

Stabilization of discrete nonholonomic chained system

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Abstract: Aimed at the stabilization of the nonholonomic chained system under fixed sample control, two control laws were proposed. The discrete model of the nonholonomic chained system under zero-hold was obtained through the integrate method to the continuous model. And the discrete model was transformed to the form with two linear subsystems through coordinate transformation. Two feedback control laws, time-invariant control law and time-varying control law, were proposed; and the local stabilization and global stabilization were realized respectively. The simulation results show the effectiveness of the proposed control laws. The discrete nonholonomic chained system can converge to zero from any initial state exponentially, and the convergence rate can be changed through changing the parameters of the control laws.

Key words: nonholonomic chained system; discrete control; exponential stabilization

In the past decades, the stabilization problem of the nonholonomic system has been studied widely. However, when the computer is used in the stabilization of the chained system, some proposed control laws^[1-4] for the differential chained systems cannot be used directly to stabilize the chained system based on the sampled data control. So it is needed to reevaluate the conditions of parameter selection of some proposed control laws on such control. In Ref. [5], a method through the pole placement for the stabilization of the chained systems based on the sampled data control has been proposed. However, for the method proposed in Ref. [5], it is necessary to compute the exponent of the matrix J , e^{JT} and usually, this item is calculated with difficulty.

In this paper, based on the proposed control laws in Refs. [1, 2], two control laws are proposed for the stabilization of the chained system on the sampled data control. It is proved that these control laws can exponentially stabilize the chained system to the origin over the sampled data control with the sample time T . And some simulation results show the efficacy of the proposed control laws.

1 Problem Formulation

Firstly, consider the following chained system described by the differential equations:

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_2 u_1, \quad \dots, \quad \dot{x}_n = x_{n-1} u_1 \quad (1)$$

where $\mathbf{x} = \{x_1, x_2, x_3, \dots, x_n\}^T \in \mathbf{R}^n$ denotes the

state vector, and $\mathbf{u} = \{u_1, u_2\}^T \in \mathbf{R}^2$ denotes the input vector.

Based on the sampled data control system for the chained systems proposed in Ref. [3], the input u_i should be a constant in the k -th sampled interval T_k , i. e. $u_i(t) = u_i(k), t \in [t_k, t_{k+1}]$.

Using the integral method, the discrete model of the nonholonomic chained system can be expressed with the difference equations:

$$\left. \begin{aligned} x_1(k+1) &= x_1(k) + u_1(k)T \\ \mathbf{x}_2(k+1) &= \mathbf{A}\mathbf{x}_2(k) + \mathbf{B}u_2(k) \end{aligned} \right\} \quad (2)$$

where $\mathbf{x}(k) = \{x_1(k), \mathbf{x}_2^T(k)\}^T$ denotes the state vector, $\mathbf{x}_2(k) = \{x_2(k), x_3(k), \dots, x_n(k)\}^T$; $\mathbf{u}(k) = \{u_1(k), u_2(k)\}^T$ denotes the input vector. $\mathbf{A} = [a_{ij}]_{(n-1) \times (n-1)}$ denotes an $(n-1) \times (n-1)$ matrix, each item can be calculated as follows: If $i < j$, then $a_{ij} = 0$; if $i \geq j$, then $a_{ij} = \frac{u_1^{i-j}(k)T^{i-j}}{(i-j)!}$; $\mathbf{B} =$

$$[b_i]_{n-1}, \text{ where } b_i = \frac{u_1^{i-1}T^i}{i!}.$$

The main problem addressed in this paper can be described as follows:

Design the state feedback control law:

$$u_i(k) = f_i(k, \mathbf{x}(k)) \quad i = 1, 2 \quad (3)$$

Such that every state of system (2) can converge to zero, i. e. $\lim_{k \rightarrow \infty} |x_i| = 0, i = 1, 2, \dots, n$.

2 Main Results

In this section, based on the control laws presented in Refs. [5, 6], two feedback control laws are proposed for the above problem. The first control law is time-invariant and can make system (2) converge to the origin from any initial state except for $x_1(0) = 0$,

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and the second control law can make system (2) converge to the origin from any initial state $x \in \mathbf{R}^n$ but it is time-variant.

2.1 The time-invariant state feedback control law

It can be easily proved that the following subsystem (4) of system (2) can be stabilized through the state feedback control law (5).

$$x_1(k+1) = x_1(k) + u_1(k)T \quad (4)$$

$$u_1(k) = -K_1 x_1(k) \quad |1 - K_1 T| < 1 \quad (5)$$

And there is a coordinate transformation

$$\left. \begin{aligned} \hat{x}_2(k) &= x_2(k) \\ \hat{x}_3(k) &= x_3(k)/x_1(k) \\ &\vdots \\ \hat{x}_n(k) &= x_n(k)/x_1^{n-2}(k) \end{aligned} \right\} \quad (6)$$

for the subsystem $x_2(k+1) = Ax_2(k) + Bu_2(k)$ of system (2) such that when the feedback control law (5) is used to $u_1(k)$, this subsystem can be expressed as a discrete time-invariant linear system:

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u_2(k) \quad (7)$$

where $\hat{x}(k) = \{\hat{x}_2(k), \hat{x}_3(k), \dots, \hat{x}_n(k)\}^T$ denotes the state vector, and $u_2(k)$ denotes the control input. $\hat{A} = [\hat{a}_{i,j}]_{(n-1) \times (n-1)}$ denotes an $(n-1) \times (n-1)$ matrix and each item can be calculated as follows: If $i < j$,

then $\hat{a}_{i,j} = 0$; if $i \geq j$, then $\hat{a}_{ij} = \frac{(-k_1 T)^{i-j}}{(i-j)!}$; each item of $\hat{B} = [\hat{b}_i]_{n-1}$ can be expressed as $\hat{b}_i = \frac{(-k_1)^{i-1} T^i}{i! (1 - k_1 T)^{i-1}}$.

Now, we can obtain the following theorem for the stabilization of system (2).

Theorem 1 Consider the discrete model (2) of chained system (1), the following time-invariant state feedback control laws (8a) and (8b) can stabilize it to the origin from any initial state except that $x_1(0) = 0$.

$$u_1(k) = -K_1 x_1(k) \quad (8a)$$

$$u_2(k) = -\sum_{i=2}^n K_i \frac{x_i(k)}{x_1^{i-2}(k)} \quad (8b)$$

where K_i , $i = 1, 2, \dots, n$ satisfy the conditions: ① $|1 - K_1 T| < 1$; ② $|\lambda_i| < 1$, $i = 1, 2, \dots, n$, where λ_i denote the eigvalues of $(\hat{A} - \hat{B}K)$, $K = [K_i]_{(n-1)}$, $i = 2, 3, \dots, n$.

Proof Firstly, it is easily proved that the state $x_1(k)$ exponentially converges to zero under (8a) when the first condition $|1 - K_1 T| < 1$ is held, and its analytic solution is $x_1(k) = (1 - K_1 T)^k x_1(0)$.

And then, (8b) can also be expressed as

$$u_2(k) = -\sum_{i=2}^n K_i \hat{x}_i(k) \quad (9)$$

Every state of the discrete time-invariant linear system (7) exponentially converges to zero through

(9), if $|\lambda_i| < 1$. λ_i can be completely assigned through the feedback gains K_i .

Finally, the following equation can be deduced from Eq. (7): $x_i(k) = \hat{x}_i(k) x_1^{i-2}(k)$. So every state of the subsystem $x_2(k+1) = Ax_2(k) + Bu_2(k)$ can exponentially converge to zero under the control law (8b), when all the conditions in theorem 1 are held.

Remark 1 The control law proposed in theorem 1 is similar to that proposed in Ref. [3]. Compared with the control law in Ref. [3], when calculating the parameters of the proposed control law in this paper, every item of the matrices \hat{A} and \hat{B} can be easily calculated. It is not necessary to calculate the exponent of the matrix J . So based on the conditions of parameter selection proposed in this paper, the parameters of the control law can be easily calculated.

Remark 2 It is noted that the proposed control law in theorem 1 only guarantees exponential convergence for all the initial states in the open and dense set $\{(x_1(0), \dots, x_n(0)) \in \mathbf{R}^n \mid x_1 \neq 0\}$; see Ref. [3] for further details.

2.2 The time-variant state feedback control law

Although the proposed control law can stabilize system (2) to the origin, the initial state should belong to an open and dense set $\{(x_1(0), \dots, x_n(0)) \in \mathbf{R}^n \mid x_1 \neq 0\}$. In this subsection, a time-variant state feedback control law is proposed, so that the control law can stabilize system (2) to the origin from any initial state.

Similar to subsection 2.1, firstly, a time-variant control law (11) is proposed to stabilize the subsystem (4).

$$u_1(k) = -K_1 x_1(k) + \alpha_0 \alpha^k \quad |1 - K_1 T| < 1, \alpha < 1 \quad (10)$$

Based on the control law (10), the state $x_1(k)$ of subsystem (2) has the analytic solution (11):

$$x_1(k) = (x_1(0) - C)(1 - K_1 T)^k + C\alpha^k \quad (11)$$

where $C = T\alpha^0/(\beta + K_1 T - 1)$.

It also means that $\lim_{k \rightarrow \infty} |x_1(k)| = 0$.

It is similar to the method in subsection 2.1 that there exists a coordinate transformation independent of $x_1(k)$:

$$\left. \begin{aligned} \hat{x}_2(k) &= x_2(k) \\ \hat{x}_3(k) &= x_3(k)/(\alpha_0 \alpha^k) \\ &\vdots \\ \hat{x}_n(k) &= x_n(k)/(\alpha_0 \alpha^k)^{n-2} \end{aligned} \right\} \quad (12)$$

such that the subsystem $x_2(k+1) = Ax_2(k) + Bu_2(k)$ can be transformed into a discrete time-variant linear system:

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u_2(k) \quad (13)$$

where $\hat{A} = [\hat{a}_{ij}]_{(n-1) \times (n-1)}$ denotes an $(n-1) \times (n-1)$ matrix, and each item can be calculated as follows: If $i < j$, then $\hat{a}_{ij} = 0$; else if $i \geq j$, then $\hat{a}_{ij} = \frac{[(c_1 + c_2(k))T]^{i-j}}{((i-j)! \alpha^{i-1})}$; each item of $\hat{B} = [\hat{b}_i]_{n-1}$ can be described by $\hat{b}_i = \frac{(c_1 + c_2(k))^{i-1} T^i}{i!}$, where $c_1 = (\alpha - 1)/(\alpha + K_1 T - 1)$, $c_2(k) = -K_1[x_1(0) - C](1 - K_1 T)^k / (\alpha_0 \alpha^{k+1})$.

In order to prove the following theorem, a lemma was presented first.

Lemma 1^[4] Consider the discrete time-variant linear system:

$$x(k+1) = (A_0 + \tilde{A}(k))x(k) \quad (14)$$

where $x \in \mathbf{R}^n$, A_0 is an $n \times n$ constant matrix, and $\tilde{A}(k) = [\tilde{a}_{i,j}]_{n \times n}$ denotes an $n \times n$ matrix dependent with k , each item can be obtained as follows: If $|\lambda_i(A_0)| < 1$, $\lim_{k \rightarrow \infty} |\tilde{a}_{i,j}(k)| = 0$ and $\sum_{k=0}^{\infty} \|\tilde{A}(k)\| < \infty$, then system (14) is locally exponential stability, i. e. $\lim_{k \rightarrow \infty} |x_i(k)| = 0$.

Based on the above lemma, the following theorem can be easily proved.

Theorem 2 Consider the discrete model (2) of chained system (1), the following time-variant state feedback control laws (15a) and (15b) can stabilize it to the origin from any initial state:

$$u_1(k) = -K_1 x_1(k) + \alpha_0 \alpha^k \quad (15a)$$

$$u_2(k) = -\sum_{i=2}^n K_i \frac{x_i(k)}{(\alpha_0 \alpha^k)^{i-2}} \quad (15b)$$

where $K_i, i = 1, 2, \dots, n$ satisfies the conditions: ① $0 < |1 - K_1 T| < \alpha < 1$; ② $|\lambda_i| < 1, i = 1, 2, \dots, n$, where λ_i is the eigvalue of $(\hat{A}_0 - \hat{B}_0 K)$, $K = [K_i]_{(n-1)}, i = 2, 3, \dots, n$; $\hat{A}_0 = [\hat{a}_{ij}]_{(n-1) \times (n-1)}$ denotes an $n \times n$ matrix, each item can be calculated as: If $i < j$, then $\hat{a}_{ij} = 0$; else if $i \geq j$, then $\hat{a}_{ij} = \frac{(c_1 T)^{i-j}}{(i-j)! \alpha^{i-1}}$; each

item of $\hat{B}_0 = [\hat{b}_i]_{n-1}$ can be described by $\hat{b}_i = \frac{c_1^{i-1} T^i}{i!}$,

where $c_1 = (\alpha - 1)/(\alpha + K_1 T - 1)$.

Proof Similar to the proof of theorem 1, the state $x_1(k)$ of system (2) converge to zero exponentially via the control law (15a), and the analytic solution is shown in (12).

And then the control law (15b) can also be shown as follows:

$$u_2(k) = -\sum_{i=2}^n K_i \hat{x}_i(k) \quad (16)$$

The closed-loop systems (13) to (16) can be ex-

pressed as

$$\hat{x}(k+1) = (\hat{A} - \hat{B}K)\hat{x}(k) = [\hat{A}_0 + \tilde{A}(k) - \hat{B}_0 K - \tilde{B}(k)K]\hat{x}(k) = [\hat{A}_0 - \hat{B}_0 K + \tilde{A}(k) - \tilde{B}(k)K]\hat{x}(k)$$

Based on (13), it is known that through selecting the appropriate parameters K_1 and α based on the sampled time T , it can make $\tilde{A}(k)$ and $\tilde{B}(k)$ satisfy the conditions: $\lim_{k \rightarrow \infty} |\tilde{a}_{i,j}(k)| = 0$, $\sum_{k=0}^{\infty} \|\tilde{A}(k)\| < \infty$ and

$$\lim_{k \rightarrow \infty} |\tilde{b}_i(k)| = 0, \quad \sum_{k=0}^{\infty} \|\tilde{B}(k)\| < \infty.$$

Since (\hat{A}_0, \hat{B}_0) is controllable, the second condition can be held through selecting the feedback gains K .

Finally, based on lemma 1, the closed-loop systems (13) to (16) are locally exponentially stable when the conditions ① and ② are held. In addition, it can be deduced that $x_i(k) = \hat{x}_i(k) (\alpha_0 \alpha^k)^{i-2}$, so the control law (16) can make system (2) converge to the origin from any initial state.

3 Simulation Results

In this section, the proposed methods are used to a discrete 4-dimensional chained system:

$$\left. \begin{aligned} x_1(k+1) &= x_1(k) + u_1(k)T \\ x_2(k+1) &= \begin{bmatrix} 1 & 0 & 0 \\ u_1(k)T & 1 & 0 \\ \frac{1}{2}u_1(k)T^2 & u_1(k)T & 1 \end{bmatrix} \cdot \\ x_2(k) &+ \begin{bmatrix} T \\ \frac{1}{2}u_1(k)T^2 \\ \frac{1}{6}u_1(k)T^3 \end{bmatrix} u_2(k) \end{aligned} \right\} \quad (17)$$

where $x_2(k) = \{x_2(k), x_3(k), x_4(k)\}^T$.

Case 1 Based on the time-invariant control law proposed in subsection 2.1, the parameters of control law can be selected as $K_1 = 2$, $K_2 = 9.9421$, $K_3 = -28.6656$, $K_4 = 38.3686$. When the initial state is $\{1, 1, 1, 1\}^T$, the response curves of every state of system (2) and the control inputs are shown as solid lines in Fig. 1 through the time-invariant control laws (8a) and (8b).

Case 2 Based on the proposed time-variant control law in subsection 2.2, the parameters of control law can be selected as $K_1 = 2$, $K_2 = 7.717$, $K_3 = -29.193$, $K_4 = 47.924$. When the initial state is $\{1, 1, 1, 1\}^T$, the response curves of each state of system (2) and the control inputs are also shown as dot lines in Fig. 1 through the time-variant control law (15).

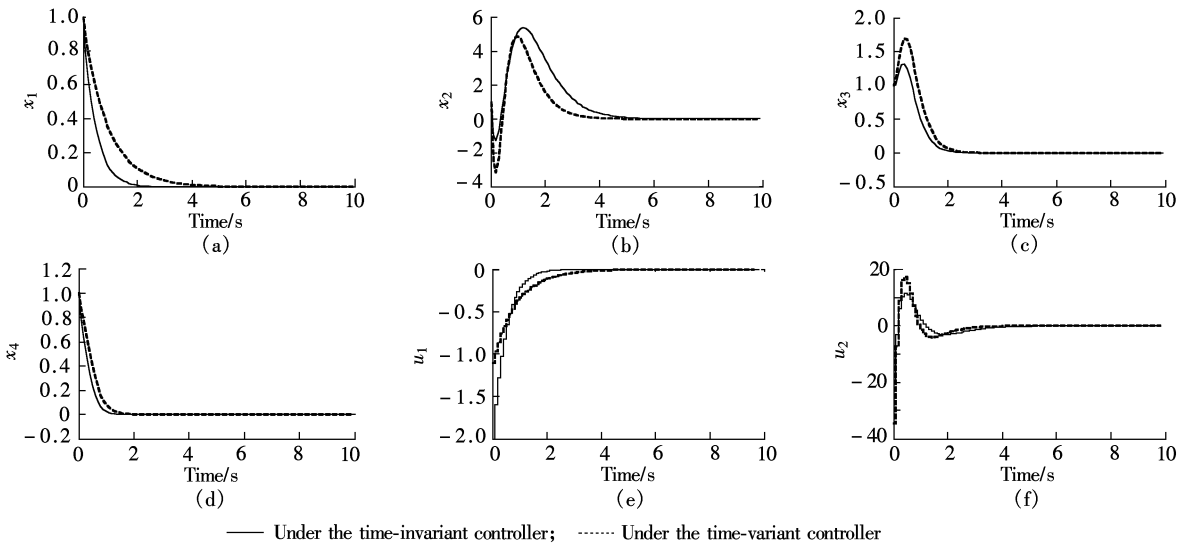


Fig. 1 Response of every state and the control inputs with time

4 Conclusion

In this paper, two state feedback control laws have been proposed for the stabilization of the discrete chained system. The discrete model of the chained system is obtained through the integral method. And it is proved that the proposed control law can stabilize the discrete chained system to the origin exponentially. Finally, two examples show the efficacy of the proposed control law. In future, based on the discrete model of the chained system, we will focus on many interesting problems such as optimal control and stabilization over networks for the discrete chained system etc.

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离散非完整链式系统的镇定

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摘要: 针对定常采样过程中非完整链式系统的镇定问题, 提出了 2 个反馈控制律. 首先通过对连续模型的积分运算, 得到了零阶保持器下非完整链式系统的离散模型, 然后利用坐标变换, 将其转换为 2 个线性子系统的形式, 并提出了 2 种反馈镇定控制律: 时不变反馈控制律和时变反馈控制律, 分别实现了对离散非完整链式系统的局部和全局镇定. 仿真实验表明所提反馈控制律是行之有效的. 在它们的作用下, 离散非完整链式系统能从任意的初始状态指数收敛到原点, 并且通过改变反馈控制律的参数能改变系统各状态的收敛速度.

关键词: 非完整链式系统; 离散控制; 指数镇定
中图分类号: TP391