the initial condition $u(x,0)=u_0(x)$, $u_t(x,0)=u_1(x,0)$ in bounded domain $\Omega\subset\mathbf{R}^n$, $n\geqslant 1$. The energy decay rate of the global solution is estimated by the multiplier method. The blow-up result of the solution in finite time is established by the ideal of a potential well theory, and the existence of the solution is gotten by the Galekin approximation method.

Key words: nonlinear wave equation; uniqueness; energy decay estimate; blow up

Recently, Varlamo[1] considered the first initial boundary value problem for the damped Boussinesq equation in a plane:

$$u_{tt}+\alpha\Delta^2 u-2b\Delta u_t+\beta\Delta u^2=0 \qquad r\in(0,1);\ t>0 \tag{1}$$

$$u(r,0)=\varepsilon^2\varphi(r),\quad u_t(r,0)=\varepsilon^2\psi(r) \qquad r\in(0,1) \tag{2}$$

$$u_r(0,t)=0,\quad u(1,t)=\Delta u(1,t)=0 \qquad t>0 \tag{3}$$

Since the initial data are radially symmetric, the solution depends only on the polar radius r and the time t. By the theory of Fourier-Bessel series, Varlamo established the existence, the uniqueness and the long-time asymptotic behavior for the solution of (1) to (3).

In this paper we are concerned with the existence and the nonexistence, the uniqueness and the energy decay estimate of the global solution for the wave equation of the fourth-order with nonlinear damping and source terms of the type:

$$u_{tt}+\alpha\Delta^2 u-b\Delta u_t-\beta\Delta u+u_t|u_t|^r+g(u)=0 \qquad \text{in } \Omega\times(0,\infty) \tag{4}$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x,0) \qquad \text{in } \Omega \tag{5}$$

$$u=\frac{\partial u}{\partial \boldsymbol{v}}=0 \qquad \text{on } \partial\Omega\times[0,\infty) \tag{6}$$

where Ω is a bounded domain in $\mathbf{R}^n$, $n\geqslant 1$ with the smooth boundary $\partial\Omega$; $\boldsymbol{v}$ is the outward unit normal vector to $\partial\Omega$; $\alpha,\beta,b>0$; $r\geqslant 0$.

If we take into account the effects of damping and sources, Eq. (4) can be looked at as a perturbation of the wave equation (1) in which the term $\beta\Delta u^2$ is replaced by $u_t|u_t|^r+g(u)$. Eq. (4) is also a generalization of the Pertrovsky equation. Guesmia[2,3] studied the existence, uniqueness and energy decay of the solution for the Petrovsky equation with the initial boundary value conditions (5) and (6).

In this paper we will use the multiplier method in Ref. [4] to obtain the energy decay estimate of the global solution for (4) to (6). The blow-up result of the solution in finite time will be obtained from the ideal of potential well theory introduced by Payne and Sattinger[5]. The methods for the existence and uniqueness are standard.

If $u(t)$ is a solution of the problem (4) to (6), then we define its energy $E(t)$ by the following formula:

$$E(t)=\frac{1}{2}\int_\Omega(u_t^2(t)+\alpha(\Delta u(t))^2+\beta|\nabla u(t)|^2)\mathrm{d}x+\int_\Omega G(u(t))\mathrm{d}x \qquad t\geqslant 0 \tag{7}$$

with $G(u)=\int_0^u g(s)\mathrm{d}s$, $u\in\mathbf{R}^1=(-\infty,+\infty)$. Our main results are as follows.

Theorem 1 Suppose that

(H_1) $g(u)$ is a continuous function with $g(u)u\geqslant 0$, and there exist the constants $k,p>0$ such that $|g(u)|\leqslant k|u|^p$, $|g'(u)|\leqslant k|u|^{p-1}$, where p satisfies $p>1$ if $1\leqslant n\leqslant 4$, $1<p\leqslant n/(n-4)$ if $n>4$.

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(H_2) r satisfies $r \geqslant 0$ if $1 \leqslant n \leqslant 4$, $0 \leqslant r \leqslant 2n/(n-4)$ if $n > 4$.

Then, for the given $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$, the initial boundary value problem (4) to (6) admits a unique solution $u(t)$ satisfying

$$u(t) \in L_{\mathrm{loc}}^{\infty}(\mathbf{R}^+; H^4 \cap H_0^2) \cap C(\mathbf{R}^+; H_0^2), \quad u_t \in L_{\mathrm{loc}}^{\infty}(\mathbf{R}^+; H_0^2), \quad u_{tt}^{\infty} \in L_{\mathrm{loc}}(\mathbf{R}^+; L^2) \tag{8}$$

where $\mathbf{R}^+ = [0, \infty)$. Furthermore, the energy function $E(t)$ is nonnegative and nonincreasing and has the decay estimate

$$E(t) \leqslant C(1+t)^{-2/r} \qquad \forall\, t \geqslant 0 \text{ if } r > 0 \tag{9}$$

with some constant $C > 0$ depending on the initial energy $E(0)$, and

$$E(t) \leqslant C\mathrm{e}^{-\lambda t} \qquad \forall\, t \geqslant 0 \text{ if } r = 0 \tag{10}$$

with some $\lambda > 0$ independent of the initial data.

Theorem 2 Assume that $g(u) = -u\,|u|^{p-1}$ and (H_2) hold, where $p > 1$ if $1 \leqslant n \leqslant 4$; $1 < p \leqslant n/(n-4)$ if $n > 4$. Then there exists $d > 0$ such that if $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$ and $K(u_0) > 0$, $J(u_0) < d$, $E(0) < d$, the problem (4) to (6) admits a unique solution $u(t)$ satisfying (8) to (10), where

$$K(u) = \alpha \|\Delta u\|_2^2 + \beta \|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1} \qquad \forall\, u \in H_0^2(\Omega)$$

$$J(u) = 2^{-1}(\alpha \|\Delta u\|_2^2 + \beta \|\nabla u\|_2^2) - (p+1)^{-1} \|u\|_{p+1}^{p+1} \qquad \forall\, u \in H_0^2(\Omega)$$

$$E(0) = 2^{-1}(\|u_1\|_2^2 + \alpha \|\Delta u_0\|_2^2 + \beta \|u_0\|_2^2) - (p+1)^{-1} \|u_0\|_{p+1}^{p+1}$$

If $K(u_0) < 0$, we have the following blow-up theorem for the problem (4) to (6).

Theorem 3 Let $g(u) = -u\,|u|^{p-1}$, $p > r+1$ and (H_2) holds, and $u_0 \in H_0^{\ 4} \cap H_0^2$, $u_1 \in H_0^2$. Then there exists $d > 0$ such that if $K(u_0) < 0$, $J(u_0) < d$, $E(0) < d$, the problem (4) to (6) does not admit a global solution $u(t)$ satisfying

$$u(t) \in L^{\infty}(\mathbf{R}^+; H_0^2(\Omega)), \quad u_t \in L^{\infty}(\mathbf{R}^+; H_0^2(\Omega)) \tag{11}$$

For the proof of the above results, we need the following lemmas.

Lemma 1 Let $y(t)$ be a nonnegative differential and nonincreasing function on $\mathbf{R}^+$ satisfying

$$\int_s^t y^{1+\frac{r}{2}}(t)\,\mathrm{d}t \leqslant \lambda y(s)$$

for all $0 \leqslant s < t < +\infty$ with the constants $r, \lambda > 0$. Then for any $t \geqslant 0$, we have

$$y(t) \leqslant C(1+t)^{-2/r} \quad \text{if } r > 0; \qquad y(t) \leqslant C\mathrm{e}^{-\mu t} \quad \text{if } r = 0$$

where C is a constant depending only on $y(0)$ and $\mu > 0$ is independent of $y(0)$. Its proof can be found in Ref. [4].

From Sobolev's embedding theorems, we have

Lemma 2[3,6] Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. Then there exist positive constants C_1 and C_2 depending only on the geometry of Ω such that

$$\|u\|_{H^2(\Omega)} \leqslant C_1 \|\Delta u\|_2 \qquad \forall\, u \in H_0^2(\Omega); \qquad \|u\|_{H^1(\Omega)} \leqslant C_2 \|\nabla u\|_2 \qquad \forall\, u \in H_0^1(\Omega)$$

1 Proof of Theorem 1

We suppose that all the assumptions hold in theorem 1. Let $u(t)$ be a weak solution of (4) to (6). As in Ref. [6], we can use the Galekin approximation method to get the existence of the solution $u(t)$ satisfying formula (8). Since it is a standard procedure, here we omit the proof of the existence. In the following, we give the proof of the uniqueness and the decay estimate for the energy $E(t)$.

We first consider the uniqueness. Let $u(t)$ and $v(t)$ be two solutions of (4) to (6) which satisfying formula (8). Denote $w(t) = u(t) - v(t)$. Then $w(t)$ satisfies

$$w_{tt} + \alpha\Delta^2 w - b\Delta w_t - \beta\Delta w + u_t\,|u_t|^r - v_t\,|v_t|^r + g(u) - g(v) = 0 \qquad \text{in } \Omega \times (0, \infty) \tag{12}$$

$$w(x,0) = w_t(x,0) \qquad \text{in } \Omega; \quad w = \frac{\partial w}{\partial \upsilon} = 0 \qquad \text{on } \partial\Omega \times [0, \infty) \tag{13}$$

Multiplying Eq. (12) by w_t and integrating over Ω, we have

$$y'(t) = -h(t) - H(t) - b\|\nabla w_t\|_2^2 \qquad t \geqslant 0 \tag{14}$$

where

$$y(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}[\,|w_t(t)|^2 + \alpha\,|\Delta w_t|^2 + \beta\,|\nabla w(t)|^2]\,\mathrm{d}x$$

$$h(t) = \int_{\Omega}(u_t(t)\,|u_t(t)|^r - v_t(t)\,|v_t(t)|^r)\,w_t(t)\,\mathrm{d}x$$

$$H(t) = \int_\Omega (g(u) - g(v)) w_t \mathrm{d}x$$

Since the function $f(s) = s|s|^r$ is nondecreasing, we have $(f(\zeta) - f(\eta))(\zeta - \eta) \geqslant 0$ for all $\zeta, \eta \in \mathbf{R}^1$ and hence $h(t) \geqslant 0$ in $\mathbf{R}^+$.

On the other hand, by the assumption (H_1) and the Hölder inequality, we have

$$|H(t)| \leqslant \int_\Omega |g(u) - g(v)| \, |w_t| \mathrm{d}x \leqslant C\int_\Omega (|u|^{p-1} + |v|^{p-1})|w||w_t|\mathrm{d}x \leqslant C(\|u\|^{p-1}_{\lambda(p-1)} + \|v\|^{p-1}_{\lambda(p-1)})\|w\|_\mu \|w_t\|_2$$

where

$$\frac{1}{\lambda} + \frac{1}{\mu} = \frac{1}{2} \qquad \lambda, \mu > 1$$

If $1 \leqslant n \leqslant 4$, we then see from the assumption $u(t), v(t) \in L^\infty_{\mathrm{loc}}(\mathbf{R}^+; H^2_0(\Omega))$ and the Sobolev's embedding theorem that for any $T > 0$, and $\lambda, \mu > 1$,

$$\|u(t)\|_{\lambda(p-1)} \leqslant K_1 \|u(t)\|_{C(\Omega)} \leqslant K_1 \|\Delta u(t)\|_2 \leqslant K_2(T) \qquad 0 \leqslant t \leqslant T$$

$$\|v(t)\|_{\lambda(p-1)} \leqslant K_1 \|\Delta v(t)\|_2 \leqslant K_2(T), \quad \|w(t)\|_\mu \leqslant K_1 \|\Delta w(t)\|_2 \qquad 0 \leqslant t \leqslant T$$

where the constant K_1 is independent of T and K_2 depends on T.

If $n > 4$, let $\lambda = n/2, \mu = n^* = 2n/(n-4)$ and we obtain

$$\|u(t)\|_{n(p-1)/2} \leqslant K_1 \|u(t)\|_{n^*} \leqslant K_1 \|\Delta u(t)\|_2 \leqslant K_2(T) \qquad 0 \leqslant t \leqslant T$$

$$\|v(t)\|_{n(p-1)/2} \leqslant K_2(T), \quad \|w(t)\|_{n^*} \leqslant K_1 \|\Delta w(t)\|_2 \qquad 0 \leqslant t \leqslant T$$

Hence we have

$$|H(t)| \leqslant K_2(T)\|\Delta w\|_2 \|w_t\|_2 \leqslant K_2(T)(\|\Delta w(t)\|^2_2 + \|w_t(t)\|^2_2) \quad 0 \leqslant t \leqslant T; \quad y'(t) \leqslant K_2(T) y(t) \quad 0 \leqslant t \leqslant T \tag{15}$$

For the decay estimate of the energy $E(t)$, by lemma 1, it is sufficient to prove that the energy $E(t)$ of the solution $u(t)$ is nonnegative and nonincreasing in $\mathbf{R}^+$ and satisfies the inequality

$$\int_s^\tau E^{1+\frac{r}{2}}(t)\,\mathrm{d}t \leqslant CE(s) \tag{16}$$

for all $0 \leqslant s < \tau < \infty$. In what follows we will denote by C, a generic positive constant independent of s, t. In the following, we consider the case $r > 0$, and the case $r = 0$ can be treated similarly.

First, multiplying Eq. (4) by u_t and integrating over Ω, we have

$$E'(t) = -\|u_t(t)\|^{r+2}_{r+2} - b\|\nabla u_t(t)\|^2_2 \tag{17}$$

This shows that the energy $E(t)$ is nonincreasing. It is obvious that $E(t)$ is nonnegative since $g(u)u \geqslant 0$ in $\mathbf{R}^1$.

Next, multiplying Eq. (4) by $E^{\frac{r}{2}}(t)u(t)$ and integrating over $\Omega \times [s, \tau]$, we obtain

$$2\int_s^\tau E^{\frac{r}{2}+1}(t)\,\mathrm{d}t = A_1 + A_2 + A_3 + A_4 + A_5 \tag{18}$$

where $A_1 = -\left[E^{\frac{r}{2}}(t)\int_\Omega u\, u_t \mathrm{d}x\right]^\tau_s$, $A_2 = \frac{r}{2}\int_s^\tau E^{\frac{r}{2}-1}(t)E'(t)\int_\Omega uu_t \mathrm{d}x\mathrm{d}t$, $A_3 = -\frac{b}{2}[E^{\frac{r}{2}}(t)\|\nabla u(t)\|^2_2]^\tau_s$, $A_4 = \frac{br}{4}\int_s^\tau E^{\frac{r}{2}-1}(t)E'(t)\|\nabla u(t)\|^2_2 \mathrm{d}t$, $A_5 = \int_s^\tau E^{\frac{r}{2}}\int_\Omega (2u_t^2 - uu_t|u_t|^r)\mathrm{d}x\mathrm{d}t$.

Then, from the definition of $E(t)$ and the Sobolev inequality, we observe that

$$\|u_t(t)\|_2, \|u(t)\|_2, \|\nabla u(t)\|_2, \|\Delta u(t)\|_2 \leqslant CE^{\frac{1}{2}}(t) \tag{19}$$

Now, we can derive from (19) that

$$|A_1|, |A_2|, |A_3|, |A_4| \leqslant CE^{1+\frac{r}{2}}(s) \tag{20}$$

for all $0 \leqslant s < \tau < \infty$. Note that

$$\|u_t(t)\|^2_2 \leqslant C\|u_t(t)\|^2_{r+2} \leqslant C(-E'(t))^{\frac{2}{r+2}} \tag{21}$$

Then for any $\varepsilon > 0$,

$$2E^{\frac{r}{2}}(t)\|u_t(t)\|^2_2 \leqslant \varepsilon E^{1+\frac{r}{2}}(t) + C_\varepsilon(-E'(t)) \tag{22}$$

$$2\int_s^\tau E^{\frac{r}{2}}(t)\|u_t(t)\|^2_2 \mathrm{d}t \leqslant \varepsilon\int_s^\tau E^{1+\frac{r}{2}}(t)\,\mathrm{d}t + C_\varepsilon E(s) \tag{23}$$

Using the assumption (H_2), Sobolev's embedding inequality and lemma 2, yield

$$\|u\|_{r+2} \leqslant C\|\Delta u\|_2 \qquad \forall\, u \in H^2_0(\Omega) \tag{24}$$

with some $C > 0$.

Furthermore, we obtain from (19) and (24) that

$$\|u(t)\|_{r+2}^{r+2} \leqslant CE^{1+\frac{r}{2}}(t) \tag{25}$$

Hence, we have for any $\varepsilon > 0$ that

$$\left|\int_\Omega uu_t\,|u_t|^r \mathrm{d}x\right| \leqslant \varepsilon\|u\|_{r+2}^{r+2} + C_\varepsilon\|u_t\|_{r+2}^{r+2} \leqslant C_\varepsilon E^{1+\frac{r}{2}}(t) - C_\varepsilon E'(t) \tag{26}$$

Finally, (18) to (26) give that

$$(2 - \varepsilon - C_\varepsilon E^{\frac{r}{2}}(0))\int_s^T E^{1+\frac{r}{2}}(t)\mathrm{d}t \leqslant C_\varepsilon(E^{1+\frac{r}{2}}(s) + E(s)) \leqslant C_\varepsilon(1 + E^{\frac{r}{2}}(0))E(s)$$

If we choose small $\varepsilon > 0$, then there exists $C > 0$ such that (16) holds and the decay estimate (9) follows from lemma 1.

2 Proof of Theorem 2

To get the proof of theorem 2, we need some preparations. Let $u(t)$ be a solution of the equation

$$u_{tt} + \alpha\Delta^2 u - b\Delta u_t - \beta\Delta u + u_t\,|u_t|^r = u\,|u|^{p-1} \qquad \text{in } \Omega\times(0, T_{\max}) \tag{27}$$

with the initial boundary data (5) and (6), $T_{\max} \leqslant \infty$. Following the ideal of the potential well theory (see Refs. [5,7]), we define

$$d = \inf\{\sup_{\lambda>0} J(\lambda\varphi) : \varphi \in H_0^2(\Omega)\setminus\{0\}\} \tag{28}$$

By Sobolev's inequality, we know that $d > 0$. We denote

$$S_1 = \{\varphi \in H_0^2(\Omega) \mid K(\varphi) > 0, J(\varphi) < d\} \cup \{0\}, \quad S_2 = \{\varphi \in H_0^2(\Omega) \mid k(\varphi) < 0, J(\varphi) < d\} \tag{29}$$

For the set S_1, S_2, we have

Lemma 3 Let $u(t)$ be a solution of (27) with the initial boundary data (5) and (6) in $[0, T_{\max})$. If there is a $t_0 \in [0, T_{\max})$ such that $u(t_0) \in S_1(S_2)$ and $E(t_0) < d$, then $u(t)$ remains in $S_1(S_2)$ for any $t \in [0, T_{\max})$.

Proof We only consider $u(t_0) \in S_1$, the other case can be treated similarly. Assume that there is $t_1 > t_0$ such that $u(t) \in S_1$ for $t \in [t_0, t_1)$ and $u(t_1) \notin S_1$. From the definition of S_1 and the continuity in t of $J(u(t))$ and $K(u(t))$, we have

Case 1 $J(u(t_1)) = d_1$ or Case 2 $K(u(t_1)) = 0$

From (17), $J(u(t_1)) \leqslant E(t_1) \leqslant E(t_0) < d$. So case 1 is impossible. Let case 2 hold, that is

$$\alpha\|\Delta u(t_1)\|_2^2 + \beta\|\nabla u(t_1)\|_2^2 = \|u(t_1)\|_{p+1}^{p+1} \tag{30}$$

On the other hand, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u(t_1)) = \lambda\beta\|\nabla u(t_1)\|_2^2 + \lambda\alpha\|\Delta u(t_1)\|_2^2 - \lambda^p\|u(t_1)\|_{p+1}^{p+1} \tag{31}$$

Then (30) and (31) imply that $\lambda = 1$ and $\sup_{\lambda>0} J(\lambda u(t_1)) = J(u(t_1)) < d$, which contradicts to the definition of d. Therefore, case 2 is impossible as well.

Similarly, we can prove that:

Lemma 4 Let $u(t)$ be a solution of (4) to (6) for $t \in [0, T_{\max})$. If there exists $t_0 \in [0, T_{\max})$ such that $u(t_0) \in S_2$ and $E(t_0) < d$, then

$$\beta\|\nabla u(t)\|_2^2 + \alpha\|\Delta u(t)\|_2^2 > 2d\frac{p+1}{p-1} \qquad t \in [t_0, T_{\max}) \tag{32}$$

The proof of (32) is similar to that of lemma 2.2 in Ref. [8] and is omitted.

Proof of theorem 2

From lemma 3, we know that for $t \in [t_0, T_{\max})$, $u(t)$ satisfies

$$K(u(t)) \geqslant 0 \quad \text{or} \quad \alpha\|\Delta u(t)\|_2^2 + \beta\|\nabla u(t)\|_2^2 \geqslant \|u(t)\|_{p+1}^{p+1} \tag{33}$$

$$J(u(t)) = \frac{1}{2}(\alpha\|\Delta u(t)\|_2^2 + \beta\|\nabla u(t)\|_2^2) - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} \geqslant \frac{p-1}{2(p+1)}(\alpha\|\Delta u(t)\|_2^2 + \beta\|\nabla u(t)\|_2^2) \tag{34}$$

Furthermore, from (17) and (34), we have that for $t \in [t_0, T_{\max})$

$$d > E(t_0) \geqslant E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)) \geqslant \frac{1}{2}\|u_t(t)\|_2^2 + \frac{p-1}{2(p+1)}(\alpha\|\Delta u(t)\|_2^2 + \beta\|\nabla u(t)\|_2^2) \tag{35}$$

Inequality (35) and the continuation principle lead to the global existence of the solution, i. e., $T_{\max} = \infty$. Similar to the proof of theorem 1, we are able to derive the integral inequality (16) under the assumptions of theorem 2, so the energy $E(t)$ decay estimate (9) is brought out.

3 Proof of Theorem 3

We suppose that the global solution $u(t)$ of (4) to (6) exists for all $t \geqslant 0$. Since $u(t)$ satisfies (11), $E(t)$ is bounded. Then we have from (17) that

$$\int_0^t (\|u_t(s)\|_{r+2}^{r+2} + b\|\nabla u_t(s)\|_2^2)\mathrm{d}s = E(0) - E(t) \leqslant M \qquad t \geqslant 0 \tag{36}$$

for some constants $M > 0$. This gives that $\int_0^\infty \|u_t(s)\|_{r+2}^{r+2}\mathrm{d}s \leqslant M$. Hence, by the Hölder inequality, we have

$$\int_0^t \|u_t(s)\|_2^2 \mathrm{d}s \leqslant \lambda t^{\frac{r}{r+2}} \qquad t \geqslant 0 \tag{37}$$

with some $\lambda > 0$. Moreover, by lemma 4, we have $K(u(t)) < 0$, for $t \geqslant 0$. This implies that there exists $C_1 > 0$ such that

$$\|u(t)\|_2^2 \leqslant C_1 \|u(t)\|_{p+1}^{p+1} \qquad t \geqslant 0 \tag{38}$$

To estimate the integral $\int_\Omega uu_t|u_t|^r \mathrm{d}x$, we use the Hölder inequality and so-called interpolation inequality and get

$$\int_\Omega |uu_t||u_t|^r \mathrm{d}x \leqslant \|u(t)\|_{r+2}\|u_t(t)\|_{r+2}^{r+1} \leqslant \|u(t)\|_2^\lambda \|u(t)\|_{p+1}^{1-\lambda}\|u_t(t)\|_{r+2}^{r+1}$$

with $\lambda = 2(p-r-1)/(p-1)(r+2) > 0$. Then, by using inequality (38), we obtain

$$\|u(t)\|_2^\lambda \|u(t)\|_{p+1}^{1-\lambda} \leqslant C\|u(t)\|_{p+1}^{(p+1)/(r+2)}\|u(t)\|_{p+1}^\theta \leqslant C\|u(t)\|_{p+1}^{(p+1)/(r+2)}$$

where $\theta = (p-1)\lambda/2 + 1 - (p+1)/(r+2) = 0$. Then we have from the Young's inequality that

$$\int_\Omega |uu_t||u_t|^r \mathrm{d}x \leqslant \varepsilon\|u(t)\|_{p+1}^{p+1} + C_\varepsilon \|u_t(t)\|_{r+2}^{r+2} \tag{39}$$

Now we denote $F(t) = \|u(t)\|_2^2$ and we will prove that $F(t)$ must blow up in finite time. Note that

$$F''(t) = 2\|u_t(t)\|_2^2 - 2K(u(t)) - 2\int_\Omega uu_t|u_t|^r \mathrm{d}x + 2b\int_\Omega u_t \Delta u \mathrm{d}x \tag{40}$$

For any $\sigma > 2$, $t > 0$, we have the following estimate by lemma 4 and Young's inequality,

$$\begin{aligned} -K(u(t)) + b\int_\Omega u_t\Delta u\mathrm{d}x &\geqslant -K(u(t)) + \sigma(E(t) - E(0)) + b\int_\Omega u_t \Delta u \mathrm{d}x \geqslant -K(u(t)) + \\ &\sigma(E(t) - E(0)) - \alpha\varepsilon\int_\Omega (\Delta u)^2 \mathrm{d}x - \frac{b^2}{\alpha\varepsilon}\int_\Omega u_t{}^2 \mathrm{d}x \geqslant \left(1 - \frac{\sigma}{p+1}\right)\|u\|_{p+1}^{p+1} + \\ &(\alpha\|\Delta u\|_2^2 + \beta\|\nabla u\|_2^2)\left(\frac{\sigma}{2} - 1 - \varepsilon\right) - \sigma E(0) + \left(\frac{\sigma}{2} - \frac{b^2}{\varepsilon\alpha}\right)\|u_t\|_2^2 \geqslant \\ &\left(1 - \frac{\sigma}{p+1}\right)\|u\|_{p+1}^{p+1} + \left(\frac{\sigma}{2} - 1 - \varepsilon\right)\frac{2d(p+1)}{p-1} - \varepsilon E(0) - \mu\|u_t\|_2^2 \end{aligned} \tag{41}$$

where $\mu = |b^2(\alpha\varepsilon)^{-1} - \sigma/2|$ and $\varepsilon > 0$ is arbitrary and will be determined later.

If we choose the small ε and $\sigma > 2$ so that

$$0 < \varepsilon < \frac{(p-1)(d - E(0))}{d(p+3) - (p-1)E(0)}, \quad \frac{\sigma}{2} - 1 - \varepsilon > 0, \quad \frac{2d(p+1)(1+\varepsilon)}{d(p+1) - (p-1)E(0)} < \sigma < (p+1)(1-\varepsilon)$$

then we have from (39) to (41) that

$$\frac{1}{2}F''(t) + C_\varepsilon\|u_t(t)\|_{r+2}^{r+2} \geqslant -\mu\|u_t(t)\|_2^2 + \mu_1\|u(t)\|_{p+1}^{p+1} + \mu_2 \tag{42}$$

with $\mu_1 = 1 - \varepsilon - \sigma/(p+1)$, $\mu_2 = d(p+1)(\sigma - 2 - 2\varepsilon)/(p-1) - \sigma E(0) > 0$.

Integrating inequality (42) twice and applying (36) and (37), we get the result that there exist $C_1 > 0$ and $t_1 > 0$ such that

$$F(t) \geqslant C_1{}^2 t^2 \qquad \text{for } t \geqslant t_1 \tag{43}$$

This shows that

$$\|u(t)\|_2 \geqslant C_1 t \qquad \text{for } t \geqslant t_1 \tag{44}$$

By an iterative procedure, we obtain from (44) that

$$F''(t) + C_\varepsilon\|u_t(t)\|_{r+2}^{r+2} \geqslant -\mu\|u_t(t)\|_2^2 + \mu_1\|u(t)\|_{p+1}^{p+1} + \mu_2 \geqslant -\mu\|u_t(t)\|_2^2 + \mu_2 + \mu_3 t^2 \qquad \text{for } t \geqslant t_1 \tag{45}$$

This implies that

$$F(t) = \|u(t)\|_2^2 \geqslant \lambda t^4 \qquad \text{for } t \geqslant t_2 \geqslant t_1 \tag{46}$$

with some $\lambda > 0$.

On the other hand, we have from (36) that

$$\|u(t)\|_2 = \|u(t_0)\|_2 + \int_{t_0}^{t} \|u_t(s)\|_2 \mathrm{d}s \leqslant \|u(t_0)\|_2 + C_1 \int_{t_0}^{t} \|u_t(s)\|_{r+2} \mathrm{d}s \leqslant$$

$$\|u(t_0)\|_2 + C_1 \int_{t_0}^{t} \|u_t(s)\|_{r+2}^{r+2} \mathrm{d}s + C_2(t - t_0) \leqslant C_3 t + C_4 \qquad \text{for } t > t_3 \geqslant t_2$$

This contradicts (46). Hence the solution $u(t)$ must blow up in finite time.

References

[1] Varlamo V. On the damped Boussinesq equation in a circle [J]. *Nonlinear Analysis*, 1999, **38**(4): 447 − 470.

[2] Guesmia A. Existence globale et stabilisation interne nonlinéaire d'un systeme de Petrovsky [J]. *Bull Belg Math Soc Simon Stevin*, 1998, **5**(4): 583 − 594.

[3] Guesmia A. Energy decay for a damped nonliner coupled system [J]. *J of Math Anal and Appl*, 1999, **239**(1): 38 − 48.

[4] Komornik V. *Exact controllability and stabilization: the multiplier method* [M]. Paris: Masson-Wiley, 1994.

[5] Payne L, Sattinger D. Saddle points and instability on nonlinear hyperbolic equations [J]. *Israel Math J*, 1991, **22**: 273 − 303.

[6] Lions J L. Quelques Méthodes de Résolution des Problèmes aux Limites Nonlinéaires [A]. In: *Gauthier-Villars* [C]. Paris: Dunod, 1969.

[7] Todorola G. Stable and unstable sets for the Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms [J]. *J of Math Anal and Appl*, 1999, **239**(2): 213 − 226.

一类四阶波动方程的弱解

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摘要:研究下列初边值问题:$u_{tt} + \alpha\Delta^2 u - b\Delta u_t + u_t|u_t|^r + g(u) = 0$ in $\Omega \times (0, \infty)$; $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x,0)$, $x \in \Omega$; $u = \dfrac{\partial u}{\partial \boldsymbol{v}}$, $x \in \partial\Omega$ 的整体解的存在性和不存在性,以及整体解的惟一性和能量估计. 这里 Ω 是 $\mathbf{R}^n (n \geqslant 1)$ 中的有界区域. 借助于乘子方程,推出了整体解的能量衰减率. 借助于势井理论,得到了在有限时刻内爆破的充分条件. 由 Galekin 近似方法得到解的存在性.

关键词:非线形波动方程;惟一性;能量衰减估计;爆破

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