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Global exponential periodicity of a class of impulsive neural networks

Liang Jinling

(Department of Mathematics, Southeast University, Nanjing 210096, China)

Abstract: By the Lyapunov function method, combined with the inequality techniques, some criteria are established to ensure the existence, uniqueness and global exponential stability of the periodic solution for a class of impulsive neural networks. The results obtained only require the activation functions to be globally Lipschitz continuous without assuming their boundedness, monotonicity or differentiability. The conditions are easy to check in practice and they can be applied to design globally exponentially periodic impulsive neural networks.

Key words: global exponential periodicity; impulsive neural networks; Lyapunov function; Lipschitz activation function

1 Preliminaries

In the past twenty years, many types of neural networks have been extensively analyzed. The results have been applied in signal processing, optimization computation and knowledge acquisition, etc. [1 – 7]. Most neural networks are classified into two categories: continuous-time or discrete-time. Recently there has been a somewhat new category of neural networks, which is neither a purely continuous-time nor a purely discrete-time one; these are called neural networks with impulses. The third category of neural networks displays a combination of characteristics of both the continuous-time and the discrete-time systems[8 – 12]. Impulses can make an unstable system stable, so it has been widely used in physics, chemistry, biology, population dynamics, industrial robotics, and so on. In this paper, the following class of impulsive neural networks will be studied:

$$\left.\begin{aligned} &\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i x_i(t) + g_i\left[\sum_{j=1}^{n} b_{ij}x_j(t) + I_i(t)\right] && 0 \leqslant t \neq t_k + r\omega \\ &x_i(t+0) = \delta_{ikr}x_i(t) && t = t_k + r\omega \end{aligned}\right\} \tag{1}$$

where $i=1,2,\ldots,n$; $k=1,2,\ldots,m$; $r\in\mathbf{Z}_+=\{0,1,2,\ldots\}$ and n denotes the number of neurons in the networks; $x_i(t)$ denotes the average membrane potential of the i-th neuron at time t; $a_i>0$ denotes the rate at which the i-th neuron resets the state when it is isolated from the system; b_{ij} represents the synaptic connection strengths among the neurons; the function g_i represents the response of the i-th neuron to its membrane potential and is known as the activation function; $I_i(t)$ denotes a constant external input current to the i-th neuron, $I_i(\cdot)$ is an ω-periodic and Lipschitz continuous function, $0=t_1<t_2<\ldots<t_m<\omega$; δ_{ikr} is constant for fixed i,k,r.

Throughout this paper, the activation functions $g_i(\cdot)$ $(i=1,2,\ldots,n)$ are assumed to possess the following property:

There exist constant scalars $l_i>0$ such that for any $u,v\in\mathbf{R}$

$$|g_i(u)-g_i(v)|\leqslant l_i|u-v| \qquad i=1,2,\ldots,n$$

Definition 1 Function $\boldsymbol{x}(t)=\{x_1(t),x_2(t),\ldots,x_n(t)\}^{\mathrm{T}}\in\mathbf{R}^n$ is said to be a solution of impulsive neural networks (1) if for all $i=1,2,\ldots,n$; $k=1,2,\ldots,m$; $r\in\mathbf{Z}_+$:

① $x_i(\cdot)$ is piecewise continuous on (t_1,β) for some $\beta>t_1$ such that $x_i(t_k+r\omega+)$ and $x(t_k+r\omega-)$ exist and $x_i(\cdot)$ is differentiable on intervals of the form $(t_{k-1}+r\omega, t_k+r\omega)\subseteq(t_1,\beta)$;

② $x_i(t)$ satisfies system (1) and $x_i(t)$ is right continuous at point $t=t_k+r\omega$.

Definition 2 The impulsive neural networks (1) is said to be globally exponentially periodic, if system (1) has a periodic solution $x^*(t)$ and there exist positive constants α,β such that any solution $x(t)$ of (1) satisfies

$$\|x(t)-x^*(t)\|\leqslant\alpha\|x(0)-x^*(0)\|\mathrm{e}^{-\beta t} \qquad t\geqslant 0$$

Lemma 1[7] Suppose $f(t)$ is a differentiable function defined on $\mathbf{R}_+$. Then the upper right Dini derivative

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Biography: Liang Jinling (1974—), female, lecturer, jinlliang@seu.edu.cn.

$D^+|f(t)|$ exists and has the following form:

$$D^+|f(t)| = \overline{\lim_{h\to 0+}}\frac{|f(t+h)|-|f(t)|}{h} = S(f(t))\dot{f}(t) \qquad t\in\mathbf{R}_+$$

in which

$$S(f(t)) = \begin{cases} f(t)/|f(t)| & \text{if } f(t)\neq 0 \\ 1 & \text{if } f(t)=0 \text{ and } \dot{f}(t)>0 \\ -1 & \text{if } f(t)=0 \text{ and } \dot{f}(t)<0 \\ 0 & \text{if } f(t)=0 \text{ and } \dot{f}(t)=0 \end{cases}$$

2 Main Results

In this section, it will be shown that under certain conditions system (1) has a unique periodic solution which is globally exponentially stable.

Theorem 1 For every periodic input $\boldsymbol{I}(t)=\{I_1(t), I_2(t), \dots, I_n(t)\}^{\mathrm{T}}, \boldsymbol{I}(t+\omega)=\boldsymbol{I}(t)$, the impulsive neural networks model (1) has a unique globally exponentially stable periodic solution $x^*(t)$, if there exist constants $\lambda_i>0, i=1,2,\dots,n$ such that

$$a_i > \sum_{j=1}^{n}\frac{\lambda_i}{\lambda_j}l_i|b_{ij}| \tag{2}$$

and

$$|\delta_{ikr}|\leqslant 1 \qquad i=1,2,\dots,n; k=1,2,\dots,m; r\in\mathbf{Z}_+ \tag{3}$$

Proof Let $u(t)$ and $v(t)$ be two solutions of system (1), it follows from (2) that there exists an $\varepsilon>0$ such that

$$\min_{1\leqslant i\leqslant n}\left\{a_i-\varepsilon-\sum_{j=1}^{n}\frac{\lambda_i}{\lambda_j}l_i|b_{ij}|\right\} =: \eta_1 > 0 \tag{4}$$

We consider a Lyapunov function $V(t)=V(t,u(t)-v(t))$ defined by

$$V(t) = \mathrm{e}^{\varepsilon t}\|u(t)-v(t)\|_{(\lambda,\infty)} = \mathrm{e}^{\varepsilon t}\max_{1\leqslant i\leqslant n}|\lambda_i(u_i(t)-v_i(t))| = \mathrm{e}^{\varepsilon t}\lambda_{i_0}|u_{i_0}(t)-v_{i_0}(t)|$$

where $i_0\in\{1,2,\dots,n\}$ is a function of the time variable t.

We first calculate the derivative of $V(t)$ along the solutions of system (1). At $t\neq t_k+r\omega$,

$$\begin{aligned} D^+V(t) &= \varepsilon\mathrm{e}^{\varepsilon t}\lambda_{i_0}|u_{i_0}(t)-v_{i_0}(t)| + \mathrm{e}^{\varepsilon t}\lambda_{i_0}D^+|u_{i_0}(t)-v_{i_0}(t)| \leqslant \varepsilon V(t) + \\ &\lambda_{i_0}\mathrm{e}^{\varepsilon t}\left[-a_{i_0}|u_{i_0}(t)-v_{i_0}(t)| + \sum_{j=1}^{n}\frac{l_{i_0}}{\lambda_j}|b_{i_0 j}|\lambda_j|u_j(t)-v_j(t)|\right] \leqslant \\ &\left[-a_{i_0}+\varepsilon+\sum_{j=1}^{n}\frac{\lambda_{i_0}}{\lambda_j}l_{i_0}|b_{i_0 j}|\right]V(t) \leqslant -\eta_1 V(t) \end{aligned} \tag{5}$$

Next, at $t=t_k+r\omega$,

$$\begin{aligned} V(t+0)-V(t) &= \mathrm{e}^{\varepsilon t}\lambda_{i_0}[|u_{i_0}(t+0)-v_{i_0}(t+0)| - |u_{i_0}(t)-v_{i_0}(t)|] = \\ &\lambda_{i_0}\mathrm{e}^{\varepsilon t}(|\delta_{ikr}|-1)|u_{i_0}(t)-v_{i_0}(t)| = (|\delta_{ikr}|-1)V(t) \leqslant 0 \end{aligned} \tag{6}$$

From (5) and (6) we have $V(t)=\mathrm{e}^{\varepsilon t}\|u(t)-v(t)\|_{(\lambda,\infty)}\leqslant V(0)=\|u(0)-v(0)\|_{(\lambda,\infty)}$, i. e.

$$\|u(t)-v(t)\|_{(\lambda,\infty)} \leqslant \mathrm{e}^{-\varepsilon t}\|u(0)-v(0)\|_{(\lambda,\infty)} \qquad \forall t\geqslant 0 \tag{7}$$

To prove that the impulsive neural networks model (1) possesses an ω-periodic solution, we define a mapping P: $\mathbf{R}^n\to\mathbf{R}^n$ by $Pu(0)=u(\omega)=u_\omega$. Choose a positive number m such that $m\geqslant \ln 3/(\varepsilon\omega)$, where ε is the number defined in (4). It follows from (7) that

$$\|P^m u(0)-P^m v(0)\|_{(\lambda,\infty)} = \|u(m\omega)-v(m\omega)\|_{(\lambda,\infty)} \leqslant \mathrm{e}^{-\varepsilon m\omega}\|u(0)-v(0)\|_{(\lambda,\infty)} \leqslant \frac{1}{3}\|u(0)-v(0)\|_{(\lambda,\infty)}$$

The above formula says that P^m is a contraction mapping on $\mathbf{R}^n$. According to the Banach contraction mapping principle, P^m has a unique fixed point $\tilde{x}\in\mathbf{R}^n$. Since

$$P^m(P\tilde{x}) = P(P^m\tilde{x}) = P\tilde{x}$$

it means that $P\tilde{x}$ is also a fixed point of mapping P^m. From the uniqueness of the fixed point, we know $P\tilde{x}=\tilde{x}$, i. e., $\tilde{x}$ is also a fixed point of mapping P. Let $x^*(t)$ denote the solution of (1) with $\tilde{x}$ as its starting point, then

$$x^*(0)=\tilde{x}=P\tilde{x}=x^*(\omega), \quad x^*(t+\omega)=x^*(t) \qquad \forall t\geqslant 0 \tag{8}$$

That is to say, $x^*(t)$ is an ω-periodic solution of (1), and also from (7), we know it is globally exponentially stable.

Remark In the proof of theorem 1,

$$D^+\max_{1\leqslant i\leqslant n}|\lambda_i(u_i(t)-v_i(t))| = \lim_{h\to 0+}\frac{1}{h}\left(\max_{1\leqslant i\leqslant n}|\lambda_i(u_i(t+h)-v_i(t+h))| - \max_{1\leqslant i\leqslant n}|u_i(t)-v_i(t)|\right)$$

Now suppose $\max_{1\leqslant i\leqslant n} |\lambda_i(u_i(t+h)-v_i(t+h))| = \lambda_k |u_k(t+h)-v_k(t+h)|$, since $|\lambda_i(u_i(t)-v_i(t))|$ $(i=1,2,\ldots,n)$ are all continuous functions, we have

$$\max_{1\leqslant i\leqslant n} |\lambda_i(u_i(t)-v_i(t))| = \lambda_{i_0}|u_{i_0}(t)-v_{i_0}(t)| = \lambda_k |u_k(t)-v_k(t)|$$

although the above may be not true for $k=i_0$. At this time, we only need to take $i_0=k$, and the proof in theorem 1 about the Dini derivative still holds.

Theorem 2 For every periodic input $\boldsymbol{I}(t)=\{I_1(t), I_2(t), \ldots, I_n(t)\}^{\mathrm{T}}$, $\boldsymbol{I}(t+\omega)=\boldsymbol{I}(t)$, the impulsive neural networks model (1) has a unique globally exponentially stable periodic solution $x^*(t)$, if there exist constants $\lambda_i>0$, $i=1,2,\ldots,n$ such that

$$\min_{1\leqslant i\leqslant n} a_i > \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda_i}{\lambda_j}l_i^2 b_{ij}^{\ 2}} \tag{9}$$

and

$$|\delta_{ikr}| \leqslant 1 \qquad i=1,2,\ldots,n;\ k=1,2,\ldots,m;\ r\in\mathbf{Z}_+ \tag{10}$$

Proof Let $u(t)$ and $v(t)$ be two solutions of system (1), it follows from (9) that there exists an $\varepsilon>0$ such that

$$2\min_{1\leqslant i\leqslant n} a_i - \varepsilon - 2\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda_i}{\lambda_j}l_i^{\ 2} b_{ij}^2} \eqqcolon \eta_2 > 0$$

We consider a Lyapunov function $V(t)=V(t, u(t)-v(t))$ defined by

$$V(t) = \frac{1}{2}\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\lambda_i(u_i(t)-v_i(t))^2$$

Calculating the derivative of $V(t)$ along the solutions of system (1) and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \varepsilon V(t) + \mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\lambda_i\Big\{-a_i(u_i(t)-v_i(t))^2 + (u_i(t)-v_i(t))\Big[g_i\Big(\sum_{j=1}^{n}b_{ij}u_j(t)+I_i(t)\Big) - \\
&\quad g_i\Big(\sum_{j=1}^{n}b_{ij}v_j(t)+I_i(t)\Big)\Big]\Big\} \leqslant \varepsilon V(t) - \mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\lambda_i a_i(u_i(t)-v_i(t))^2 + \\
&\quad \mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\sqrt{\lambda_i}\,|u_i(t)-v_i(t)|\sum_{j=1}^{n}\sqrt{\frac{\lambda_i}{\lambda_j}}\,|b_{ij}|\,l_i\sqrt{\lambda_j}\,|u_j(t)-v_j(t)| \leqslant (\varepsilon - 2\min_{1\leqslant i\leqslant n}a_i)V(t) + \\
&\quad \mathrm{e}^{\varepsilon t}\sqrt{\sum_{i=1}^{n}\lambda_i|u_i(t)-v_i(t)|^2}\sqrt{\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\sqrt{\frac{\lambda_i}{\lambda_j}}\,|b_{ij}|\,l_i\sqrt{\lambda_j}\,|u_j(t)-v_j(t)|\Big)^2} \leqslant \\
&\quad (\varepsilon - 2\min_{1\leqslant i\leqslant n}a_i)V(t) + \mathrm{e}^{\frac{\varepsilon t}{2}}\sqrt{2V(t)}\sqrt{\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\frac{\lambda_i}{\lambda_j}b_{ij}^2 l_i^2\Big)\Big(\sum_{j=1}^{n}\lambda_j(u_j(t)-v_j(t))^2\Big)} = \\
&\quad \Big(\varepsilon - 2\min_{1\leqslant i\leqslant n}a_i + 2\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda_i}{\lambda_j}b_{ij}^2 l_i^{\ 2}}\Big)V(t) = -\eta_2 V(t) \qquad t\neq t_k + r\omega
\end{aligned} \tag{11}$$

Next, when $t=t_k+r\omega$, then

$$\begin{aligned}
V(t+0) - V(t) &= \frac{1}{2}\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\lambda_i(u_i(t+0)-v_i(t+0))^2 - \frac{1}{2}\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\lambda_i(u_i(t)-v_i(t))^2 = \\
&\quad \frac{1}{2}\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\lambda_i(\delta_{ikr}^2-1)(u_i(t)-v_i(t))^2 \leqslant 0
\end{aligned} \tag{12}$$

From (11) and (12), we have $V(t)=\mathrm{e}^{\varepsilon t}\|u(t)-v(t)\|_{(\lambda,2)} \leqslant V(0) = \|u(0)-v(0)\|_{(\lambda,2)}$, i.e.,

$$\|u(t)-v(t)\|_{(\lambda,2)} \leqslant \mathrm{e}^{-\varepsilon t}\|u(0)-v(0)\|_{(\lambda,2)} \tag{13}$$

The remaining part of the proof is similar to that of theorem 1 and is omitted.

Corollary 1 For every periodic input $\boldsymbol{I}(t)=\{I_1(t), I_2(t), \ldots, I_n(t)\}^{\mathrm{T}}$, $\boldsymbol{I}(t+\omega)=\boldsymbol{I}(t)$, the impulsive neural networks model (1) has a unique globally exponentially stable periodic solution $x^*(t)$, if

$$a_i > \sum_{j=1}^{n} l_i|b_{ij}|,\quad |\delta_{ikr}| \leqslant 1 \qquad i=1,2,\ldots,n;\ k=1,2,\ldots,m;\ r\in\mathbf{Z}_+$$

Corollary 2 For every periodic input $\boldsymbol{I}(t)=\{I_1(t), I_2(t), \ldots, I_n(t)\}^{\mathrm{T}}$, $\boldsymbol{I}(t+\omega)=\boldsymbol{I}(t)$, the impulsive neural networks model (1) has a unique globally exponentially stable periodic solution $x^*(t)$, if

$$\min_{1\leqslant i\leqslant n} a_i > \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} l_j^{\ 2} b_{ij}^2},\quad |\delta_{ikr}| \leqslant 1 \qquad i=1,2,\ldots,n;\ k=1,2,\ldots,m;\ r\in\mathbf{Z}_+$$

When $\delta_{ikr}=1$ for all $i=1,2,\dots,n; k=1,2,\dots,m; r\in \boldsymbol{Z}_+$, the impulsive neural networks (1) will become continuous neural networks:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}=-a_i x_i(t)+g_i\left[\sum_{j=1}^{n} b_{ij}x_j(t)+I_i(t)\right]\qquad t\geqslant 0;\ i=1,2,\dots,n \tag{14}$$

and the dynamics of system (14) have been extensively studied in Refs. [12 – 14]. From theorem 1 and theorem 2, we have the following corollaries.

Corollary 3 For every periodic input $\boldsymbol{I}(t)=\{I_1(t), I_2(t),\dots, I_n(t)\}^{\mathrm{T}}$, $\boldsymbol{I}(t+\omega)=\boldsymbol{I}(t)$, the neural networks model (14) has a unique globally exponentially stable periodic solution $x^*(t)$, if there exist constants $\lambda_i>0$, $i=1, 2,\dots,n$ such that

$$a_i>\sum_{j=1}^{n}\frac{\lambda_i}{\lambda_j}l_i\mid b_{ij}\mid$$

Corollary 4 For every periodic input $\boldsymbol{I}(t)=\{I_1(t), I_2(t),\dots, I_n(t)\}^{\mathrm{T}}$, $\boldsymbol{I}(t+\omega)=\boldsymbol{I}(t)$, the neural networks model (14) has a unique globally exponentially stable periodic solution $x^*(t)$, if there exist constants $\lambda_i>0$, $i=1, 2,\dots,n$ such that

$$\min_{1\leqslant i\leqslant n} a_i>\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda_i}{\lambda_j}l_i^2 b_{ij}^2}$$

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一类脉冲神经网络的全局指数周期解

梁金玲

（东南大学数学系，南京 210096）

摘要：利用 Lyapunov 函数并结合不等式的技巧，给出了一些充分判据来确保一类脉冲神经网络系统具有全局指数稳定性的周期解. 给出的充分判据仅仅要求激励函数是李普希兹连续的，而对它的有界性、单调性及可微性都不再要求. 这些充分判据在实际应用中非常容易验证，也可利用这些判据来设计全局指数周期的脉冲神经网络.

关键词：全局指数周期；脉冲神经网络；Lyapunov 函数；Lipschitz 激励函数

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