Journal of Southeast University (English Edition) Vol. 21, No. 4, pp. 480 – 485 Dec. 2005 ISSN 1003—7985

New tangent stiffness matrix for geometrically nonlinear analysis of space frames

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Abstract: A three-dimensional beam element is derived based on the principle of stationary total potential energy for geometrically nonlinear analysis of space frames. A new tangent stiffness matrix, which allows for high order effects of element deformations, replaces the conventional incremental secant stiffness matrix. Two deformation stiffness matrices due to the variation of axial force and bending moments are included in the tangent stiffness. They are functions of element deformations and incorporate the coupling among axial, lateral and torsional deformations. A correction matrix is added to the tangent stiffness matrix to make displacement derivatives equivalent to the commutative rotational degrees of freedom. Numerical examples show that the proposed element is accurate and efficient in predicting the nonlinear behavior, such as axial-torsional and flexural-torsional buckling, of space frames even when fewer elements are used to model a member.

Key words: beam elements; space frames; tangent stiffness matrix; flexural-torsional buckling; second-order effects; geometric nonlinearity

Geometric nonlinearity is important for investigating the ultimate strength of space frame members that fail due to axial-torsional and flexural-torsional buckling. The conventional beam-column approach cannot predict the flexural-torsional buckling because coupling terms among axial, flexural and torsional displacements are lost in the tangent stiffness matrix. Thus, the development of efficient formulations for nonlinear analysis of space frame structures has attracted the study of many researchers.

Bathe and Bolourchi[1] developed a large deflection finite element formulation and pointed out that the updated Lagrangian formulation is computationally more effective. Yang and McGuire[2] presented a comprehensive formulation of the thin-walled beam element starting from the principle of virtual displacements. This formulation was derived using the updated Lagrangian approach and allowed the inclusion of all types of loading. Chan and Kitipornchai[3] presented a formulation for beam-column elements with asymmetric thin-walled sections. The coupling effect of axial forces, bending moments and torsional moments is also included.

In most research, the cubic Hermite element is extended to the nonlinear analysis of space beams. The incremental secant stiffness matrix including the geometric stiffness with the linear stiffness matrix is often linearised into the tangent stiffness matrix. In order to accurately predict the nonlinear performance of a beam, some higher order terms should be taken into account. Meek and Tan[4] allowed for higher order terms, due to axial force, in their element formulations. Al-Bermani and Kitipornchai[5] proposed an improved analysis technique using fewer elements to model a member via the introduction of the deformation stiffness matrix. Yang and Leu[6] accounted for higher order nonlinear effects in the force recovery procedure in two-dimensional nonlinear analysis and obtained better results than using conventional equations. Liew et al. [7] developed a mixed element for improving nonlinear analysis of space frame structures. Their method is clever in that the merits of the stability functions, cubic Hermite functions and the pointwise equilibrium polynomial (PEP) functions were fully used. In their formulations, the stability functions were employed for the effect due to the axial force, the cubic element for the effect due to bending moments and the PEP element for bowing and member initial out-of-straightness. Thus, the higher order effects were included in their matrix.

This paper describes a new tangent stiffness matrix, which allows for high order effects of element deformations, for geometrically nonlinear analysis of space frames. The following assumptions are made in this study: ① The beam element is slender, the Euler-Bernoulli hypothesis is valid and warping is neglected; ② The beam element is doubly symmetric and prismatic; ③ The material remains elastic within the loading range; ④ The load is conservative, nodal and shear deformations are negligible; ⑤ Cross sections

Received 2005-08-23.

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are compact and rigid and do not distort; ⑥ Small strains but moderately large displacements and rotations are assumed.

1 Element Stiffness Matrix

1.1 Total potential energy of an element

The element stiffness matrix can be derived by applying the principle of stationary potential energy. The total potential energy of a general element is given by

$$\Pi = U - V \tag{1}$$

in which U is the strain energy stored in the element, and V is the external work done.

In the updated Lagrangian formulation, incremental nodal force and incremental nodal displacement vectors $\boldsymbol{f}$ and $\boldsymbol{u}$ at the two ends of the element are given by

$$\boldsymbol{f} = \{f_{x1}, f_{y1}, f_{z1}, m_{x1}, m_{y1}, m_{z1}, f_{x2}, f_{y2}, f_{z2}, m_{x2}, m_{y2}, m_{z2}\}^{\mathrm{T}} \tag{2}$$

$$\boldsymbol{u} = \{u_1, v_1, w_1, \theta_{x1}, \theta_{y1}, \theta_{z1}, u_2, v_2, w_2, \theta_{x2}, \theta_{y2}, \theta_{z2}\}^{\mathrm{T}} \tag{3}$$

The strain energy of a space beam element can be determined as[8]

$$\begin{aligned} U = {} & \frac{1}{2}\int_0^L \left[EA(u')^2 + EI_z(v'')^2 + EI_y(w'')^2 + GJ(\theta_x')^2 \right] \mathrm{d}x + \\ & \int_0^L \frac{F_x}{2}(v'^2 + w'^2)\,\mathrm{d}x + \int_0^L \frac{F_x r_1^2}{2}(\theta_x')^2 \mathrm{d}x + \\ & \int_0^L [F_y(\theta_x w' - u'v') - F_z(\theta_x v' + u'w')]\,\mathrm{d}x - \\ & \int_0^L M_y(v'\theta_x')\,\mathrm{d}x - \int_0^L M_z(w'\theta_x')\,\mathrm{d}x + \\ & \frac{1}{2}\int_0^L M_x(v''w' - v'w'')\,\mathrm{d}x \end{aligned} \tag{4}$$

where E is the elastic modulus; G is the shear modulus; L is the length of element; A is the cross-sectional area; I_y and I_z are the second moments of area; $r_1^2 = (I_y + I_z)/A$ is the polar radius of gyration about the shear center; J is the torsional constant; u is the axial displacement; F_x, F_y, F_z, M_x, M_y and M_z are the nodal forces in and moments about x, y and z axes; θ_x is the angle of twist; v, w are the lateral displacements; a prime represents a derivative with respect to x.

The element nodal force vector in which the above nodal forces at the element ends are contained for the three-dimensional beam element is as follows:

$$\boldsymbol{F} = \{F_{x1}, F_{y1}, F_{z1}, M_{x1}, M_{y1}, M_{z1}, F_{x2}, F_{y2}, F_{z2}, M_{x2}, M_{y2}, M_{z2}\}^{\mathrm{T}} \tag{5}$$

In Eq. (4), bending moments are assumed to be distributed linearly, torques are constant. So forces F_i and M_i at the internal cross-section x can be expressed in terms of those at the element ends using

$$F_x = F_{x2} = P \tag{6}$$

$$F_y = -(M_{z1} + M_{z2})/L \tag{7}$$

$$F_z = (M_{y1} + M_{y2})/L \tag{8}$$

$$M_x = M_{x2} \tag{9}$$

$$M_y = -M_{y1}(1 - x/L) + M_{y2}(x/L) \tag{10}$$

$$M_z = -M_{z1}(1 - x/L) + M_{z2}(x/L) \tag{11}$$

The work done by element nodal force under nodal displacement increments is given by

$$V = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f} \tag{12}$$

Linear interpolation functions are adopted for the axial displacement u and the angle of twist θ_x. Cubic interpolation functions are used for the lateral displacements v and w. Substituting these functions into Eqs. (4) and (1), the expression for the total potential energy may be defined in terms of the incremental nodal displacements at the two ends of the element.

1.2 Incremental tangent stiffness matrix

The beam element secant equilibrium equation can be formulated from the principle of stationary potential energy

$$\delta\Pi = 0 \tag{13}$$

which leads to the element secant stiffness matrix in updated Lagrangian formulation. It can be written as

$$\boldsymbol{f} = (\boldsymbol{k}_{\mathrm{L}} + \boldsymbol{k}_{\mathrm{G}})\boldsymbol{u} \tag{14}$$

where $\boldsymbol{k}_{\mathrm{L}}$ is the linear stiffness matrix, and $\boldsymbol{k}_{\mathrm{G}}$ is the geometric stiffness matrix. In general, the derived incremental secant stiffness matrix is linearised into the tangent stiffness matrix, which is then used as both the predictor and the corrector in the incremental-iterative analysis. Here, the incremental tangent stiffness matrix can be obtained by the differentiation of incremental forces with respect to incremental displacements.

In the co-rotational formulation, the six element basic forces can be obtained in terms of total element basic deformations from Eq. (13) as

$$\begin{aligned} P = EA\Big[& \frac{e}{L} + \frac{2}{30}(\Theta_{z1}^2 + \Theta_{z2}^2) - \frac{1}{30}\Theta_{z1}\Theta_{z2} + \\ & \frac{2}{30}(\Theta_{y1}^2 + \Theta_{y2}^2) - \frac{1}{30}\Theta_{y1}\Theta_{y2} + \frac{r_1^2}{2L^2}\Theta_x^2 \Big] \end{aligned} \tag{15}$$

$$M_{z1} = \left(\frac{4EI_z}{L} + \frac{2PL}{15}\right)\Theta_{z1} + \left(\frac{2EI_z}{L} - \frac{PL}{30}\right)\Theta_{z2} + \frac{M_{y1} + M_{y2}}{6}\Theta_x \tag{16}$$

$$M_{z2} = \left(\frac{2EI_z}{L} - \frac{PL}{30}\right)\Theta_{z1} + \left(\frac{4EI_z}{L} + \frac{2PL}{15}\right)\Theta_{z2} - \frac{M_{y1} + M_{y2}}{6}\Theta_x \tag{17}$$

$$M_{y1} = \left(\frac{4EI_y}{L} + \frac{2PL}{15}\right)\Theta_{y1} + \left(\frac{2EI_y}{L} - \frac{PL}{30}\right)\Theta_{y2} - \frac{M_{z1} + M_{z2}}{6}\Theta_x \tag{18}$$

$$M_{y2} = \left(\frac{2EI_y}{L} - \frac{PL}{30}\right)\Theta_{y1} + \left(\frac{4EI_y}{L} + \frac{2PL}{15}\right)\Theta_{y2} + \frac{M_{z1} + M_{z2}}{6}\Theta_x \tag{19}$$

$$M_x = \left(\frac{GJ}{L} + \frac{Pr^2}{L}\right)\Theta_x + \frac{M_{z1} + M_{z2}}{6}(-\Theta_{y1} + \Theta_{y2}) + \frac{M_{y1} + M_{y2}}{6}(\Theta_{z1} - \Theta_{z2}) \tag{20}$$

where e is the relative axial stretch; $\Theta_x = \Theta_{x2} - \Theta_{x1}$ is the total twist; Θ_{y1}, Θ_{y2}, Θ_{z1} and Θ_{z2} are the total end rotations. It should be noted that the above secant equations are different from those given by Meek and Tan[4]. These equations allow for the coupling between bending and torsion.

Making use of chain differentiation, the terms in the incremental tangent stiffness matrix can be derived from the following equation,

$$K_{ij} = \frac{\partial f_i}{\partial u_j} + \frac{\partial f_i}{\partial P}\frac{\partial P}{\partial u_j} + \sum_{k=1}^{4}\frac{\partial f_i}{\partial M_k}\frac{\partial M_k}{\partial u_j} \tag{21}$$

where M_k represents the bending moment, and $k = y1$, $y2$, $z1$ and $z2$.

It should be noted that the derivatives of axial force and internal bending moments are the same irrespective of the total or the incremental displacements. For example, the differentiation of the axial force with respect to a total rotation is identical to one with respect to an incremnetal rotation as $\partial P/\partial\Theta_{y1} = \partial P/\partial\theta_{y1}$. Only the linear parts are retained in terms $\partial M_k/\partial u_j$.

Eq. (21) can be written in matrix form as[8]

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{k}\mathrm{d}\boldsymbol{u} \tag{22}$$

$$\boldsymbol{k} = \boldsymbol{k}_{\mathrm{L}} + \boldsymbol{k}_{\mathrm{G}} + \boldsymbol{k}_{\mathrm{P}} + \boldsymbol{k}_{\mathrm{M}} \tag{23}$$

in which $\boldsymbol{k}$ is the incremental tangent stiffness matrix, $\boldsymbol{k}_{\mathrm{P}}$ the component due to the variation of axial force, and $\boldsymbol{k}_{\mathrm{M}}$ the component due to the variation of bending moments. They are all obtained by taking variation on the geometric stiffness matrix $\boldsymbol{k}_{\mathrm{G}}$. It is interesting to note that the element deformations in matrix $\boldsymbol{k}_{\mathrm{P}}$ are the total element deformations and those in matrix $\boldsymbol{k}_{\mathrm{M}}$ are deformation increments during iterations. The matrices $\boldsymbol{k}_{\mathrm{P}}$ and $\boldsymbol{k}_{\mathrm{M}}$ are symmetrical and the non-zero components are given in appendix.

2 Consideration of Rotations and Moments in Space

The non-commutative nature of rotations in space makes the three-dimensional large displacement analysis more complicated than two-dimensional analysis. Argyris et al. [9] first proposed the concept of a semitangential rotation. Semitangential rotations are independent of the order of the rotations and can be added like true vectors if their components are about orthogonal axes.

Argyris et al. [9] identified bending moments as the quasitangential moments and St. Venant torques as the semitangential moment. Either of these moment types is generated as a stress resultant at each cross section of the structural members. In the formulation by Yang and Kuo[10], the potential energy associated with all six stress components are considered and the quasitangential properties of the bending moments M_y and M_z and the semitangential property of the torque M_x can be revealed naturally.

Argyris et al. [11] further pointed out that an apparent lack of equilibrium could occur when a joint, which is initially in equilibrium, is subjected to a finite rotation, even though the error associated with the vectorial assumption for three-dimensional small rotations is minimal. The assumption based on the behavior of moments undergoing finite rotations was demonstrated to affect the response of a structural system significantly. In the literature, the problem associated with the lack of equilibrium at rotated structural joints has been solved through the use of semitangential moments and rotations for modeling the nodal moments and rotations. For example, Argyris et al. [9] modified the properties of nodal bending moments from their original quasitangential to a semitangential nature, by adding a correction matrix to the geometric stiffness matrix.

Argyris et al. [11] and Yang and Kuo[10] enforced the conditions of satisfaction of equilibrium for structural joints of a deformed configuration by imposing the joint moment matrix, i. e. a correction matrix superimposed upon the geometric stiffness matrix. In such a situation the nature of quasitangential moment need not be changed, and displacement derivatives are still selected as the rotational degrees of freedom. In this case, the semitangential rotation concept need not be used.

For the spatial frame element, a correction matrix should be added to the incremental tangent stiffness matrix to yield the true equilibrium condition that satisfies the rigid body test. So the incremental tangent stiffness matrix $\boldsymbol{k}$ in Eq. (23) becomes

$$\boldsymbol{k} = \boldsymbol{k}_{\mathrm{L}} + \boldsymbol{k}_{\mathrm{G}} + \boldsymbol{k}_{\mathrm{P}} + \boldsymbol{k}_{\mathrm{M}} + \boldsymbol{k}_{\mathrm{i}} \tag{24}$$

where $\boldsymbol{k}_{\mathrm{i}}$ is the induced moment matrix[10].

$\boldsymbol{k}_{\mathrm{i}}$ is asymmetry for an individual element due to the lack of conjugateness between bending moments and rotation degrees of freedom. The asymmetrical parts will be canceled at the joints when the element is connected to the other elements. When the induced moment matrix is assembled into the structural tangent stiffness matrix, only the symmetrical part is required because of the enforcement of the equilibrium conditions for interconnected elements at structural nodes. Therefore, the symmetrical part of the induced moment matrix, which is referred to as the joint moment matrix $\boldsymbol{k}_{\mathrm{j}}$ (see Ref. [10]), needs to be assembled to form the structure tangent stiffness matrix.

3 Nonlinear Analysis Procedures

In the nonlinear load-deflection analysis of a spatial frame, $\boldsymbol{k}_L$, $\boldsymbol{k}_G$, $\boldsymbol{k}_P$ and $\boldsymbol{k}_j$ are used in the "predictor" phase. This phase involves the solution of structural displacement increments from the total equations of equilibrium. All the components of the tangent stiffness matrix $\boldsymbol{k}$ are used in the "corrector" phase. This phase is concerned with the recovery of the element forces from the element displacement increments obtained in the predictor phase.

The natural deformation approach proposed by Gattass and Abel[12] is adopted in the "corrector" phase. The element incremental displacements can be conceptually decomposed into two parts: the rigid body displacements serve to rotate the initial forces acting on the element from the previous configuration to the current configuration; whereas the natural deformations constitute the only source for generating the incremental forces. The element forces at the current configuration can be calculated as the summation of the incremental forces and the forces at the previous configuration. Once the element natural deformations are obtained, the element force increments can be evaluated mathematically as

$$\boldsymbol{f} = (\boldsymbol{k}_e + \boldsymbol{k}_g + \boldsymbol{k}_i + \boldsymbol{k}_P + \boldsymbol{k}_m)\ \boldsymbol{u}_n \tag{25}$$

in which $\boldsymbol{u}_n$ denotes the natural deformation obtained by excluding the rigid body motions from the element displacement vector $\boldsymbol{u}$ defined by Eq. (3).

The updated internal forces in an element can thus be obtained as

$${}^{i+1}\boldsymbol{F} = {}^{i}\boldsymbol{F} + \boldsymbol{f} \tag{26}$$

in which ${}^{i}\boldsymbol{F}$ are internal forces at the i-th configuration.

4 Numerical Examples

4.1 Axial-torsional buckling of a column

Fig. 1 shows a column of doubly symmetric cross-sections under the action of an axial force. The two ends of the bar are rigidly built-in. The material and section properties are $E = 200$ GPa, $G = 76.9$ GPa, $A = 4.706 \times 10^{-3}$ m^2, $I_y = 6.992 \times 10^{-6}$ m^4, $I_z = 1.949 \times 10^{-5}$ m^4, $J = 2.045 \times 10^{-7}$ m^4, $L = 4.0$ m. The theoretical torsional buckling force can be predicted as $P_{cr} = GJ/r_1^{\,2} = 2.795$ MN. In the geometrical nonlinear analysis, a disturbing torque of $M_t = 2.5 \times 10^{-4}\, PL$ is applied at mid-span of the column to initiate the displacement and the twist. The column is modeled by two or four proposed elements, and, for comparison, by the conventional cubic elements without the inclusion of the present matrices. The same results are obtained by two and four present elements. They are also in good agreement with the results from Timoshenko's theory[13] by the following equation

$$(GJ + Pr_1^2)\frac{\partial \theta_x}{\partial x} = {}^{2}M_x - {}^{1}M_x = 2.5 \times 10^{-4} P \frac{L}{2} \tag{27}$$

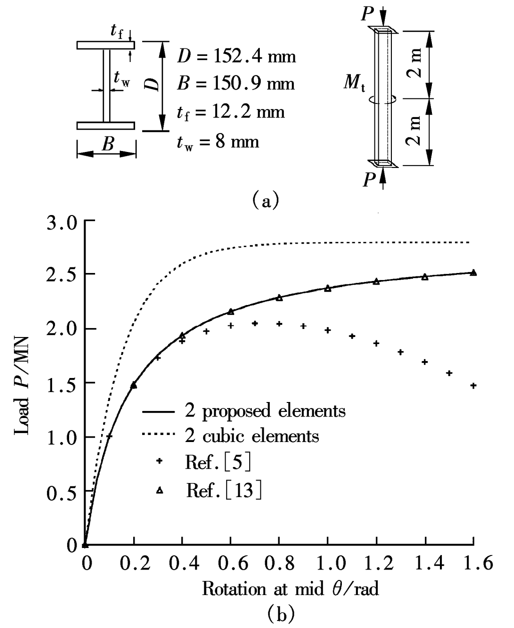

Fig. 1 Load-displacement curves of axially compressed column. (a) Axially compressed column; (b) Load-displacement curves

The results by two or more conventional cubic elements cannot accurately predict the nonlinear behavior because the incremental secant equation does not allow for higher order terms in twist deformations. The results by two elements proposed by Al-Bermani and Kitipornchai[5] are also plotted in Fig. 1, which show that their results are excellent in small and moderately large deformation ranges with twist less than 0.4 rad. The above results are based upon small twist rotation theory in which the effect of an axial force is taken into account. If large twist rotation theory is adopted and the nonlinear component of longitudinal strain is allowed for, the column may be stiffened to prevent helical shortening of the longitudinal fibres of the member during large twist[14].

4.2 Williams' toggle frame

The two-member frame shown in Fig. 2 has been solved both experimentally and analytically by Williams[15]. Recently, the problem was also analyzed by a number of researchers including Liew et al.[7] and Teh and Clarke[16]. Material and section properties are $E = 0.3 \times 10^6$ psi (71 GPa), $A = 0.183$ in^2 (1.181 cm^2) and $I = 9.00 \times 10^{-4}$ in^4 (0.0375 cm^4). This frame is analyzed by using one proposed element and the results are compared with those obtained using the cubic elements and an exact element by Chan and Gu[17].

As shown in Fig. 2, the load-deflection behavior of the frame cannot be predicted accurately by using two or four cubic elements for each frame member.

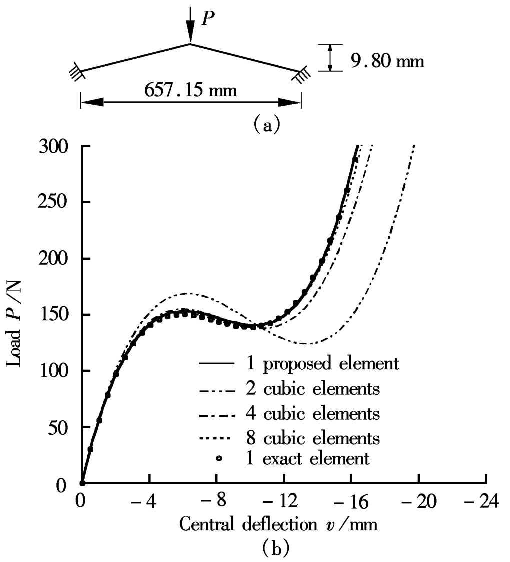

Fig. 2 Load-deflection curves of Williams' toggle frame. (a) Williams' toggle; (b) Load-deflection curves

The necessary accuracy can only be obtained when eight cubic elements are used and the equilibrium path is then close to the results predicted by an exact element. Only one proposed element per member is needed and the load-deflection curve can be predicted with good accuracy when compared with that of the exact element.

Teh and Clarke[16] noticed this situation and claimed that the error using cubic elements is attributed to the updated Lagangrian formulation in which a straight configuration at the last known state is assumed instead of a deformed configuration. They also observed that good accuracy can be obtained by using co-rotational cubic elements. But in this study, the element is still developed in the updated Lagangrian formulation. The conventional cubic element in the updated Lagangrian formulation is based on the element secant stiffness instead of tangent stiffness. The tangent stiffness includes the high order effect of the element deformations as well as the initial stress. This is why the proposed element and co-rotational cubic elements are successful. It can be seen that the proposed element can take the $P-\delta$ effect and member bowing effects into account.

5 Conclusion

This paper describes a refined beam element for geometrically nonlinear analysis of space frames in an updated Lagrangian framework. A new incremental tangent stiffness matrix includes two deformation matrices due to the variation of axial force and bending moments. These proposed matrices are believed to be original. They allow for high order effects of element deformations and incorporate the coupling among axial, lateral and torsional deformations. An induced moment matrix is used together with the tangent stiffness matrix for beam elements to analyze the deflection behavior of space frames. Numerical examples demonstrate that the proposed element is accurate and efficient in predicting the nonlinear behavior, such as axial-torsional and flexural-torsional, of space frames even when fewer elements are used to model a member.

Appendix Deformation Tangent Matrices

- Components of deformation tangent matrix $\boldsymbol{k}_{\mathrm{P}}$

$$K_P(1,2) = -\frac{S_1}{10}(\Theta_{z1}+\Theta_{z2})$$
$$K_P(1,3) = \frac{S_1}{10}(\Theta_{y1}+\Theta_{y2})$$
$$K_P(1,4) = -\frac{r_1^2 S_1}{L}(\Theta_{x1}-\Theta_{x2})$$
$$K_P(1,5) = \frac{S_1 L}{30}(-4\Theta_{y1}+\Theta_{y2})$$
$$K_P(1,6) = \frac{S_1 L}{30}(-4\Theta_{z1}+\Theta_{z2})$$
$$K_P(1,8) = \frac{S_1}{10}(\Theta_{z1}+\Theta_{z2})$$
$$K_P(1,9) = -\frac{S_1}{10}(\Theta_{y1}+\Theta_{y2})$$
$$K_P(1,10) = \frac{r_1^2 S_1}{L}(\Theta_{x1}-\Theta_{x2})$$
$$K_P(1,11) = \frac{S_1 L}{30}(\Theta_{y1}-4\Theta_{y2})$$
$$K_P(1,12) = \frac{S_1 L}{30}(\Theta_{z1}-4\Theta_{z2})$$
$$K_P(7,J) = -K_P(1,J) \quad J=1,2,\ldots,12$$
$$K_P(I,J) = \frac{K_P(1,I)\,K_P(1,J)}{S_1}$$
$$I=2,3,\ldots 6,8,\ldots,12;\ J=2,3,\ldots,6,8,\ldots,12$$

- Components of deformation tangent matrix $\boldsymbol{k}_{\mathrm{M}}$

$$K_M(2,3) = \frac{6(-S_2+S_3)}{L}(\theta_{x1}+\theta_{x2})$$
$$K_M(2,4) = 2S_3(\theta_{y1}-\theta_{y2})$$
$$K_M(2,5) = 2S_2(2\theta_{x1}+\theta_{x2})+2S_3(\theta_{x1}-\theta_{x2})$$
$$K_M(2,9) = \frac{6(S_2-S_3)}{L}(\theta_{x1}+\theta_{x2})$$
$$K_M(2,10) = -2S_3(\theta_{y1}-\theta_{y2})$$
$$K_M(2,11) = 2S_2(\theta_{x1}+2\theta_{x2})-2S_3(\theta_{x1}-\theta_{x2})$$
$$K_M(3,4) = 2S_2(\theta_{z1}-\theta_{z2})$$
$$K_M(3,6) = 2S_2(\theta_{x1}-\theta_{x2})+2S_3(2\theta_{x1}+\theta_{x2})$$
$$K_M(3,8) = \frac{6(S_2-S_3)}{L}(\theta_{x1}+\theta_{x2})$$
$$K_M(3,10) = -2S_2(\theta_{z1}-\theta_{z2})$$
$$K_M(3,12) = -2S_2(\theta_{x1}-\theta_{x2})+2S_3(\theta_{x1}+2\theta_{x2})$$
$$K_M(4,5) = -S_2 L(\theta_{z1}-\theta_{z2})$$
$$K_M(4,6) = S_3 L(\theta_{y1}-\theta_{y2})$$
$$K_M(4,8) = -2S_3(\theta_{y1}-\theta_{y2})$$
$$K_M(4,9) = -2S_2(\theta_{z1}-\theta_{z2})$$
$$K_M(4,11) = -S_2 L(\theta_{z1}-\theta_{z2})$$
$$K_M(4,12) = S_3 L(\theta_{y1}-\theta_{y2})$$
$$K_M(5,6) = (-S_2 L+S_3 L)(\theta_{x1}-\theta_{x2})$$

$$K_M(5,8) = -2S_2(2\theta_{x1}+\theta_{x2}) - 2S_3(\theta_{x1}-\theta_{x2})$$
$$K_M(5,10) = S_2L(\theta_{z1}-\theta_{z2})$$
$$K_M(5,12) = (S_2L+S_3L)(\theta_{x1}-\theta_{x2})$$
$$K_M(6,9) = -2S_2(\theta_{x1}-\theta_{x2}) - 2S_3(2\theta_{x1}+\theta_{x2})$$
$$K_M(6,10) = -S_3L(\theta_{y1}-\theta_{y2})$$
$$K_M(6,11) = -(S_2L+S_3L)(\theta_{x1}-\theta_{x2})$$
$$K_M(8,9) = \frac{6(-S_2+S_3)}{L}(\theta_{x1}+\theta_{x2})$$
$$K_M(8,10) = 2S_3(\theta_{y1}-\theta_{y2})$$
$$K_M(8,11) = -2S_2(\theta_{x1}+2\theta_{x2}) + 2S_3(\theta_{x1}-\theta_{x2})$$
$$K_M(9,10) = 2S_2(\theta_{z1}-\theta_{z2})$$
$$K_M(9,12) = 2S_2(\theta_{x1}-\theta_{x2}) - 2S_3(\theta_{x1}+2\theta_{x2})$$
$$K_M(10,11) = S_2L(\theta_{z1}-\theta_{z2})$$
$$K_M(10,12) = -S_3L(\theta_{y1}-\theta_{y2})$$
$$K_M(11,12) = (S_2L-S_3L)(\theta_{x1}-\theta_{x2})$$

in which

$$S_1 = \frac{EA}{L}, \quad S_2 = \frac{EI_y}{L^2}, \quad S_3 = \frac{EI_z}{L^2}; \quad r_1^2 = \frac{I_y^2+I_z^2}{A}$$

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一种用于空间框架结构几何非线性分析的切线刚度矩阵

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摘要:为了改善空间框架结构的几何非线性分析,根据总势能驻值原理推导了一个三维梁单元,用增量切线刚度矩阵代替传统的增量割线刚度矩阵.新的切线刚度矩阵除了常用的线性刚度和几何刚度矩阵外,还包含2个变形刚度矩阵,这2个矩阵是由每个增量步中轴力和弯距的变化引起的,是单元变形的函数,包含轴向、横向和扭转变形的耦合项.考虑空间转动和弯距的特征,增加一个修正矩阵,使提出的单元通过刚体测试,达到良好的收敛性能.实例显示使用较少的单元模拟构件仍可有效、准确地得到空间框架的轴向扭转和弯扭屈曲的非线性性能.

关键词:梁单元;空间框架;切线刚度矩阵;弯扭屈曲;二阶效应;几何非线性

中图分类号:TU323.5