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Note on a diffusive ratio-dependent predator-prey model

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Abstract: Subject to the homogeneous Neumann boundary condition, a ratio-dependent predator-prey reaction diffusion model is discussed. An improved result for the model is derived, that is, the unique positive constant steady state is the global stability. This is done using the comparison principle and establishing iteration schemes involving positive solutions supremum and infimum. The result indicates that the two species will ultimately distribute homogeneously in space. In fact, the comparison argument and iteration technique to be used in this paper can be applied to some other models. This method deals with the not-existence of a non-constant positive steady state for some reaction diffusion systems, which is rather simple but sufficiently effective.

Key words: ratio-dependent predator-prey model; global stability; comparison principle; iteration

1 Background

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$, and $u(x,t)$ and $v(x,t)$ represent the densities of prey and predator at spatial $x \in \Omega$ and time t, respectively. In Ref. [1], subject to the homogeneous Neumann boundary condition, the authors studied the following diffusive ratio-dependent predator-prey model:

$$\left.\begin{aligned} &u_t - d_1 \Delta u = u(1-u) - \frac{buv}{u+mv} \\ &v_t - d_2 \Delta v = rv\left(\frac{u}{u+mv} - k\right) \end{aligned}\right\} \qquad (1)$$

with the corresponding initial $u(x,0) = u_0(x) \geqslant 0 (\not\equiv 0)$, $v(x,0) = v_0(x) \geqslant 0 (\not\equiv 0)$ are continuous functions. All the parameters appearing in model (1) are assumed to be positive constants. The constants d_1 and d_2 are the diffusion rates corresponding to u and v. Prey is assumed to grow logistically in absence of predator. Here the homogeneous Neumann boundary condition means that model (1) is self-contained and has no population flux across the boundary $\partial\Omega$. For the more detailed biological background of the model, readers can refer to Refs. [1 – 3] and the references therein.

Since the variables u and v represent the densities of prey and predator, they are required to be non-negative. It is clear that model (1) has a unique global solution (u, v). In addition, in virtue of $u_0 \not\equiv 0$, , $v_0 \not\equiv 0$, the solution is positive; i. e., $u(x,t) > 0$, $v(x,t) > 0$ on $\overline{\Omega}$ for all $t > 0$. We note that model (1) has at most one positive constant steady state $(\tilde{u}, \tilde{v})$ given by

$$\tilde{u} = 1 - \frac{b(1-k)}{m}, \quad \tilde{v} = \frac{1-k}{mk}\tilde{u} \qquad \text{if } k<1,\, b(1-k)<m$$

Pang et al. [1] gave some qualitative descriptions of solutions to (1) and its corresponding steady state problem. In particular, they discussed the non-existence of a non-constant positive steady state, that is, theorem 4. 1 of Ref. [1] roughly states that when d_2 is not too small and d_1 is large enough, then (1) has no non-constant positive steady state. In Ref. [4], the authors constructed a Liapunov function and claimed that $(\tilde{u}, \tilde{v})$ was globally asymptotically stable when $k<1$ and $b(1+k-k^2) < m$ (theorem 2. 1 in Ref. [4]). Furthermore, for the steady state problem of (1), in addition to the condition $k<1$ and $b<m$, under some assumptions, they obtained some improved non-existence results for non-constant positive classical solutions (theorem 4. 1 and theorem 4. 2 in Ref. [4]). Based on the implicit function theorem, their proofs are complicated.

In this paper, by the comparison argument and the iteration technique, we will prove that if $k<1$ and $b<m$, then $(\tilde{u}, \tilde{v})$ is globally asymptotically stable (see theorem 1 for details). The result gives an improved non-existence result for the non-constant positive steady state of Ref. [1] and covers all the results of Ref. [4].

In the following, we discuss the global stability of $(\tilde{u}, \tilde{v})$ for (1). In fact, the comparison argument and iteration technique to be used in this paper can be applied to some other models, and for the details, please see the remarks in section 2.

2 Global Stability of $(\tilde{u}, \tilde{v})$

This section is devoted to the global stability of

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$(\tilde{u}, \tilde{v})$ for (1). We first state a result which has been verified in Ref. [1].

Proposition 1 Assume that $k<1$ and $b<m$. For any solution (u, v) of (1), there holds

$$\left.\begin{aligned} &\limsup_{t\to\infty} \max_{\bar{\Omega}} u(\cdot, t) \leqslant \eta_1 \\ &\limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot, t) \leqslant a\eta_1 \\ &\limsup_{t\to\infty} \min_{\bar{\Omega}} u(\cdot, t) \geqslant \xi_1 \\ &\limsup_{t\to\infty} \min_{\bar{\Omega}} u(\cdot, t) \geqslant a\xi_1 \end{aligned}\right\} \tag{2}$$

where $\eta_1 = 1, \xi_1 = 1 - b/m$ and $a = (1-k)/(mk)$.

Proposition 2 Assume that $k<1$ and $b<m$. Let a be the constant defined in (2). The problem

$$\left.\begin{aligned} &y = 1 - \frac{abz}{y+amz} \\ &z = 1 - \frac{aby}{z+amy} \\ &1 - \frac{b}{m} < y \leqslant z < 1 \end{aligned}\right\} \tag{3}$$

has a unique solution $y = z = \tilde{u}$.

Proof It is obvious that $(y, z) = (\tilde{u}, \tilde{u})$ is a solution of (3), and if $y = z$ then the unique solution of (3) is $(\tilde{u}, \tilde{u})$.

Let (y, z) be a solution of (3). We prove $y \equiv z$ by contradiction. If $y \neq z$, a direct computation gives

$$y + z = 1 - ab - am$$

As a result, it is necessary that $ab + am < 1$. By (3), we have

$$\left.\begin{aligned} &(1-am)y^2 + a(ab+am-1)(m-b) + \\ &\quad [am(2-am-ab) - 1 - ab]y = 0 \\ &(1-am)z^2 + a(ab+a\ m-1)(m-b) + \\ &\quad [am(2-am-ab) - 1 - ab]z = 0 \end{aligned}\right\} \tag{4}$$

It is obvious that there at most exists a unique solution (y, z) satisfying (3) and (4). Moreover, with $z = y$ contradicting $z \neq y$, the proof is complete.

Now, we can obtain the global stability of $(\tilde{u}, \tilde{v})$ as the following theorem. This result indicates that the two species of prey and predator will ultimately distribute homogeneously in space.

Theorem 1 Let $k<1$ and $b<m$, then $(\tilde{u}, \tilde{v})$ is globally asymptotically stable for (1). In particular, this implies that (1) has no non-constant positive steady state if $k<1$ and $b<m$ hold.

Proof Let (u, v) be any solution of (1). In virtue of (2), for any $\varepsilon > 0$ small, there exists $T>0$ such that $u(x, t) \leqslant \eta_1 + \varepsilon$ and $v(x, t) \geqslant a\xi_1 - \varepsilon$ for all $x \in \bar{\Omega}$ and $t \geqslant T$. By the first equation of (1), we have that if $(x, t) \in (\Omega \times [T, \infty))$, then under the homogeneous Neumann boundary condition

$$u_t - d_1 \Delta u \leqslant u(1-u) - \frac{bu(a\xi_1 - \varepsilon)}{\eta_1 + \varepsilon + m(a\xi_1 - \varepsilon)}$$

The standard comparison argument shows that

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} u(\cdot, t) \leqslant \eta_2(\varepsilon)$$

where $\eta_2(\varepsilon) = 1 - ab\xi_1/(\eta_1 + am\xi_1) + o(\varepsilon)$. In view of the arbitrariness of ε, we have

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} u(\cdot, t) \leqslant \eta_2$$

where $\eta_2 = 1 - ab\xi_1/(\eta_1 + am\xi_1)$. Then, for any $\varepsilon > 0$ small, there exists $T>0$ such that $u(x, t) \leqslant \eta_2 + \varepsilon$ for all $x \in \bar{\Omega}$ and $t \geqslant T$. Hence if $(x, t) \in (\Omega \times [T, \infty))$, v solves

$$v_t - d_2 \Delta v \leqslant rv\left(\frac{\eta_2 + \varepsilon}{\eta_2 + \varepsilon + mv} - k\right)$$

with the homogeneous Neumann boundary condition. Note that $k<1$, let $z(t)$ be a solution of the ordinary different equation (ODE) problem

$$\left.\begin{aligned} z'(t) &= rz\left(\frac{\eta_2 + \varepsilon}{\eta_2 + \varepsilon + mz} - k\right) = \\ &\quad rz\,\frac{(1-k)(\eta_2 + \varepsilon) - mkz}{\eta_2 + \varepsilon + mz} \\ z(T) &= \max_{\bar{\Omega}} v(\cdot, T) > 0 \end{aligned}\right\}$$

where $t \geqslant T$. Then

$$\lim_{t\to\infty} z(t) = \frac{(1-k)(\eta_2 + \varepsilon)}{mk}$$

Thanks to a comparison argument and the arbitrariness of ε, we yield that

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot, t) \leqslant a\eta_2 \tag{5}$$

Hence, due to (2) and (5), for any $\varepsilon > 0$ small, there exists $T>0$ such that $u(x, t) \leqslant \xi_1 - \varepsilon$ and $v(x, t) \geqslant a\eta_2 + \varepsilon$ for all $x \in \bar{\Omega}$ and $t \geqslant T$. If $(x, t) \in (\Omega \times [T, \infty))$, it then follows that u satisfies

$$u_t - d_1 \Delta u \geqslant u(1-u) - \frac{bu(a\eta_2 + \varepsilon)}{\xi_1 - \varepsilon + m(a\eta_2 + \varepsilon)}$$

with the homogeneous Neumann boundary condition. Again, by the comparison argument and the arbitrariness of ε, we have

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} u(\cdot, t) \geqslant \xi_2$$

where $\xi_2 = 1 - ab\eta_2/(\xi_1 + am\eta_2)$. As a result, applying the equation for $v(x, t)$, as above, we have

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} v(\cdot, t) \geqslant a\xi_2$$

It is clear to see that $\xi_1 < \xi_2 < \eta_2 < \eta_1$. Repeating the above arguments, inductively, for $i \geqslant 1$, there exists an increasing sequence $\{\xi_i\}$ and a decreasing sequence $\{\eta_i\}$ satisfying

$$\xi_{i+1} = 1 - \frac{ab\eta_{i+1}}{\xi_i + am\eta_{i+1}}, \quad \eta_{i+1} = 1 - \frac{ab\xi_i}{\eta_i + a\ m\xi_i}$$

$$1 - \frac{b}{m} = \xi_1 < \xi_2 \ldots < \xi_i < \xi_{i+1} < \ldots < \eta_{i+1} < \eta_i < \ldots < \eta_1 = 1$$

Hence, we have $\lim_{i\to\infty}(\xi_i, \eta_i) = (\tilde{\xi}, \tilde{\eta})$. Moreover, $(\tilde{\xi}, \tilde{\eta})$ satisfies (3) with $(y, z) = (\tilde{\xi}, \tilde{\eta})$. From proposition 2, we have $\tilde{\xi} = \tilde{\eta} = \tilde{u}$. This shows that $u \to \tilde{u}$ uniformly on $\bar{\Omega}$ as $t \to \infty$. Owing to the comparison principle, $v \to \tilde{v}$ uniformly on $\bar{\Omega}$ as $t \to \infty$, which ends the proof.

Remark 1 Our comparison argument and iteration technique can also be applied in some more models, for example, the classical Lotka-Volterra competition model with diffusion

$$\left.\begin{aligned} u_t - d_1\Delta u = u(a_1 - b_1 u - c_1 v) \\ v_t - d_2\Delta v = v(a_2 - b_2 u - c_2 v) \end{aligned}\right\} \quad (6)$$

and the diffusive predator-prey model with Holling-Ⅱ functional response

$$\left.\begin{aligned} u_t - d_1\Delta u = u\left(a - u - \frac{bv}{m+u}\right) \\ v_t - d_2\Delta v = v\left(d - v - \frac{cv}{m+u}\right) \end{aligned}\right\} \quad (7)$$

By the argument above, we can obtain the same result for (6) as in theorem 3. 1 of Ref. [5] and for (7) as in theorem 1 of Ref. [6]. However, their methods are invalid for model (1). From the proof of theorem 1, it is not hard to see that for all the models mentioned above, our method is possible to be applied when one specie is assumed to grow logistically in absence of the other in population dynamics.

Remark 2 We would like to point out, it is clear that the method can also be applied to the ODE model. For example, the following ODE predator-prey model[7]

$$\left.\begin{aligned} u_t = r_1 u - b_1 u^2 - \frac{a_1 uv}{k_1 + u} \\ v_t = r_2 v - \frac{a_2 v^2}{k_2 + u} \end{aligned}\right\} \quad (8)$$

By giving a positive invariant attracting set and constructing a suitable Liapunov function, the authors show that the interior equilibrium (u^*, v^*) is globally asymptotically stable if

$$2a_1 L < r_1 k_1,\ 4(r_1 + b_1 k_1) < a_1$$
$$k_1 < 2k_2,\ a_1 r_2 k_2 < a_2 r_1 k_1$$

where $4a_2 b_1 L = a_2 r_1 (r_1 + 4) + (r_2 + 1)^2 (r_1 + b_1 k_2)$ (see Ref. [7], theorem 4, proposition 5, theorem 6). By virtue of our method, by a series of computations, we can obtain an improved result, that is, the interior equilibrium (u^*, v^*) of (8) is globally asymptotically stable if $a_1 r_2 k_2 < a_2 r_1 k_1$ and $2a_1 M < r_1 k_1$, where

$$4a_2 b_1 \frac{M}{r_2} = (2b_1 k_2 + r_1 - b_1 k_1) + \left[(r_1 - b_1 k_1)^2 + \frac{4}{a_2}(a_2 r_1 k_1 - a_1 r_2 k_2)\right]^{\frac{1}{2}}$$

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比率依赖型捕食扩散模型的一个注记

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摘要:讨论了在齐次 Neumann 边界条件下具有比率依赖型捕食反应扩散模型. 应用比较原理和建立与正解的上下确界相关的迭代格式,得到了一些改进的结果,即惟一的正常数平衡态是全局渐近稳定的. 该结果说明了 2 种群最终在空间上均匀分布. 所提出的方法也适用于其他一些模型. 应用于讨论一些反应扩散系统非正常数平衡态的不存在性,该方法相当简单但是十分有效.

关键词:比率依赖型捕食模型;全局渐近稳定;比较原理;迭代

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