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Finite element solution based on fast numerical technique for large-scale electromagnetic computation

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Abstract: A numerical technique of the target-region locating (TRL) solver in conjunction with the wave-front method is presented for the application of the finite element method (FEM) for 3-D electromagnetic computation. First, the principle of TRL technique is described. Then, the availability of TRL solver for non-linear application is particularly discussed demonstrating that this solver can be easily used while still remaining great efficiency. The implementation on how to apply this technique in FEM based on magnetic vector potential (MVP) is also introduced. Finally, a numerical example of 3-D magnetostatic modeling using the TRL solver and FEMLAB is given. It shows that a huge computer resource can be saved by employing the new solver.
Key words: finite element method; electromagnetic computation; numerical technique; fast solver

1 Principle of Target-Region Locating (TRL) Method

1.1 Wave-front principle

Because of the flexibility of finite element processing and easy implementation in optimizing large-scale equation solutions, the wave-front (W-F) method is selected and studied in our research for the reason that there will be no cumbersome convergent solution problems. The W-F method is a special direct method, or called a special Gauss elimination method[1-2]. Different from regular Gauss method, the W-F method carries the job of loading an element matrix into a global stiffness matrix, eliminating element node variables out of the matrix, and storing equation coefficients of the eliminated variables in a dynamic way. It is known that for the usual direct Gauss method, usually, at first, all the finite element equations are processed and loaded into a general matrix one by one to finalize a global or general stiffness matrix equation. Then the Gauss elimination processing is done for each node variable according to the sequence order in the stiffness matrix and the elimination coefficients are stored in a one-dimensional array. Finally back-substitution is done to solve all node variables in the whole field region. Because in this regular Gauss method all node variables have to be stored in a global matrix which may generate a huge memory requirement and also cost a huge amount of calculating work, thus it is not an efficient solver for large-scale FEM especially for 3-D electromagnetic analysis. But when utilizing the W-F method, a dynamic stiffness matrix defined as a "front matrix" instead of a global stiffness matrix is adopted, which means that when one element equation is processed and loaded into this "front matrix" and, meantime, if some of node variables are judged to be the last in the loading process or saying they are judged to be last in this "front", then they are eliminated out of the "front" because they are no longer in use and will not make any contributions to next new-inputting element node variables. Also, their elimination coefficients should be stored at this time. In this way, only limited node variables exist in the "front matrix" in FEM process during the time when some new elements being loaded into "front" and some old element variables being eliminated out of "front". Since this phenomenon looks like a wave moving-forward, it is called the wave-front method and the size of "front matrix" defined as front width is changeable in equations processing.

1.2 Target-region locating method

Though the W-F method does not need to store a general global matrix, it still has to store all the variables elimination coefficients in a one-dimensional array during the W-F process, and if it is for a large field analysis problem, this will surely require another huge memory size for storing elimination coefficients. For this reason, the target-region locating technique is

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presented in conjunction with the W-F method. The idea is that usually in field analysis only a small part of the field region named as the target-region or solving region needs to be known and it often occupies a very small percentage of the total space, say for example 5%. So actually we just need to solve these target-region unknowns and do not need to solve all others. This implies that in modifying the W-F method, it is only necessary to store those coefficients of the target-region variables and abandon the storing of all other variables in non-target-regions. Consequently, before numerical computation begins, all target-regions have to be located in advance and their involved elements are to be renumbered so as to be loaded in the "front" as late as possible while ensuring that non-target-region elements are loaded in the "front" as early as possible. Then in the computation, non-target-region elements will be dealt with first and eliminated out of "front" without storing their elimination coefficients, and the target-region elements will be inputted and their coefficients will be stored in a 1-D array which will be solved by back-substitution. Therefore, by employing the TRL technique, it not only avoids storing all the elimination coefficients resulting in a huge memory saved, but also keeps the bandwidth of "front" (or front width) to a very small size. This may be explained by, on one hand, the target-region elements are loaded into "front" and thus they are not encircled or combined together with widely-located non-target-region elements in the "front matrix". On the other hand, it is ruled that in the element loading process of FEM, any target-region element cannot be eliminated until all the elements of target-regions have been inputted and, therefore, finally, only those variables of target-regions are left in "front" and they can then be eliminated and stored together. As a result the target-region variables will not stay in "front" for a long time and also will not occupy any additional space in "front", so front width is kept small.

The following is the analysis of the TRL solver in conjunction with the W-F method. As shown in Fig. 1, assume that $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ are target-regions, and $\boldsymbol{\Omega}$ is a regular region or non-target-region. $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ are joint surfaces between target regions and non-target-regions ($\boldsymbol{T}_1$, $\boldsymbol{T}_2/\boldsymbol{\Omega}$). Then at first, regular region elements $\boldsymbol{\Omega}$ are loaded into "front", and the W-F matrix equation is

$$\boldsymbol{K}_{\Omega,\Omega}\boldsymbol{X}_{\Omega} = \boldsymbol{F}_{\Omega} \tag{1}$$

During the time of element loading into "front", the element node variables within the regular region are eliminated at the same time because they are in the last appearance in "front" and will be no use and only those variables of $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ on the joint surface are left in the "front". Then Eq. (1) becomes

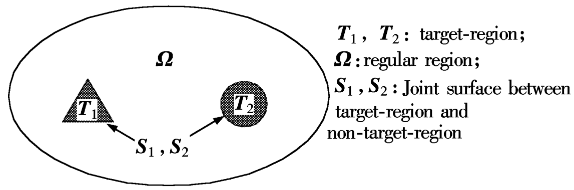

Fig. 1 Illustration of TRL technique for solver

$$\begin{bmatrix} \boldsymbol{K}_{S_1,S_1} & \boldsymbol{K}_{S_1,S_2} \\ \boldsymbol{K}_{S_2,S_1} & \boldsymbol{K}_{S_2,S_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{S_1} \\ \boldsymbol{X}_{S_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_{S_1} \\ \boldsymbol{F}_{S_2} \end{bmatrix} \tag{2}$$

Now the new target-region elements of $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ are continuously loaded into "front" and Eq. (2) is changed into

$$\begin{bmatrix} \boldsymbol{K}_{T_1,T_1} & \boldsymbol{K}_{T_1,T_2} \\ \boldsymbol{K}_{T_2,T_1} & \boldsymbol{K}_{T_2,T_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{T_1} \\ \boldsymbol{X}_{T_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_{T_1} \\ \boldsymbol{F}_{T_2} \end{bmatrix} \tag{3}$$

where the variables $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ on the joint surface are already included in the target-region variables $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ and thus not written in Eq. (3). Eq. (3) can be rewritten as

$$\boldsymbol{K}_{T,T}\boldsymbol{X}_T = \boldsymbol{F}_T \tag{4}$$

where $\boldsymbol{T}$ is the total target-region variable. Up to now it is obviously shown that only Eq. (4) needs to be solved by back-substitution and the number of these solved unknowns is reduced to only n_T, which is small enough compared with the non-target-region's variable number n_{Ω}. Therefore, the TRL technique obtains the result of a great saving of memory used for storing elimination coefficients, which, for example, may be only less than 1% of regular W-F use, and accordingly much CPU time can be saved.

1.3 Application of new solver in non-linear problem

When there are non-linear materials in the field such as non-linear permeability μ_r of magnetic materials in magnetostatic modeling, usually it is a very time-consuming process and is also a complex and cumbersome problem in a numerical scheme since it needs much iteration time to have convergence solutions approaching the exact solution. However in the non-linear case, the TRL solver is also available which still retains the advantage of great small memory use; and one more important thing is that because it can save calculating time in each iteration, the TRL solver can thereby also greatly save the total calculating time. This is readily realized if we regard the non-linear region as an additional target-region and the non-linear region variables are together solved with target-region variables in each iteration of calculation. Finally, by judging whether the following condition of convergence criterion

$$\sum_{k=1}^{M} \left| \frac{\mu_k^{(n)} - \mu_k^{(n-1)}}{\mu_k^{(n-1)}} \right| \to \varepsilon \tag{5}$$

is met or not then the convergent solution can be achieved. In formula (5) M is the number of sub-regions of non-linear permeability materials.

The non-linear analysis is given below. Assuming that beside the exact target-region $\boldsymbol{T}$ and the regular region $\boldsymbol{\Omega}$ (non-target-region), there are two new sub-regions of non-linear materials $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$. Now since non-linear material regions $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ are already treated as additional target-regions and have been located before computation, thus, regular region $\boldsymbol{\Omega}$ is first loaded into "front" and its W-F matrix equation is the same as Eq. (1), which is similar to the usual TRL method. Then by eliminating non-target-region elements out of "front", the left variables staying in "front" will be those on the joint surface between regular region $\boldsymbol{\Omega}$ (i. e., non-target region) and equivalent target-regions organized by exact target-region $\boldsymbol{T}$ adding non-linear regions $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ together. Assuming this jointed surface variable is $\boldsymbol{\Omega}_c$, then the front equation is

$$\boldsymbol{K}_{\Omega_c,\Omega_c}\boldsymbol{X}_{\Omega_c} = \boldsymbol{F}_{\Omega_c} \tag{6}$$

Now loading the equivalent target-region elements $\boldsymbol{T}$ (exact target-region) and $\boldsymbol{N}_1, \boldsymbol{N}_2$ (non-linear regions) into the "front matrix" of Eq. (6), then Eq. (6) changes into

$$\begin{bmatrix} \boldsymbol{K}_{\Omega_c,\Omega_c} & \boldsymbol{K}_{\Omega_c,T} & \boldsymbol{K}_{\Omega_c,N_1} & \boldsymbol{K}_{\Omega_c,N_2} \\ \boldsymbol{K}_{T,\Omega_c} & \boldsymbol{K}_{T,T} & \boldsymbol{K}_{T,N_1} & \boldsymbol{K}_{T,N_2} \\ \boldsymbol{K}_{N_1,\Omega_c} & \boldsymbol{K}_{N_1,T} & \boldsymbol{K}_{N_1,N_1} & \boldsymbol{K}_{N_1,N_2} \\ \boldsymbol{K}_{N_2,\Omega_c} & \boldsymbol{K}_{N_2,T} & \boldsymbol{K}_{N_2,N_1} & \boldsymbol{K}_{N_2,N_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_{\Omega_c} \\ \boldsymbol{X}_T \\ \boldsymbol{X}_{N_1} \\ \boldsymbol{X}_{N_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}'_{\Omega_c} \\ \boldsymbol{F}'_{T_1} \\ \boldsymbol{F}'_{N_1} \\ \boldsymbol{F}'_{N_2} \end{bmatrix} \tag{7}$$

Considering that the joint-surface variable $\boldsymbol{\Omega}_c$ also belongs to the equivalent target-regions ($\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{N}_2$) and can be involved in these target-regions as done before, then Eq. (7) is rewritten in terms of only equivalent target-region variables ($\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{N}_2$) as

$$\begin{bmatrix} \boldsymbol{K}_{T,T} & \boldsymbol{K}_{T,N_1} & \boldsymbol{K}_{T,N_2} \\ \boldsymbol{K}_{N_1,T} & \boldsymbol{K}_{N_1,N_1} & \boldsymbol{K}_{N_1,N_2} \\ \boldsymbol{K}_{N_2,T} & \boldsymbol{K}_{N_2,N_1} & \boldsymbol{K}_{N_2,N_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_T \\ \boldsymbol{X}_{N_1} \\ \boldsymbol{X}_{N_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_T \\ \boldsymbol{F}_{N_1} \\ \boldsymbol{F}_{N_2} \end{bmatrix} \tag{8}$$

From Eq. (8) it is clearly seen that though some non-linear regions are added in the field region, finally the total number of variables existing in the "front matrix" and to be solved is just equal to

$$n_{\text{non-linear}} = n_T + n_{N_1} + n_{N_2} \tag{9}$$

It is often true that these non-linear regions often still occupy a small percentage of total space. For example, in our project research there is a non-linear iron in a coil for magnetic field generation and the space of non-linear iron is very small compared with the total space, and, therefore, there should be a very limited increase of total solving variables of the parameter $n_{\text{non-linear}}$ in Eq. (9), since the number of target-region variables, either n_T or ($n_T + n_{N_1} + n_{N_2}$), is greatly less than the total number of field variables ($n_T + n_{N_1} + n_{N_2} + n_\Omega$). In other words, it can be said that, compared with non-target-region variable number n_Ω, both exact target-region variable number n_T and equivalent target-region variable number ($n_T + n_{N_1} + n_{N_2}$) can be ignored. So this is still a very small calculating work just like the TRL solver application in the linear case.

Furthermore the TRL solver is even more powerful in the non-linear case, because, if N is iteration time and ΔT is the saved time in each iteration compared to the usual Gauss direct method, then the total saved calculating time compared to the Gauss method is $N \cdot \Delta T$! This will represent a huge saving of time.

2 Implementation of TRL Solver in Finite Element Method

The purpose of the electromagnetic computation in this work is for the design of a magnet wrapped by a DC coil for generating an appropriate magnetic field for non-invasive detection of strut failures in the Bjork-Shiley convexo-concave (BSCC) prosthetic heart valve[3]. This is a 3-D magnetostatic modeling problem with exciting current source. After considering the magnetic vector potential (MVP), a finite element is adopted[4-5], and magnetic field $\boldsymbol{B}$ is derived from solved variable $\boldsymbol{A}$ (i. e., MVP) which is

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{10}$$

Then the governing equation is

$$\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{A} = \boldsymbol{Js} \tag{11}$$

The corresponding finite element formula by the Galerkin's weighted residual method is obtained as

$$\int_{\Omega_c} \boldsymbol{N} \cdot \nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{A} \mathrm{d}\boldsymbol{\Omega} = \int_{\Omega_c} \boldsymbol{N} \cdot \boldsymbol{Js} \mathrm{d}\boldsymbol{\Omega} \tag{12}$$

Using vector identities, Eq. (12) is transferred into

$$\int_{\Omega_c} \nabla \times \boldsymbol{N} \cdot \frac{1}{\mu} \nabla \times \boldsymbol{A} \mathrm{d}\boldsymbol{\Omega} = \int_{\Omega_c} \boldsymbol{N} \cdot \boldsymbol{Js} \mathrm{d}\boldsymbol{\Omega} + \int_{\Gamma} \boldsymbol{N} \cdot \left(\frac{1}{\mu} \nabla \times \boldsymbol{A} \times \boldsymbol{n}_c \right) \mathrm{d}\boldsymbol{\Gamma} \tag{13}$$

Rewriting Eq. (13) into matrix equation form, the element matrix equation is

$$\begin{bmatrix} K^{e}_{Ax,Ax} & K^{e}_{Ax,Ay} & K^{e}_{Ax,Az} \\ K^{e}_{Ay,Ax} & K^{e}_{Ay,Ay} & K^{e}_{Ay,Az} \\ K^{e}_{Az,Ax} & K^{e}_{Az,Ay} & K^{e}_{Az,Az} \end{bmatrix}_{e} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}_{e} = \begin{bmatrix} F_{Ax} \\ F_{Ay} \\ F_{Az} \end{bmatrix}_{e} \tag{14a}$$

or

$$K_{ij\,e} X_{e} = F_{e} \tag{14b}$$

which is a 24×24-dimensional matrix equation. Now substituting Eq. (14) into Eqs. (1) to (4), the TRL solver can be easily implemented in the MVP finite element method.

3 Numerical Example

A numerical example of 3-D magnetostatic problems is given to verify the accuracy and efficiency of the TRL solver with the W-F technique in the application of the FEM, and the result comparison between the TRL solver and FEMLAB® [6] indicates that the new solver is more efficient which may be a good candidate for use.

An iron cube with relative permeability $\mu_r = 10$ is immersed in the field of a cube coil[7]. The current density is assumed to be 1 A/mm^2. Fig. 2 is the illustration of model geometry and its mesh generation of iron and coil. To compare the results between the new solver and FEMLAB®, the z-components of the flux density along the z-axis at the center of iron are plotted in Fig. 3. It is found from Fig. 3 that the two results are close to each other. Moreover, these results are also consistent with the theoretical solution in a similar model[8] and the error is within an acceptable range, which again proves the validity of the TRL solver.

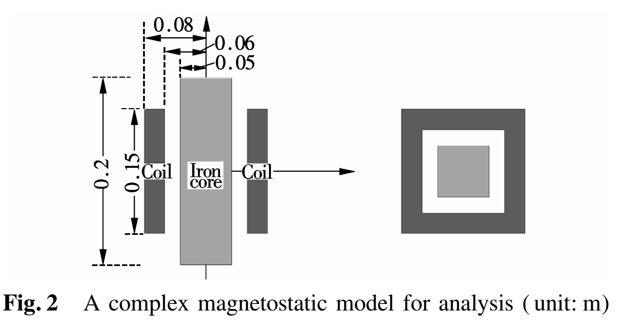

Fig. 2 A complex magnetostatic model for analysis (unit: m)

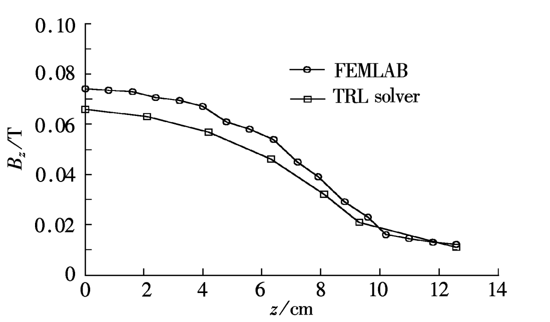

Fig. 3 Results comparison between FEMLAB® and the TRL solver

Tab. 1 is the comparison of used computer resource and other essential data between FEMLAB® and the TRL solver. It is found that great computer resource can be saved by the new solver. This is because, by using the new solver, the number of solved unknowns decreases from 38 462 to only 96, which is only about 0.2% of the FEMLAB®. Thus both memory used and CPU time decrease to only about 0.009% and 1.7% that used by FEMLAB®! Additionally, it is shown that, the number of equations to be processed reaches 10 584 for the TRL solver, but the actual maximum number of equations to be processed in "wave-front" is only 732. This means that the relative space occupation ratio of the TRL solver in total space is just equal to (732/10 584) × 100% = 6.9% and there is still much spare space for TRL solver use. This result comes from the fact of utilizing the TRL solver to control the front width kept at limited small value. Moreover, it is still observed in a similar example that when the permeability is changed to high, the CPU time of FEMLAB® may even be increased to almost 60 times of original time, but the time of the TRL solver does not change. It should be mentioned that though the TRL solver is implemented on a high-performance computer and FEMLAB® is on a regular computer; however, if considering the memory used and the maximum equation number between the TRL solver and FEMLAB® as shown in Tab. 1, this comparison is reasonable since the calculating time is directly related to the maximum number of equations and the memory used.

Tab. 1 Comparison of used computer resource between TRL and FEMLAB®

Items	TRL solver	FEMLAB®	Ratio (TRL/FEMLAB®)/%
Number of elements	2 873	28 264	10
Number of nodes	3 528	—	—
Number of unknowns	10 584	38 462	28
Number of solved unknowns	96 (with 7 solved elements)	38 462	0.2
Maximum number of equations	732	38 462	1.9
Memory used	6.7×10^4	7.25×10^8	0.009
CPU time/s	5	287	1.7

Up to now the conclusion can be drawn that though FEMLAB® has the advantage of flexibility in use for various EM problems, the fast solver proposed in this paper is more powerful for large scale computation with small solving regions.

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一种基于快速数值技术的电磁场分析有限元算法

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摘要:提出了一种用于三维电磁场有限元分析的数值技术,即与波阵法配合使用的目标区域定位(TRL)算子.首先对波阵法进行了分析,然后对TRL算子原理进行了详细论述,并讨论了TRL算子在非线性场合应用的可行性,指出其仍能够适用并保持原来的高效率.还介绍了TRL算子在采用矢量磁位函数有限元公式中的实现.最后给出了一个三维静磁场建模的算例,分别采用TRL算子和有限元软件FEMLAB.结果表明:与FEMLAB相比,TRL算子可以节省可观的计算机资源.

关键词:有限元方法;电磁场计算;数值技术;快速算子

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