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Notes on pure projective modules

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Abstract: Let R be an associated ring with identity. A new equivalent characterization of pure projective left R-modules is given by applying homological methods. It is proved that a left R-module P is pure projective if and only if for any pure epimorphism $E \to M \to 0$, where E is pure injective, $\mathrm{Hom}_R(P, E) \to \mathrm{Hom}_R(P, M) \to 0$ is exact. Also, we obtain a dual result of pure injective left R-modules. Furthermore, it is shown that every pure projective left R-module is closed under pure submodule if and only if every pure injective left R-module is closed under pure epimorphic image.

Key words: pure projective left R-module; pure injective left R-module; pure submodule; pure exact sequence

Throughout this paper, R means an associated ring with identity and R-modules are all unital. Purity of modules has been extensively studied by many authors[1-5]. In this paper, we mainly discuss pure projective modules and pure injective modules.

A submodule T of a left R-module N is said to be a pure submodule if $0 \to A \otimes T \to A \otimes N$ is exact for any right R-module A[2]. While $0 \to T \to N$ is said to be a pure monomorphism if T is a pure submodule of N, also, $0 \to T \to N \to N/T \to 0$ is called pure exact[2]. For left R-modules A, C, an epimorphism $f: A \to C$ is called a pure epimorphism if $\mathrm{Hom}(M, f)$: $\mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, C) \to 0$ is exact for any finitely presented left R-module M[1]. A left R-module P is called pure projective if every pure epimorphism $M \to P$ splits for any left R-module M[1]. A left R-module E is called pure injective if every pure monomorphism $E \to M$ splits for any left R-module M[3].

For projective modules and injective modules, we recall the following well known facts: a left R-module M is projective if and only if for every exact sequence $N \to A \to 0$, where N is any injective module, $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, A) \to 0$ is exact; a left R-module N is injective if and only if for every exact sequence $0 \to A \to M$, where M is any projective R-module, $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(A, N) \to 0$ is exact. Naturally, we want to know whether there are similar properties for pure projective modules and pure injective modules? Here, we say yes!

The next two lemmas are well known, so we omit proofs.

Lemma 1[3] Let A, B, C be left R-modules satisfying $A \subseteq B \subseteq C$. If A pure in B, B pure in C, then A pure in C.

Lemma 2[2] Every left R-module is a pure submodule of a pure injective R-module.

Theorem 1 A left R-module P is pure projective if and only if for any pure injective left R-module E and any pure epimorphism $E \to M \to 0$, $\mathrm{Hom}_R(P, E) \to \mathrm{Hom}_R(P, M) \to 0$ is exact.

Proof ($\Rightarrow$) Trivial.

($\Leftarrow$) For any pure exact $0 \to A' \to A \to A'' \to 0$ (where suppose $\lambda: A' \to A$; $\mu: A \to A''$) and any homomorphism $f: P \to A''$. We need to prove that there exists a homomorphism $g: P \to A$ such that $\mu g = f$. Indeed, for the left R-module A, by lemma 2, there exists a pure monomorphism $i: A \to E$ where E is pure injective. By lemma 1, $i\lambda: A' \to E$ is a pure monomorphism. So we have an exact sequence $0 \to A' \to E \to E/A' \to 0$, where $i\lambda: A' \to E$, $\pi: A \to E/A'$. From Five Lemma, we see that there exists a monomorphism $h: A'' \to E/A'$ such that $h\mu = \pi i$. Hence there is a homomorphism $hf: P \to E/A'$. Note that $\pi: E \to E/A'$ is a canonical epimorphism. By assumption, there exists a homomorphism $k: P \to E$ such that $\pi k = hf$. For any $p \in P$, if $k(p) = 0$, there exists unique $a = 0 \in A$ such that $i(a) = 0 = k(p)$ since i is a monomorphism. If $k(p) \neq 0$, then $\pi k(p) \in E/A'$. Since μ is an epimorphism, we have $\mu(a) = f(p)$ for some a

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$\in A$. It follows that

$$\pi k(p) = hf(p) = h\mu(a) = \pi i(a)$$

Hence

$$k(p) - i(a) \subseteq \ker\pi = \mathrm{Im}\, i\lambda \subseteq \mathrm{Im}\, i$$

It is easy to see that $k(p) \subseteq \mathrm{Im}\, i$. Then there exists a unique $a \in A$ such that $i(a) = k(p)$. So we may define a map g: $P \to A$ by $g(p) = a$. From the above, we see that g is a well-defined homomorphism and $ig(p) = i(a) = k(p)$ for any $p \in P$. It follows that

$$h\mu g = \pi i g = \pi k = hf$$

Since h is a monomorphism, it follows that $\mu g = f$.

Now, we prove a dual result of theorem 1. First, we also have two lemmas.

Lemma 3 Let A, B, C be left R-modules. f: $A \to B$ and g: $B \to C$ are both pure epimorphisms, then gf: $A \to C$ is a pure epimorphism.

Proof For any finitely presented module left R-module F and any homomorphism h: $F \to C$. Since g: $B \to C$ is a pure epimorphism, there exists a homomorphism λ: $F \to B$ such that $g\lambda = h$. However, f: $A \to B$ is also a pure epimorphism, then there is a homomorphism k: $F \to A$ satisfying $fk = \lambda$. Hence we obtain that $gfk = g\lambda = h$, i. e., gf is a pure epimorphism.

Lemma 4[4] For every R-module M, there exists a pure projective R-module P such that f: $P \to M$ is a pure epimorphism.

Theorem 2 A left R-module E is pure injective if and only if for any pure projective left R-module P and any pure monomorphism $M \to P$, $\mathrm{Hom}_R(P, E) \to \mathrm{Hom}_R(M, E) \to 0$ is exact.

Proof ($\Rightarrow$) Trivial.

($\Leftarrow$) For any pure exact $0 \to A' \to A \to A'' \to 0$ (where suppose i: $A' \to A$, π: $\to A''$) and any homomorphism f: $A' \to E$. We need to prove there exists a homomorphism g: $A \to E$ such that $gi = f$. By lemma 4, we see that there exists a pure projective left R-module P such that μ: $P \to A$ is a pure epimorphism for a left R-module A . It follows from lemma 3 that $\pi\mu$: $P \to A''$ is also a pure epimorphism. Hence there is a pure monomorphism λ: $K \to P$ where $K = \ker\pi\mu$. From diagram chasing and the Five Lemma, it has an epimorphism φ: $K \to A'$ such that $\mu\lambda = i\varphi$. By assumption, there exists a homomorphism h: $P \to E$ such that $h\lambda = f\varphi$. If $\ker\mu \subseteq \ker h$, then from Factor Theorem, there exists a homomorphism g: $A \to E$ such that $g\mu = h$. Hence

$$gi\varphi = g\mu\lambda = h\lambda = f\varphi$$

Note that φ is an epimorphism, it follows that $gi = f$.

Now, we prove $\ker\mu \subseteq \ker h$. For any $p \in P$ and $\mu(p) = 0$, then $\pi\mu(p) = 0$, which implies $p \in \ker\pi\mu = \mathrm{Im}\lambda$. Hence $\lambda(k) = p$ for some $k \in K$, and $f\varphi(k) = h\lambda(k) = h(p)$. Since $0 = \mu(p) = \mu\lambda(k) = i\varphi(k)$ and i is a monomorphism, it follows that $\varphi(k) = 0$. Therefore, $h(p) = f\varphi(k) = 0$. The proof is complete.

Theorem 3 For a ring R, the following statements are equivalent:

① Every pure submodule of pure projective left R-module is pure projective;

② Every pure epimorphic image of pure injective left R-module is pure injective.

Proof ①$\Rightarrow$② Assume E is a pure injective left R-module, and π: $E \to M$ is any pure epimorphism. We need to prove M is pure injective. For any pure monomorphism i: $A \to P$ with P pure projective and any homomorphism f: $A \to M$. Since P is a pure projective left R-module, A is also a pure projective left R-module by assumption. There exists a homomorphism g: $A \to E$ such that $\pi g = f$. However E is a pure injective left R-module, it follows that there is a homomorphism h: $P \to E$ such that $g = hi$. Put $k = \pi h$. Then we have

$$ki = \pi hi = \pi g = f$$

From theorem 2, M is pure injective.

②$\Rightarrow$① Suppose P is a pure projective left R-module, T is a pure submodule of P. We claim T is pure projective. For any pure epimorphism f: $E \to M$ where E is a pure injective left R-module and any homomorphism g: $T \to M$. By assumption, M is pure injective since E is pure injective. Hence there exists a homomorphism h: $P \to M$ such that $hi = g$. Note that P is pure projective, then there exists a homomorphism k: $P \to E$ such that $fk = h$. Let $\varphi = ki$. Then we have

$$f\varphi = fki = hi = g$$

From theorem 1, T is a pure projective left R-module.

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关于纯投射模的注记

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摘要:R 表示有单位元的结合环. 通过同调的方法,给出了纯投射左 R-模的一个新的等价刻画. 证明了左 R-模 P 是纯投射的当且仅当对任意纯满射 $E\to M\to 0$,其中 E 是纯内射的, $\mathrm{Hom}_R(P,E)\to\mathrm{Hom}_R(P,M)\to 0$ 是正合的. 同时,关于纯内射模的对偶结果也是成立的. 最后,作为应用,证明了每一纯投射左 R-模在纯子模下封闭当且仅当每一纯内射左 R-模在纯满像下封闭.

关键词:纯投射左 R-模;纯内射左 R-模;纯子模;纯正合列

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