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Auto-Bäcklund transformation and exact solutions of Wick-type stochastic Burgers equation

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Abstract: Burgers equation in random environment is studied. In order to give the exact solutions of random Burgers equation, we only consider the Wick-type stochastic Burgers equation which is the perturbation of the Burgers equation with variable coefficients by white noise $W(t)=\dot{B}_t$, where B_t is a Brown motion. The auto-Bäcklund transformation and stochastic soliton solutions of the Wick-type stochastic Burgers equation are shown by the homogeneous balance and Hermite transform. The generalization of the Wick-type stochastic Burgers equation is also studied.

Key words: Wick-type stochastic Burgers equation; auto-Bäcklund transformation; stochastic soliton solution; white noise; Hermite transform; homogeneous balance principle

This paper is devoted to the exact solutions of the Wick-type stochastic Burgers equation as the following form:

$$U_t+f(t)U\diamond U_x+g(t)U_{xx}=W(t)\diamond R^{\diamond}(t,U,U_x,U_{xx}) \tag{1}$$

which is the perturbation of the Bergers equation with variable coefficients

$$u_t+f(t)uu_x+g(t)u_{xx}=0 \tag{2}$$

by random force $W(t)\diamond R^{\diamond}(t,U,U_x,U_{xx})$. Where $f(t)$ and $g(t)$ are bounded or integrable functions on $\mathbf{R}_+$, $W(t)$ is the Gaussian white noise, i. e., $W(t)=\dot{B}_t$ and B_t is a Brown motion, $R(t,u,u_x,u_{xx})=\alpha uu_x+\beta u_{xx}$ is a function of u, u_x and u_{xx} for some constants α,β and $R^{\diamond}$ is the Wick version of the function R.

Random waves is an important subject of stochastic partial differential equation (SPDE). Many authors studied this subject, e. g., Konotop and Vázquez[1], Chen and Xie[2 – 4], Morien[5], Xie[6 – 11], and so on. In Ref. [12], Holden et al. gave white noise functional approach to research stochastic partial differential equations in Wick versions. As Chen and Xie did in Refs. [2 – 4] and Xie did in Refs. [6 – 11], we will give exact solutions of the stochastic Wick-type Burgers equation (1) by the Hermite transform and the homogeneous balance principle. When $\alpha=\beta=0$, we get the exact solutions of the variable coefficients Bergers equation (2). The homogeneous balance principle which was given by Wang in Ref. [13] has been widely applied to derive the nonlinear transformations and exact solutions (especially the solitary waves), and auto-Bäcklund transformations as well as the similarity reductions of nonlinear partial differential equations (PDEs) in mathematical physics. These subjects have been researched by many authors, such as Wang[13], et al.

1 Soliton Solutions of Stochastic Burgers Equation

In this section, we will give exact solutions of Eq. (1) by theorem 2. 1 of Chen and Xie[2] with $d=1$. Taking the Hermite transform of Eq. (1), we get

$$\widetilde{U}_t(t,x,z)+f(t)\widetilde{U}(t,x,z)\widetilde{U}_x(t,x,z)+g(t)\widetilde{U}_{xx}(t,x,z)=\widetilde{W}(t,z)[\alpha\widetilde{U}(t,x,z)\widetilde{U}_x(t,x,z)+\beta\widetilde{U}_{xx}(t,x,z)] \tag{3}$$

Let $\alpha(t,z)=(f(t)-\alpha\widetilde{W}(t,z))$ and $\beta(t,z)=(g(t)-\beta\widetilde{W}(t,z))$, Eq. (3) can be written as

$$\widetilde{U}_t(t,x,z)+\alpha(t,z)\widetilde{U}(t,x,z)\widetilde{U}_x(t,x,z)+\beta(t,z)\widetilde{U}_{xx}(t,x,z)=0 \tag{4}$$

where the Hermite transform of $W(t)$ defined by $\widetilde{W}(t,z)=\sum_{k=1}^{\infty}\eta_k(t)z_k$, $z=(z_1,z_2,\dots)\in\mathbf{C}^{\mathbf{N}+}$ is a parameter. We first solve Eq. (4).

Put $v(t,x,z)=\widetilde{U}(t,x,z)$. For any $z\in\mathbf{C}^{\mathbf{N}+}$, according to the idea of the homogeneous balance principle we sup-

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pose that the solution of Eq. (4) is the form

$$v(t,x,z)=A_1F'(\varphi(t,x,z))\varphi_x(t,x,z)+A_2 \tag{5}$$

where A_1 and A_2 are the constants, $F(\varphi)$ is a function of one variable only, $F(\varphi)$ and φ are to be determined later. From Eq. (5), we have

$$\begin{aligned} v_t+\alpha(t,z)vv_x+\beta(t,z)v_{xx}=&A_1[A_1\alpha(t,z)F'F''+\beta(t,z)F''']\varphi_x^3+A_1^2\alpha(t,z)(F')^2\varphi_x\varphi_{xx}+\\ &A_1F''[\varphi_x\varphi_t+A_2\alpha(t,z)\varphi_x^2]+3A_1\beta(t,z)F''\varphi_x\varphi_{xx}+A_1F'[\varphi_{xt}+A_2\alpha(t,z)\varphi_{xx}+\beta(t,z)\varphi_{xxx}] \end{aligned} \tag{6}$$

Let

$$A_1\alpha(t,z)F'F''+\beta(t,z)F'''=0 \tag{7}$$

Then the solution of Eq. (7) is

$$F(\varphi)=K\ln\varphi \tag{8}$$

where we suppose $KA_1\alpha(t,z)=2\beta(t,z)$ and $K\neq 0$ is any constant. Hence, we have

$$v_t+\alpha(t,z)vv_x+\beta(t,z)v_{xx}=\frac{KA_1}{\varphi^2}\{\varphi[\varphi_{xt}+A_2\alpha(t,z)\varphi_{xx}+\beta(t,z)\varphi_{xxx}]-\varphi_x[\varphi_t+A_2\alpha(t,z)\varphi_x+\beta(t,z)\varphi_{xx}]\} \tag{9}$$

For any fixed $z\in\mathbf{C}^{N+}$, we have the Bäcklund transformation of (4) as follows:

$$v(t,x,z)=KA_1(\ln\varphi)_x+A_2 \tag{10}$$

$$\varphi[\varphi_{xt}+A_2\alpha(t,z)\varphi_{xx}+\beta(t,z)\varphi_{xxx}]-\varphi_x[\varphi_t+A_2\alpha(t,z)\varphi_x+\beta(t,z)\varphi_{xx}]=0 \tag{11}$$

Using the Bäcklund transformation (10) and (11) we can obtain the solitary wave solutions of (4). In fact, suppose that Eq. (11) has the following solution:

$$\varphi(t,x,z)=p(t,z)+\sum_{j=1}^{N}\exp[\psi_j(t,x,z)] \tag{12}$$

where $\psi_j(t,x,z)=q_j(t)x+r_j(t,z)+\eta_j$, $q_j(t)$, $r_j(t,z)$, $j=1,2,\dots,N$ will be determined later, η_j, $j=1,2,\dots,N$ are arbitrary constants. Then we have

$$\begin{aligned} &\varphi[\varphi_{xt}+A_2\alpha(t,z)\varphi_{xx}+\beta(t,z)\varphi_{xxx}]-\varphi_x[\varphi_t+A_2\alpha(t,z)\varphi_x+\beta(t,z)\varphi_{xx}]=\\ &\sum_{j=1}^{N}\Big[-\frac{\partial p}{\partial t}q_j+p\Big(\frac{\partial q_j}{\partial t}+q_j\frac{\partial r_j}{\partial t}+A_2\alpha q_j^{\,2}+\beta q_j^{\,3}\Big)\Big]e^{\psi_j}+p\sum_{j=1}^{N}q_j\frac{\partial q_j}{\partial t}xe^{\psi_j}+\sum_{j,l=1}^{N}\Big(-q_j\frac{\partial q_l}{\partial t}+q_l\frac{\partial q_l}{\partial t}\Big)xe^{\psi_j+\psi_l}+\\ &\sum_{j,l=1}^{N}\Big(-q_j\frac{\partial r_l}{\partial t}-A_2\alpha q_jq_l-\beta q_jq_l^2+\frac{\partial q_l}{\partial t}+q_l\frac{\partial r_l}{\partial t}+A_2\alpha q_l^2+\beta q_j^3\Big)e^{\psi_j+\psi_l} \end{aligned} \tag{13}$$

Since e^{ψ_j}, xe^{ψ_j}, $e^{\psi_j+\psi_l}$ and $xe^{\psi_j+\psi_l}$ are linear independence, we have $\frac{\partial q_j}{\partial t}=0$ for $j=1,2,\dots,N$, that is q_j, $j=1,2,\dots,N$ are constants. These yield

$$-\frac{\partial p}{\partial t}q_j+p\Big(q_j\frac{\partial r_j}{\partial t}+A_2\alpha q_j^{\,2}+\beta q_j^{\,3}\Big)=0 \tag{14}$$

and

$$\Big(\frac{\partial r_j}{\partial t}-\frac{\partial r_l}{\partial t}\Big)(q_j-q_l)+A_2\alpha(q_j-q_l)^2+\beta(q_j-q_l)(q_j^{\,2}-q_l^{\,2})=0 \tag{15}$$

Eq. (14) implies

$$\frac{1}{p}\frac{\partial p}{\partial t}=\frac{\partial r_j}{\partial t}+A_2\alpha q_j+\beta q_j^{\,2} \tag{16}$$

and for $j\neq l$, Eq. (15) yields

$$\Big(\frac{\partial r_j}{\partial t}-\frac{\partial r_l}{\partial t}\Big)=A_2\alpha(q_l-q_j)+\beta(q_l^{\,2}-q_j^{\,2}) \tag{17}$$

Choose $r_1(t,z)=r(t)$, by (16) and (17) we have, for $j=2,3,\dots,N$,

$$p(t,z)=C_0\exp\Big\{\int_0^t[r'(s)+A_2q_1f(s)+q_1^2g(s)-(\alpha A_2q_1+\beta q_1^2)\widetilde{W}(s,z)]\,ds\Big\} \tag{18}$$

$$r_j(t,z)=\int_0^t[r'(s)+A_2q_{1j}f(s)+q_{2j}g(s)-(\alpha A_2q_{1j}+\beta q_{2j})\widetilde{W}(s,z)]\,ds+\overline{C}_j \tag{19}$$

where $q_{1j}=q_1-q_j$, $q_{2j}=q_1^{\,2}-q_j^{\,2}$, C_0 and $\overline{C}_j$, $j=1,2,\dots,N$ are constants.

From Eqs. (10), (12), (18) and (19), the solution of Eq. (4) is given by

$$v(t,x,z) = \frac{KA_1\sum_{j=1}^{N} q_j \exp[\psi_j(t,x,z)]}{p(t,z) + \sum_{j=1}^{N}\exp[\psi_j(t,x,z)]} + A_2 \tag{20}$$

where

$$\psi_j(t,x,z) = q_j x + \int_0^t [r'(s) + A_2 q_{1j} f(s) + q_{2j} g(s) - (\alpha A_2 q_{1j} + \beta q_{2j})\widetilde{W}(s,z)]\,\mathrm{d}s + \eta_j \tag{21}$$

By (18), (20), (21) and the definition of $\widetilde{W}(t,z)$, it is easy to prove that there exists a bounded open set $G \subset \mathbf{R}_+ \times \mathbf{R}$, $q>0$ and $r>0$ such that $v(t,x,z)$, $v_t(t,x,z)$, $v_x(t,x,z)$ and $v_{xx}(t,x,z)$ are uniformly bounded for $(t,x,z) \in G\times K_q(r)$, continuous with respect to $(t,x)\in G$ for all $z\in K_q(r)$ and analytic with respect to $z\in K_q(r)$ for all $(t,x)\in G$. Theorem 2.1 in Ref. [2] implies that there exists $U(t,x)\in(S)_{-1}$ such that $v(t,x,z) = (\mathscr{H}U(t,x))(z)$ for all $(t,x,z)\in G\times K_q(r)$ and that $U(t,x)$ solves Eq. (1). From the above, we have that $U(t,x)$ is the inverse Hermite transformation of $v(t,x,z)$. Hence, by (18), (20) and (21) we have a stochastic solitary solution of (1)

$$U(t,x) = \frac{KA_1\sum_{j=1}^{N} q_j \exp^{\diamond}[\Psi_j(t,x)]}{p^{\diamond}(t) + \sum_{j=1}^{N}\exp^{\diamond}[\Psi_j(t,x)]} + A_2 \tag{22}$$

where

$$\begin{aligned}\Psi_j(t,x) &= q_j x + \int_0^t [r'(s) + A_2 q_{1j} f(s) + q_{2j} g(s) - (\alpha A_2 q_{1j} + \beta q_{2j}) W(s)]\,\mathrm{d}s + \eta_j = \\ &\quad q_j x + \int_0^t [r'(s) + A_2 q_{1j} f(s) + q_{2j} g(s)]\,\mathrm{d}s - (\alpha A_2 q_{1j} + \beta q_{2j}) B_t + \eta_j \end{aligned}\tag{23}$$

$$\begin{aligned}p^{\diamond}(t) &= C_0 \exp^{\diamond}\left\{\int_0^t [r'(s) + A_2 q_1 f(s) + q_1{}^2 g(s) - (\beta q_1^2 + \alpha A_2 q_1) W(s)]\,\mathrm{d}s\right\} = \\ &\quad C_0 \exp^{\diamond}\left\{\int_0^t [r'(s) + A_2 q_1 f(s) + q_1{}^2 g(s)]\,\mathrm{d}s - (\alpha A_2 q_1 + \beta q_1{}^2) B_t\right\}\end{aligned}\tag{24}$$

Since $\exp^{\diamond}(B_t) = \exp\left(B_t - \frac{1}{2}t^2\right)$ (see lemma 2.6.16 in Ref. [12]), Eqs. (22), (23) and (24) yield the solution of Eq. (1)

$$U(t,x) = \frac{KA_1\sum_{j=1}^{N} q_j \exp[\overline{\Psi}_j(t,x)]}{\overline{p}(t) + \sum_{j=1}^{N}\exp[\overline{\Psi}_j(t,x)]} + A_2 \tag{25}$$

where

$$\overline{\Psi}_j(t,x) = q_j x + \int_0^t [r'(s) + A_2 q_{1j} f(s) + q_{2j} g(s)]\,\mathrm{d}s - (\alpha A_2 q_{1j} + \beta q_{2j})\left(B_t - \frac{1}{2}t^2\right) + \eta_j$$

and

$$\overline{p}(t) = C_0 \exp\left\{\int_0^t [r'(s) + A_2 q_1 f(s) + q_1{}^2 g(s)]\,\mathrm{d}s - (\alpha A_2 q_1 + \beta q_1{}^2)\left(B_t - \frac{1}{2}t^2\right)\right\}$$

2 Generalizations

Using the method in section 1, we can get the exact solutions of the following the Wick-type stochastic Burgers equation

$$U_t + H(t)\diamond U \diamond U_x + G(t)\diamond U_{xx} = 0$$

where $H(t)$ and $G(t)$ are white noise functions. Readers can do this easily.

When $\alpha=\beta=0$, the exact solutions of the variable coefficients Burgers equation (2) are given by Eqs. (22), (23) and (24).

In section 1, we only discussed SPDEs driven by the Gaussian white noise. From a modelling point of view one might feel that this is too special. One can easily envisage situations where the underlying noise has a different nature. It turns out, however, that there is a close mathematical connection between SPDEs driven by Gaussian and Poisson noise, at least for Wick-type equations. More precisely, there is a unitary map between the Gaussian white noise space and the Poisson white noise space, such that one can obtain the solution of the Poisson SPDE simply by applying this map to the solution of the corresponding Gaussian SPDE. A nice, concise account of this connection can also be found in section 4.9 in Ref. [12]. Hence, we can get stochastic soliton solutions if the coefficients $f(t)$ and $g(t)$ are perturbed by the Poisson white noise in Eq. (2).

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Wick 型随机 Burgers 方程的自 Bäcklund 变换和精确解

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摘要:研究了随机环境中的 Burgers 方程. 为了给出随机 Burgers 方程的精确解,只讨论变系数 Burgers 方程的系数经白噪声 $W(t) = \dot{B}(t)$ 扰动所得的 Wick 型随机 Burgers 方程($B(t)$是 Brown 运动),利用齐次平衡原理和 Hermite 变换给出了 Wick 型随机 Burgers 方程的自 Bäcklund 变换和随机孤立子解的精确表达式,同时也研究了一般情形的 Wick 型随机 Burgers 方程.

关键词: Wick 型随机 Burgers 方程;自 Bäcklund 变换;随机孤立子解;白噪声;Hermite 变换;齐次平衡原理

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