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Stochastic resonance based on correlation coefficient in parallel array of threshold devices

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Abstract: The phenomenon of stochastic resonance (SR) based on the correlation coefficient in a parallel array of threshold devices is discussed. For four representative noises: the Gaussian noise, the uniform noise, the Laplace noise and the Cauchy noise, when the signal is subthreshold, noise can improve the correlation coefficient and SR exists. The efficacy of SR can be significantly enhanced and the maximum of the correlation coefficient can dramatically approach to one as the number of the threshold devices in the parallel array increases. Two theorems are presented to prove that SR has some robustness to noises in the parallel array. These results further extend the applicability of SR in signal processing.

Key words: stochastic resonance; correlation coefficient; threshold array

Stochastic resonance (SR) is a nonlinear phenomenon, which describes the possibility of improvement in signal transmission or signal processing, thanks to the actions of noise. Most occurrences of SR involve a signal (periodic, non-periodic or non-deterministic), which is mostly subthreshold and too weak to elicit a stronger response from a single nonlinear system. Addition of noise then brings assistance to the subthreshold signal in eliciting a stronger response from the nonlinear system. SR is usually measured with signal-to-noise ratio, mutual information, input-output correlation, probability of detection[1–15]. Recently, Ref. [4] first gave two theorems to prove that SR exists in a single threshold neuron based on the measure of Shannon's mutual information. Here, we study further SR in a parallel array of threshold devices. The parallel array is of interest, since it is widely used in electric engineering, neural network and it can model many natural phenomena. SR is characterized by the correlation coefficient. Although the correlation coefficient is thought to be a measure of linearity, it is a valid measure of how similar the output signal is to the input signal or how closely the output signal matches the input signal[16]. If the correlation coefficient of two random signals approaches to one, the two signals are linearly related to each other[16–18]. This paper first discusses and simulates the effect of SR for four representative noises in the array. When the signal is subthreshold, noise can improve the correlation coefficient and SR exists. The efficacy of SR can be enhanced significantly and the maximum of the correlation coefficient can approach dramatically to one as the number of the threshold devices in the array increases. Then two general theorems are presented to prove that SR has some robustness to noises in the parallel array. These results further extend the applicability of SR in signal processing.

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1 Parallel Array of Threshold Devices and Correlation Coefficient

A parallel array consists of N threshold devices (when $N=1$, the array is a single threshold system). Each threshold device is subject to the same discrete input random signal $x \in \{0, 1\}$, but independent and identically distributed noise n_i with probability density function (PDF) $f_n(x)$. The output y_i is given by the response function

$$y_i = \begin{cases} 1 & x + n_i > u_i \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $u_i (i = 1, 2, \ldots, N)$ are the threshold levels. The response of the parallel array is obtained by maximizing the individual response of each device, i. e., $y = \max\{y_1, y_2, \ldots, y_N\}$.

We denote the mean of a random variable x as $E(x)$, the variance as $\mathrm{var}(x)$ and the cross-correlation of x and y as $E(xy)$. Then the correlation coefficient of x and y is

$$\rho(x, y) = \frac{E(xy) - E(x)E(y)}{\sqrt{\mathrm{var}(x)}\sqrt{\mathrm{var}(y)}} \tag{2}$$

Because the distribution of threshold levels has little effect on the system performance, we can assume, as in Ref. [16], that all thresholds share the same value $u_i = u$ in the parallel array. Now we assume $P(x=0) = p_0$, $P(x=1) = p_1 = 1 - p_0$, then

$$E(x) = p_1,\ \operatorname{var}(x) = p_0 p_1$$

$$E(y) = 1 - p_0 P^N(n<u) - p_1 P^N(n<u-1)$$

$$\operatorname{var}(y) = \frac{1}{4} - \left[\frac{1}{2} - p_0 P^N(n<u) - p_1 P^N(n<u-1)\right]^2$$

$$E(xy) = p_1[1 - P^N(n<u-1)]$$

The correlation coefficient $\rho(x, y)$ in Eq. (2) becomes

$$\rho(x,y) = \frac{p_1[1-P^N(n<u-1)] - p_1[1-p_0P^N(n<u) - p_1P^N(n<u-1)]}{\sqrt{p_0p_1}\sqrt{\frac{1}{4} - \left[\frac{1}{2} - p_0P^N(n<u) - p_1P^N(n<u-1)\right]^2}} = \frac{\sqrt{p_1p_0}\,[P^N(n<u) - P^N(n<u-1)]}{\sqrt{[p_0P^N(n<u) + p_1P^N(n<u-1)][1 - p_0P^N(n<u) - p_1P^N(n<u-1)]}} \tag{3}$$

Given a noise PDF, we can derive $P(n<u)$ and $P(n<u-1)$ and then discuss the variation of $\rho(x, y)$.

2 Examples of SR in Parallel Array of Threshold Devices

In this section, we discuss and simulate the phenomenon of SR in the parallel array for four representative noises: the Gaussian noise (thin-tailed PDF), the uniform noise (finite support PDF), the Laplace noise (heavy-tailed PDF) and the Cauchy noise (impulsive noise without mean and variance).

1) Gaussian noise The Gaussian PDF with zero mean and variance σ^2 has the form

$$f_n(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

then

$$P(n<u) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{u}{\sqrt{2}\sigma}\right) \tag{4}$$

$$P(n<u-1) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{u-1}{\sqrt{2}\sigma}\right) \tag{5}$$

2) Uniform noise The uniform PDF with zero mean and variance σ^2 has the form

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{12}\sigma} & -\frac{\sqrt{12}\sigma}{2} < x < \frac{\sqrt{12}\sigma}{2} \\ 0 & \text{otherwise} \end{cases}$$

then

$$P(n<u) = \begin{cases} \frac{1}{2} + \frac{u}{\sqrt{12}\sigma} & \sigma > \frac{u}{\sqrt{3}} \\ 1 & \sigma < \frac{u}{\sqrt{3}} \end{cases} \tag{6}$$

$$P(n<u-1) = \begin{cases} \frac{1}{2} + \frac{u-1}{\sqrt{12}\sigma} & \sigma > \frac{u-1}{\sqrt{3}} \\ 1 & \sigma < \frac{u-1}{\sqrt{3}} \end{cases} \tag{7}$$

3) Laplace noise The Laplace PDF with zero mean and variance σ^2 has the form

$$f_n(x) = \frac{1}{\sqrt{2}\sigma}\exp\left(-\left|\frac{\sqrt{2}x}{\sigma}\right|\right)$$

then

$$P(n<u) = 1 - \frac{1}{2}\exp\left(-\frac{\sqrt{2}u}{\sigma}\right) \tag{8}$$

$$P(n<u-1) = 1 - \frac{1}{2}\exp\left(-\frac{\sqrt{2}(u-1)}{\sigma}\right) \tag{9}$$

4) Cauchy noise The Cauchy PDF with zero location and finite dispersion γ^2 has the form

$$f_n(x) = \frac{\gamma}{\pi(x^2+\gamma^2)}$$

then

$$P(n<u) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{u}{\gamma} \tag{10}$$

$$P(n<u-1) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{u-1}{\gamma} \tag{11}$$

For $p_0 = p_1 = 1/2$, Figs. 1 and 2 give the theoretical results of Eq. (3) and the Monte Carlo computer simulations (discrete data points) for four representative noises. In the single threshold system, when the signal is subthreshold ($u > 1$), Fig. 1 shows a non-monotonic variation, $\rho(x, y)$ increases with the noise intensity (standard deviation σ or standard dispersion γ), up to the maximum where the noise intensity is optimal, and then decreases. This non-monotonic influence of the noise on $\rho(x, y)$ is the signature of SR. For a fixed threshold $u = 1.2$, Fig. 2 shows that the efficacy of SR can be enhanced significantly and the maximum of $\rho(x, y)$ can approach dramatically to one as the number of the threshold devices in the parallel array increases. It seems that when different threshold noises are independently added on the devices of the array, each device will in general produce a distinct response. When all these responses are collected over the array, it is shown that the optimal response in the array can be more efficient than the response in a single threshold system.

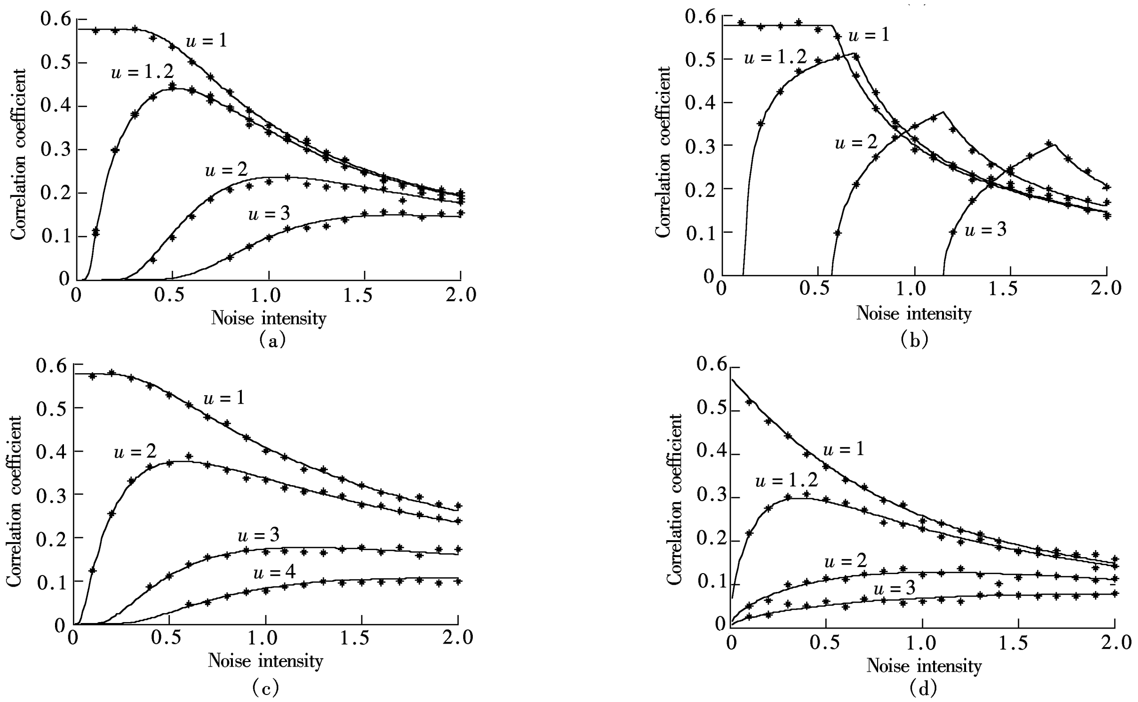

Fig. 1 Correlation coefficient $\rho(x, y)$ for different thresholds in the single threshold system and different noises. (a) Gaussian noise; (b) Uniform noise; (c) Laplace noise; (d) Cauchy noise

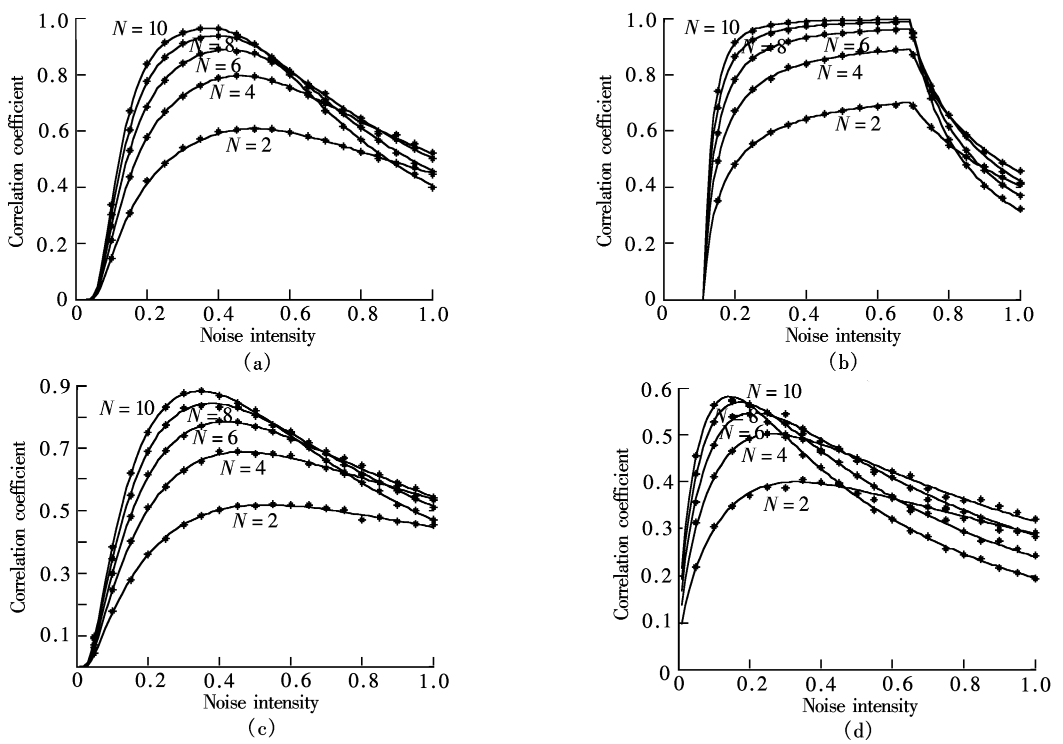

Fig. 2 Correlation coefficient $\rho(x, y)$ for different threshold numbers in the parallel array and different noises. (a) Gaussian noise; (b) Uniform noise; (c) Laplace noise; (d) Cauchy noise

3 Proof of SR in Parallel Array of Threshold Devices

We now give a theorem to prove that almost all the finite-variance densities produce SR in the parallel array with subthreshold input signal. The theorem shows that if $\rho(x, y) > 0$ then eventually the correlation coefficient $\rho(x, y)$ tends toward zero as the noise variance tends toward zero. So the correlation coefficient $\rho(x, y)$ must increase as the noise variance increases from zero. The only limiting assumption is that the noise PDF $f_n(x)$ is an even function with respect to the mean $E(n)$ and the noise mean $E(n)$ is less than $u-1$.

Theorem 1 Suppose that the noise PDF $f_n(x)$ is even function with respect to the mean $m=E(n)$ and that the input signal is subthreshold ($u<1$). Suppose that there is some statistical dependence between the input signal x and the output signal y (i. e., $\rho(x, y)>0$). Suppose that the noise mean $E(n)$ is less than $u-1$ if $f_n(x)$ has a finite variance $\sigma^2=E[(x-m)^2]$. Then the array exhibits the non-monotone SR effect in the sense that $\rho(x,y)\to 0$ as $\sigma\to 0$.

Proof Considering that the noise PDF $f_n(x)$ is an even function with respect to the mean $E(n)$ and the noise mean $E(n)$ is less than $u-1$, we have

$$\frac{1}{2}=P(n<m)\leqslant P(n<u-1)\leqslant P(n<u)\leqslant 1 \tag{12}$$

Then Eq. (3) implies that

$$\rho(x,y)\leqslant\frac{\sqrt{2^N p_0 p_1}\,[P^N(n<u)-P^N(n<u-1)]}{\sqrt{1-p_0P^N(n<u)-p_1P^N(n<u-1)}}\leqslant$$
$$\frac{\sqrt{2^N p_0 p_1}\,[1-P^N(n<u-1)+1-P^N(n<u)]}{\sqrt{p_0[1-P^N(n<u)]+p_1[1-P^N(n<u-1)]}}\leqslant$$
$$\sqrt{2^N p_1[1-P^N(n<u)]}+\sqrt{2^N p_0[1-P^N(n<u-1)]} \tag{13}$$

The result now follows if $P(n<u-1)\to 1$ and $P(n<u)\to 1$ as $\sigma\to 0$, i. e., $\rho(x,y)\to 0$ as $\sigma\to 0$. Now we pick $\varepsilon=u-1-m>0$, then

$$1\geqslant p(n<u)\geqslant p(n<u-1)=P(n-m<\varepsilon)=$$
$$1-P(n-m\geqslant\varepsilon)\geqslant 1-P(|n-m|\geqslant\varepsilon)\geqslant 1-\frac{\sigma^2}{\varepsilon^2} \tag{14}$$

The last inequality in Eq. (14) is according to the Chebyshev inequality. So we have $P(n<u-1)\to 1$ and $P(n<u)\to 1$ as $\sigma\to 0$, thus $\rho(x,y)\to 0$ as $\sigma\to 0$.

Like in Ref. [4], we now proceed to the more general (and more realistic) case where infinite-variance noise interferes with the parallel array. Many types of impulsive noise are modeled with symmetric (about a) alpha-stable bell-curve probability density functions with parameter α in the characteristic function $\varphi(\omega)=\exp\{ia\omega-\gamma|\omega|^{\alpha}\}$, here γ is the standard dispersion parameter and a is the location parameter.

Theorem 2 Suppose that $\rho(x,y)>0$ and the parallel array uses symmetric alpha-stable noise with the location parameter a. Suppose that a is less than $u-1$. Then the array exhibits the non-monotone SR effect if the input signal is subthreshold.

Proof Considering that the noise is symmetric alpha-stable noise with location parameter a and a is less than $u-1$, we also have

$$\frac{1}{2}=P(n<a)\leqslant P(n<u-1)\leqslant P(n<u)\leqslant 1 \tag{15}$$

Then Eq. (3) also implies that

$$\rho(x,y)\leqslant\frac{\sqrt{2^N p_0 p_1}\,[P^N(n<u)-P^N(n<u-1)]}{\sqrt{p_0[1-P^N(n<u)]+p_1[1-P^N(n<u-1)]}}\leqslant$$
$$\frac{\sqrt{2^N p_0 p_1\{[1-P^N(n<u)]+[1-P^N(n<u-1)]\}}}{\sqrt{p_0[1-P^N(n<u)]+p_1[1-P^N(n<u-1)]}}\cdot$$
$$\sqrt{P^N(n<u)-P^N(n<u-1)}\leqslant$$
$$\sqrt{N2^N[P(n<u)-P(n<u-1)]}\leqslant\sqrt{N2^N}\left\{\int_{u-1}^{u}f_n(x)\,dx\right\}^{\frac{1}{2}} \tag{16}$$

According to $\lim_{\gamma\to 0}\varphi(\omega)=\exp(ia\omega)$, the Fourier transformation gives the corresponding density function in the limiting case ($\gamma\to 0$) as a translated delta function $\lim_{\gamma\to 0}f_n(x)=\delta(x-a)$, then $\lim_{\gamma\to 0}\int_{u-1}^{u}f_n(x)\,dx=\int_{u-1}^{u}\delta(x-a)\,dx=0$, because a is less than $u-1$. So we have $\rho(x,y)\to 0$ as $\gamma\to 0$.

Noise affects the nonlinear system in a complex way. The proposed two theorems do not guarantee that the predicted increase in the correlation coefficient will be significant and do not especially guarantee that the maximum of the correlation coefficient can approach to one. But, they guarantee that many kinds of noises produce SR in the parallel array when the input signal is subthreshold, SR shows some robustness to noises in the array.

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并行阈值阵列中基于相关系数的随机谐振

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摘要:基于相关系数讨论了并行阈值阵列中的随机谐振现象. 对于4种典型噪声:高斯噪声、均匀噪声、拉普拉斯噪声和柯西噪声,当信号在阈下时,噪声能改善信号间的相关系数,随机谐振现象存在;随着并行阈值阵列中阈值单元数的增加,相关系数的最大值迅速地趋于1,随机谐振功效极大地提高. 给出2个定理证明在极大阈值网络中随机谐振现象对噪声具有一定的鲁棒性. 这些结果进一步拓展了随机谐振在信号处理中的应用.

关键词:随机谐振;相关系数;阈值阵列
中图分类号:TN911.7;TN911.2