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New sufficient conditions for general linear SISO Takagi-Sugeno fuzzy systems as universal approximators

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Abstract: By the best approximation theory, it is first proved that the SISO (single-input single-output) linear Takagi-Sugeno (TS) fuzzy systems can approximate an arbitrary polynomial which, according to Weierstrass approximation theorem, can uniformly approximate any continuous functions on the compact domain. Then new sufficient conditions for general linear SISO TS fuzzy systems as universal approximators are obtained. Formulae are derived to calculate the number of input fuzzy sets to satisfy the given approximation accuracy. Then the presented result is compared with the existing literature's results. The comparison shows that the presented result needs less input fuzzy sets, which can simplify the design of the fuzzy system, and examples are given to show its effectiveness.

Key words: Takagi-Sugeno (TS) fuzzy system; universal approximator; sufficient condition

A fuzzy system consists of four components: the fuzzy rule base, the fuzzy inference engine, the fuzzifier and the defuzzifier. A very significant contribution of fuzzy theory is that it provides a mapping transformation from linguistically expressed human knowledge and experience to the nonlinear functions. It has been widely used in many areas like control, signal progressing and communication systems.

There are two major types of fuzzy systems: Mamdani fuzzy systems and Takagi-Sugeno (TS) fuzzy systems. The main difference between the two is the rule consequent. The Mamdani fuzzy system has the rule consequent based on the linguistic while the TS fuzzy system's is formed by the linear combinations of the input variables.

So far, most of the universal approximation results have been about Mamdani fuzzy systems[1 – 5]. Only a few results on TS fuzzy systems have been achieved[6 – 8]. Ref. [7] constructively proved the general SISO fuzzy systems with linear rule consequent are universal approximators given the minimal upper bound of the input fuzzy sets to guarantee the desired approximation accuracy.

In this paper, new sufficient conditions for a general SISO TS fuzzy system with linear rule consequent as universal approximators are discussed. First, some mathematical preliminaries are introduced. Then based on this, our new sufficient conditions for a general SISO TS fuzzy system with linear rule consequent are given. Finally our new result is compared with the result in Ref. [7].

Without loss of generality, we assume the input variable $-1 \leqslant x \leqslant 1$.

1 Mathematical Preliminaries

Definition 1 $\| * \|_\infty$ is defined as $\|f(x)\|_\infty = \sup\limits_{x\in U} |f(x)|$, $U \subset \mathbf{R}$ is compact.

Definition 2 Chebyshev polynomials

The chebyshev polynomial (of the first kind) is defined recursively as follows:

$$\left.\begin{aligned} &T_0(x) = 1 \\ &T_1(x) = x \\ &T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad n \geqslant 1 \end{aligned}\right\} \tag{1}$$

Remark 1 The term of the highest degree in $T_n(x)$ is $2^{n-1}x^n$ for $n > 0$.

Lemma 1[9] For x in the interval $[-1, 1]$, the chebyshev polynomials have the closed form expression $T_n(x) = \cos(n\cos^{-1}x)$ $(n \geqslant 0)$.

Since $\cos(n+1)\theta = \cos\theta\cos n\theta - \sin\theta\sin n\theta$ and $\cos(n-1)\theta = \cos\theta\cos n\theta + \sin\theta\sin n\theta$, we can get $\cos(n+1)\theta = 2\cos\theta\cos n\theta - \cos(n-1)\theta$, let $\theta = \cos^{-1}x$ and $x = \cos\theta$, we can get the above result.

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Lemma 2[9] For x in the interval $[-1, 1]$, $T_n(x)$ is equal to zero when $x_i = \cos\left(i+\frac{1}{2}\right)\frac{\pi}{n}$ $(i=1, 2, \ldots, n)$.

Corollary 1 For x in the interval $[a, b]$, $T_n(x)$ is equal to zero when $x = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(i+\frac{1}{2}\right)\frac{\pi}{n}$ $(i = 1, 2, \ldots, n)$.

It is equal to lemma 2 since the change of variable $x = \frac{a+b}{2} + \frac{b-a}{2}t$ can be used to go back and forth between $[a, b]$ and $[-1, 1]$.

Theorem 1[9] Let f be a function in $C^{n+1}[a, b]$, and let $N_n(x)$ be the polynomial of degree at most n that interpolates the function f at $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in the interval $[a, b]$, then the error bound $R_n(x) = |f(x) - N_n(x)| \leqslant \frac{\max\limits_{x\in[a,b]} |f^{(n+1)}(x)|}{(n+1)!}\left(\frac{a-b}{2}\right)^{n+1}\frac{1}{2^n}$ and the equation holds when the nodes x_i of $N_n(x)$ are the roots of the chebyshev polynomial $T_{n+1}(x)$, i.e., $x_i = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(i+\frac{1}{2}\right)\frac{\pi}{n+1}$ $(i=1, 2, \ldots, n)$.

Remark 2 From theorem 1, we can get $N_n(x)$ which satisfies the equation by formulating the Newton or Lagrange interpolative polynomials. For example, if $n=1$, we can get $N_n(x) = f(x_0) + f[x_0, x_1](x - x_0)$, where $f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$, x_0, x_1 are the roots of the chebyshev polynomials $T_2(x)$.

Theorem 2[9] Weierstrass approximation theorem

If $f(x)$ is continuous on $[a, b]$ and if $\varepsilon > 0$, then there is a polynomial $p(x)$ satisfying $\|f(x) - p(x)\|_\infty \leqslant \varepsilon$ on the interval $[a, b]$.

To fuzzify x, we partition $[-1, 1]$ into $2n$ equal intervals $\left[\frac{j}{n}, \frac{j+1}{n}\right]$, based on this, we define $2n+1$ fuzzy sets $A_j (j = 0, \pm 1, \ldots, \pm n)$ and the corresponding membership functions $\mu_{A_j}(x)$, where A_{-n} is defined over $[-1, (1-n)/n]$, A_n is defined over $[(n-1)/n, 1]$ and the others $[(j-1)/n, (j+1)/n]$, $1-n \leqslant j \leqslant n-1$.

Here, we assume $\mu_{A_j}(x)$ can be any reasonable continuous functions.

For a linear SISO TS fuzzy system, the fuzzy rule is expressed as

Rule m: If x is A_{p_m}, then $y = a_m + b_m x$

where x is the input variable, A_{p_m} is the input fuzzy set $(-n \leqslant P_m \leqslant n)$, y is the output variable, and a_m and b_m are design parameters whose values are determined by the fuzzy system.

Utilizing the general defuzzifier[10], we can formulate the linear SISO TS fuzzy system:

$$F_n(x):\ [-1, 1] \to (-\infty, +\infty),\ F_n(x) = \sum_{m=1}^{N} (\mu_m)^\alpha (a_m + b_m x) \Big/ \sum_{m=1}^{N} (\mu_m)^\alpha \tag{2}$$

Different defuzzification results can be obtained by using different α, where $0 \leqslant \alpha \leqslant \infty$. Clearly, the centroid defuzzifier will be achieved when $\alpha = 1$ and the maximum defuzzifier will be gotten when $\alpha = \infty$.

We will use $F_n(x)$ to represent our general SISO TS fuzzy system in the following.

2 Sufficient Conditions for SISO TS Fuzzy System as Universal Approximators

In this section, we will prove new sufficient conditions for the general SISO TS fuzzy system as universal approximators and give out minimal upper bound of the input fuzzy sets and rules to guarantee the desired approximation accuracy. Without loss of generality, we assume $N = 2n+1$.

Lemma 3 Suppose that N fuzzy sets are assigned to input variables of a general linear SISO TS fuzzy system, let $p_h(x)$ be any given polynomial defined on $C[-1, 1]$ and $\varepsilon > 0$ be a universal approximation error bound, then there exists a linear SISO TS fuzzy system $F_n(x)$ such that $\|F_n(x) - p(x)\|_\infty \leqslant \varepsilon$ for all $n \geqslant n_0$, when $n_0 = \sqrt{M_2/(16\varepsilon)}$.

Proof For a given input variable x of a linear TS fuzzy system, without loss of generality, assume $j/n \leqslant x \leqslant (j+1)/n$, $j = 0, \pm 1, \ldots, \pm n$, then from formula (2) we have

$$|F_n(x) - p_h(x)| = \left| \sum_{m=1}^{N} (\mu_m)^\alpha (a_m + b_m x) \Big/ \sum_{m=1}^{N} (\mu_m)^\alpha - p_h(x) \right| =$$

$$\left| \sum_{m=1}^{N} (\mu_m)^{\alpha} (a_m + b_m x - p_h(x)) \Big/ \sum_{m=1}^{N} (\mu_m)^{\alpha} \right| \qquad \frac{j}{n} \leqslant x \leqslant \frac{j+1}{n}$$

where $|a_m + b_m x - p_h(x)|$ is a problem in which the polynomial $(a_m + b_m x)$ approximates a polynomial $p_h(x)$ over $[j/n, (j+1)/n]$, so from theorem 1, we have $|a_m + b_m x - p_h(x)| \leqslant \frac{M_2}{2!}\left(\frac{1}{2n}\right)^2 \frac{1}{2} = \frac{M_2}{16n^2}$, where $M_2 = \max_{x\in[-1,1]} |p''_h(x)|$, thus $|F_n(x) - p_h(x)| = \left| \sum_{m=1}^{N} (\mu_m)^{\alpha}(a_m + b_m x - p_h(x)) \Big/ \sum_{m=1}^{N} (\mu_m)^{\alpha} \right| \leqslant \frac{M_2}{16n^2} < \varepsilon$. Then we get $n \geqslant \sqrt{\frac{M_2}{16\varepsilon}}$, $M_2 = \max_{x\in[-1,1]} |p''_h(x)|$.

Hence, on condition $n \geqslant \sqrt{\frac{M_2}{16\varepsilon}}$ and $M_2 = \max_{x\in[-1,1]} |p''_h(x)|$, $\|F_n(x) - p_h(x)\|_\infty < \varepsilon$ holds.

Remark 3 From theorem 1, we know that $R_n(x) = \frac{\max_{x\in[a,b]} f^{n+1}(x)}{(n+1)!}\left(\frac{a-b}{2}\right)^{n+1}\frac{1}{2^n}$ at $x_i = \frac{a+b}{2} + \frac{b-a}{2}\cdot\cos\left(i+\frac{1}{2}\right)\frac{\pi}{n+1}$ $(i = 1, 2, \ldots, n)$, so for this problem, we can get the nodes at $x_i = \frac{j/n + (j+1)/n}{2} + \frac{1}{2n}\cos\left(i+\frac{1}{2}\right)\frac{\pi}{2} = \frac{2j+1}{2n} + \frac{1}{2n}\cos\left(i+\frac{1}{2}\right)\frac{\pi}{2}$ $(i = 1, 2)$, then we can get the approximate through $a_m + b_m x = p_h(x_0) + p_h[x_0, x_1](x - x_0)$, where $p_h[x_0, x_1] = \frac{p_h(x_0) - p_h(x_1)}{x_0 - x_1}$.

Theorem 3 Suppose that N fuzzy sets are assigned to input variables of a general linear SISO TS fuzzy system, let $g(x)$ be any given real continuous function defined on $[-1, 1]$ and let $\varepsilon > 0$ be the universal approximation error, then there exists a linear SISO TS fuzzy system $F_n(x)$ such that $\|F_n(x) - g(x)\|_\infty < \varepsilon$ if $n \geqslant \sqrt{\frac{M_2}{16(\varepsilon - \varepsilon_1)}}$, where $0 \leqslant \varepsilon_1 < \varepsilon$ and $\|p_h(x) - g(x)\|_\infty < \varepsilon_1$.

Proof From Weierstrass approximation theorem, there exists a polynomial $p_h(x)$, such that $\|p_h(x) - g(x)\|_\infty < \varepsilon_1$, then from lemma 3, there exists a linear SISO TS fuzzy system $F_n(x)$ such that $\|F_n(x) - p_h(x)\|_\infty < \varepsilon - \varepsilon_1$ if $n \geqslant \sqrt{\frac{M_2}{16(\varepsilon - \varepsilon_1)}}$. These two imply that

$$\|F_n(x) - g(x)\|_\infty \leqslant \|F_n(x) - p_h(x) + p_h(x) - g(x)\|_\infty \leqslant \|F_n(x) - p_h(x)\|_\infty + \|p_h(x) - g(x)\|_\infty < \varepsilon_1 + \varepsilon - \varepsilon_1 = \varepsilon$$

3 Comparisons

The result in Ref. [7] on general linear SISO TS fuzzy systems: Ref. [7] (theorem 3) achieved the sufficient conditions for general linear SISO TS fuzzy systems when n_0 follows the formula $n_0 = \left(|\beta_1| + \sum_{i=1}^{h} |\beta_i|(2^i - 1)\right)/(\varepsilon - \varepsilon_1)$, β_i is the coefficient of x^i in $p_h(x)$. Obviously,

$$M_2 = \max_{x\in[-1,1]} |p''_h(x)| = \max_{x\in[-1,1]} |2\beta_2 + 3\cdot 2\beta_3 x + , \ldots, + h(h-1)\beta_h x^{h-2}| \leqslant 2|\beta_2| + 6|\beta_3| + , \ldots, + h(h-1)|\beta_h|$$

Then we can get $\sqrt{\frac{M_2}{16(\varepsilon - \varepsilon_1)}} < \left(|\beta_1| + \sum_{i=1}^{h} |\beta_i|(2^i - 1)\right)/(\varepsilon - \varepsilon_1)$ easily.

Example In Ref. [7] an example has been given to illustrative the use of the formula. For comparisons, we will use the same example: How many fuzzy sets are needed for a general linear TS fuzzy system to uniformly approximate function $G(x) = e^x$ defined on $[-1, 1]$ with ① $\varepsilon = 0.1$ and ② $\varepsilon = 0.01$?

Solution In Ref. [7], $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$ are used to approximate $g(x)$ with truncation error less than 0.003 8 (i. e. $\varepsilon_1 = 0.0038$) and $n_0 = 58$ and $n_0 = 896$ are computed respectively. But through $n_0 = \sqrt{\frac{M_2}{16\varepsilon_2}}$, we get $M_2 = 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3}$, then $n_0 = 2$ and $n_0 = 6$ are computed respectively.

Although our final results (i. e. the minimal upper bound of the needed input fuzzy sets) are better than the result in Ref. [7], we should also see that the computation of ours is more complex since our results need to compute the second-order derivates of $p_h(x)$ while the result in Ref. [7] only needs to know the coefficients of it. It is difficult to say clearly which one is better, since it depends on what is more important to you, and whether there exists a method that has both good results and simple computation is a question worth of thought.

4 Conclusion

We have derived new sufficient conditions for a general SISO TS fuzzy system with linear rule consequent by employing the best approximation theory in numerical analysis, and have proved that our result is better than the result in Ref. [7] from the numerical point. Since the necessary and sufficient conditions for a fuzzy system as a universal approximation is still an open question, we hope that the study of sufficient conditions can help to achieve the target.

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线性 SISO TS 模糊系统万能逼近的一种新的充分条件

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摘要：借助于最优逼近理论，证明了线性 SISO TS 模糊系统可以逼近任意一个多项式，然后以 Weierstrass 逼近定理为桥梁，证明了该模糊系统可以以任意精度逼近一个任意的连续函数，从而得到了该模糊系统万能逼近性的一个新的充分条件. 并在证明过程中，得到了要达到所要求的逼近精度所需的输入模糊集的下确界. 然后从理论上将所得的结果与现有文献中的结果进行了比较，证明了该结果所需的输入模糊集的数目要少得多，从而可以简化模糊系统的设计. 最后举例证明了该结论的有效性.

关键词：Takagi-Sugeno（TS）模糊系统；万能逼近；充分条件

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