

Second-order difference scheme for a nonlinear model of wood drying process

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Abstract: A numerical simulation for a model of wood drying process is considered. The model is given by a couple of nonlinear differential equations. One is a nonlinear parabolic equation and the other one is a nonlinear ordinary equation. A difference scheme is derived by the method of reduction of order. First, a new variable is introduced and the original problem is rewritten into a system of the first-order differential equations. Secondly, a difference scheme is constructed for the later problem. The solvability, stability and convergence of the difference scheme are proved by the energy method. The convergence order of the difference scheme is second-order both in time and in space. A prior error estimate is put forward. The new variable is put aside to reduce the computational cost. A numerical example testifies the theoretical result.

Key words: wood drying process; model; nonlinear differential equation; difference scheme; method of reduction of order; stability; convergence

The process of moisture motion in a wood can be described by a system of two nonlinear differential equations

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left(d(x) \frac{\partial u}{\partial x} \right) - P(u, v) \quad 0 < x < 1, t > 0 \quad (1)$$

$$\frac{\partial v}{\partial t} = P(u, v) \quad 0 \leq x \leq 1, t > 0 \quad (2)$$

combined with the initial and boundary value conditions

$$u(x, 0) = \phi(x), \quad v(x, 0) = \varphi(x) \quad 0 \leq x \leq 1 \quad (3)$$

$$-d(0) \frac{\partial u}{\partial x}(0, t) + \mu_0 u(0, t) = f(t), \quad d(1) \frac{\partial u}{\partial x}(1, t) + \mu_1 u(1, t) = g(t) \quad t > 0 \quad (4)$$

where a, μ_0, μ_1 are given positive constants; d is a positive function; $u(x, t)$ is the moisture content in the gap of wood cells; $v(x, t)$ is the moisture content in the cellular wall; and $P(u, v)$ is the exchange rate. In the two environments the moisture exchange occurs and as the existence of gravity, there is also diffusion and convection between the gap and cellular wall, which have greatly influenced on the moisture motion.

The wood-drying model has been studied by some scholars^[1-2]. However, in their papers the model was usually restricted to a simple diffusion equation^[3-4]. The model (1) to (4) deals with a more comprehensive and practical problem, such as the drying and saturation processes in various porous solids and soils. In Ref. [5], the author studied this model, constructed a finite difference scheme and proved the stability as well as gave a prior error estimate. The global error is $O(\tau + h^2)$.

In this paper, a new difference scheme is constructed for the model (1) to (4) by the method of reduction of order^[6-7] and the solvability, uniqueness and convergence are proved. The global error estimate is of the second-order in both time and space. Finally, a numerical example is presented and the numerical results are accordant with the theoretical results.

1 Derivation of Difference Scheme

In this section, we restrict t in a finite interval $[0, T]$ and derive a difference scheme for (1) to (4) by the method of reduction of order. We assume that

- ① There exist two constants c_0 and c_1 such that $c_0 \leq d(x) \leq c_1$.
- ② All the functions are smooth enough and there exists a smooth solution $\{u(x, t), v(x, t)\}$ to (1) to (4).
- ③ Function $P(u, v)$ is Lipschitz continuous on the neighborhood of the solution, that is, there exist two posi-

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tive numbers ε_0 and c_2 such that

$$|P(u(x, t) + \varepsilon_1, v(x, t) + \varepsilon_2) - P(u(x, t) + \varepsilon_3, v(x, t) + \varepsilon_4)| \leq c_2(|\varepsilon_1 - \varepsilon_3| + |\varepsilon_2 - \varepsilon_4|) \quad 0 \leq x \leq 1, 0 \leq t \leq T \quad (5)$$

where $|\varepsilon_j| < \varepsilon_0 (j=1, 2, 3, 4)$.

Take two positive integers M and K . Let $\tau = \frac{T}{K}$ and $h = \frac{1}{M}$. Define

$$\Omega_h^\tau = \{(x_i, t_n) \mid x_i = ih, t_n = k\tau, 0 \leq i \leq M, 0 \leq k \leq K\}$$

Suppose that $u = \{u_i^k \mid 0 \leq i \leq M, 0 \leq k \leq K\}$ is a grid function on Ω_h^τ . Introduce the following notations:

$$\begin{aligned} u_{i-\frac{1}{2}}^k &= \frac{1}{2}(u_i^k + u_{i-1}^k), \quad \bar{u}_i^k = \frac{1}{2}(u_i^{k+1} + u_i^{k-1}), \quad \delta_x u_{i-\frac{1}{2}}^k = \frac{1}{h}(u_i^k - u_{i-1}^k) \\ \delta_x u_i^k &= \frac{1}{2h}(u_{i+1}^k - u_{i-1}^k), \quad \Delta_t u_i^k = \frac{1}{2\tau}(u_i^{k+1} - u_i^{k-1}), \quad x_{i-\frac{1}{2}} = \frac{1}{2}(x_i + x_{i-1}) \\ d_{i-\frac{1}{2}} &= d(x_{i-\frac{1}{2}}), \quad \delta_x(d_i \delta_x u_i^k) = \frac{1}{h}(d_{i+\frac{1}{2}} \delta_x u_{i+\frac{1}{2}}^k - d_{i-\frac{1}{2}} \delta_x u_{i-\frac{1}{2}}^k) \end{aligned}$$

where \bar{u}_i^k is an average of u at the points (x_i, t_{k+1}) and (x_i, t_{k-1}) , and $\Delta_t u_i^k$ is the difference quotient of u based on these two points; $u_{i-\frac{1}{2}}^k$ is an average of u at the points (x_i, t_k) and (x_{i-1}, t_k) , and $\delta_x u_{i-\frac{1}{2}}^k$ is the first-order difference quotient of u on these two points; $d_{i-\frac{1}{2}}$ means the value of function $d(x)$ at $x = x_{i-\frac{1}{2}}$. In addition, we denote $\|u^k\| =$

$$\left[h \sum_{i=1}^M (u_{i-\frac{1}{2}}^k)^2 \right]^{\frac{1}{2}}.$$

Let us introduce a new variable $w = d(x) \frac{\partial u}{\partial x}$, then the system (1) to (4) can be rewritten as a system of the first-order differential equations

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} - \frac{a}{d(x)} w - P(u, v) \quad 0 < x < 1, 0 < t \leq T \quad (6)$$

$$\frac{\partial u}{\partial x} = \frac{1}{d(x)} w \quad 0 < x < 1, 0 < t \leq T \quad (7)$$

$$\frac{\partial v}{\partial t} = P(u, v) \quad 0 \leq x \leq 1, 0 < t \leq T \quad (8)$$

$$-w(0, t) + \mu_0 u(0, t) = f(t), \quad w(1, t) + \mu_1 u(1, t) = g(t) \quad 0 < t \leq T \quad (9)$$

$$u(x, 0) = \phi(x), \quad v(x, 0) = \varphi(x) \quad 0 < x < 1 \quad (10)$$

Define the grid functions:

$$U_i^k = u(x_i, t_k), \quad V_i^k = v(x_i, t_k), \quad W_i^k = w(x_i, t_k) \quad 0 \leq i \leq M, 0 \leq k \leq K$$

By the Taylor expansion, we have

$$\Delta_t U_{i-\frac{1}{2}}^k = \delta_x \bar{W}_{i-\frac{1}{2}}^k - \frac{a}{d_{i-\frac{1}{2}}} \bar{W}_{i-\frac{1}{2}}^k - P(U_{i-\frac{1}{2}}^k, V_{i-\frac{1}{2}}^k) + \rho_{1,i-\frac{1}{2}}^k \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (11)$$

$$0 = \delta_x \bar{U}_{i-\frac{1}{2}}^k - \frac{1}{d_{i-\frac{1}{2}}} \bar{W}_{i-\frac{1}{2}}^k + \rho_{2,i-\frac{1}{2}}^k \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (12)$$

$$\Delta_t V_{i-\frac{1}{2}}^k = P(U_{i-\frac{1}{2}}^k, V_{i-\frac{1}{2}}^k) + \rho_{3,i-\frac{1}{2}}^k \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (13)$$

$$U_i^0 = \phi_i, \quad U_i^1 = \phi_i + \tau(\phi_1)_i + \rho_i \quad 0 \leq i \leq M \quad (14)$$

$$V_i^0 = \varphi_i, \quad V_i^1 = \varphi_i + \tau(\varphi_1)_i + \bar{\rho}_i \quad 0 \leq i \leq M \quad (15)$$

$$-\bar{W}_0^k + \mu_0 \bar{U}_0^k = \bar{f}^k, \quad \bar{W}_M^k + \mu_1 \bar{U}_M^k = \bar{g}^k \quad 1 \leq k \leq K-1 \quad (16)$$

where

$$\begin{aligned} \phi_1(x) &= \frac{d}{dx} \left(d(x) \frac{d\phi(x)}{dx} \right) - P(\phi(x), \varphi(x)) - a \frac{d\phi(x)}{dx}, \quad \bar{f}^k = \frac{1}{2}[f(t_{k+1}) + f(t_{k-1})] \\ \bar{g}^k &= \frac{1}{2}[g(t_{k+1}) + g(t_{k-1})], \quad \varphi_1(x) = P(\phi(x), \varphi(x)) \end{aligned} \quad (17)$$

and there exists a positive constant c_3 satisfying

$$|\rho_i| \leq c_3 \tau^2, \quad |\bar{\rho}_i| \leq c_3 \tau^2 \quad 0 \leq i \leq M \quad (18)$$

$$|\rho_{l,i-\frac{1}{2}}^k| \leq c_3(\tau^2 + h^2) \quad 1 \leq i \leq M; 1 \leq k \leq K-1; l=1, 2, 3 \quad (19)$$

Omitting the small terms in (11) to (16), we construct for system (6) to (10) the following difference scheme

$$\Delta_t u_{i-\frac{1}{2}}^k = \delta_x \bar{w}_{i-\frac{1}{2}}^k - \frac{a}{d_{i-\frac{1}{2}}} \bar{w}_{i-\frac{1}{2}}^k - P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k) \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (20)$$

$$0 = \delta_x \bar{u}_{i-\frac{1}{2}}^k - \frac{1}{d_{i-\frac{1}{2}}} \bar{w}_{i-\frac{1}{2}}^k \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (21)$$

$$\Delta_i v_{i-\frac{1}{2}}^k = P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k) \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (22)$$

$$u_i^0 = \phi_i, \quad u_i^1 = \phi_i + \tau(\phi_1)_i \quad 0 \leq i \leq M \quad (23)$$

$$v_i^0 = \varphi_i, \quad v_i^1 = \varphi_i + \tau(\varphi_1)_i \quad 0 \leq i \leq M \quad (24)$$

$$-\bar{w}_0^k + \mu_0 \bar{u}_0^k = \bar{f}^k, \quad \bar{w}_M^k + \mu_1 \bar{u}_M^k = \bar{g}^k \quad 1 \leq k \leq K-1 \quad (25)$$

For this difference scheme (20) to (25), we have the following theorem.

Theorem 1 The difference scheme (20) to (25) is equivalent to

$$\Delta_i u_{i-\frac{1}{2}}^k + a \delta_x \bar{u}_{i-\frac{1}{2}}^k = \frac{2}{h} [d_{i-\frac{1}{2}} \delta_x \bar{u}_{i-\frac{1}{2}}^k - \mu_0 \bar{u}_0^k + \bar{f}^k] - P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k) \quad 1 \leq k \leq K-1 \quad (26)$$

$$\frac{1}{2} (\Delta_i u_{i+\frac{1}{2}}^k + \Delta_i u_{i-\frac{1}{2}}^k) + a \delta_x \bar{u}_i^k = \delta_x (d_i \delta_x \bar{u}_i^k) - \frac{1}{2} [P(u_{i+\frac{1}{2}}^k, v_{i+\frac{1}{2}}^k) + P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k)] \quad 1 \leq i \leq M-1, 1 \leq k \leq K-1 \quad (27)$$

$$\Delta_i u_{M-\frac{1}{2}}^k + a \delta_x \bar{u}_{M-\frac{1}{2}}^k = \frac{2}{h} [-d_{M-\frac{1}{2}} \delta_x \bar{u}_{M-\frac{1}{2}}^k - \mu_1 \bar{u}_M^k + \bar{g}^k] - P(u_{M-\frac{1}{2}}^k, v_{M-\frac{1}{2}}^k) \quad 1 \leq k \leq K-1 \quad (28)$$

$$\Delta_i v_{i-\frac{1}{2}}^k = P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k) \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (29)$$

$$u_i^0 = \phi_i, \quad u_i^1 = \phi_i + \tau(\phi_1)_i \quad 0 \leq i \leq M \quad (30)$$

$$v_i^0 = \varphi_i, \quad v_i^1 = \varphi_i + \tau(\varphi_1)_i \quad 0 \leq i \leq M \quad (31)$$

$$\bar{w}_i^k = d_{i+\frac{1}{2}} \delta_x \bar{u}_{i+\frac{1}{2}}^k - \frac{h}{2} [\Delta_i u_{i+\frac{1}{2}}^k + a \delta_x \bar{u}_{i+\frac{1}{2}}^k + P(u_{i+\frac{1}{2}}^k, v_{i+\frac{1}{2}}^k)] \quad 0 \leq i \leq M-1, 1 \leq k \leq K-1 \quad (32)$$

$$\bar{w}_M^k = d_{M-\frac{1}{2}} \delta_x \bar{u}_{M-\frac{1}{2}}^k + \frac{h}{2} [\Delta_i u_{M-\frac{1}{2}}^k + a \delta_x \bar{u}_{M-\frac{1}{2}}^k + P(u_{M-\frac{1}{2}}^k, v_{M-\frac{1}{2}}^k)] \quad 1 \leq k \leq K-1 \quad (33)$$

Proof It follows from (21) that

$$\bar{w}_{i-\frac{1}{2}}^k = d_{i-\frac{1}{2}} \delta_x \bar{u}_{i-\frac{1}{2}}^k \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (34)$$

Substituting (34) into (20), we obtain

$$\delta_x \bar{w}_{i-\frac{1}{2}}^k = \Delta_i u_{i-\frac{1}{2}}^k + a \delta_x \bar{u}_{i-\frac{1}{2}}^k + P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k) \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (35)$$

Multiplying (35) by $\frac{1}{2}h$ and adding the result with (34), we have

$$\bar{w}_i^k = d_{i-\frac{1}{2}} \delta_x \bar{u}_{i-\frac{1}{2}}^k + \frac{h}{2} [\Delta_i u_{i-\frac{1}{2}}^k + a \delta_x \bar{u}_{i-\frac{1}{2}}^k + P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k)] \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (36)$$

Similarly, multiplying (35) by $\frac{1}{2}h$ and subtracting the result from (34), we have

$$\bar{w}_i^k = d_{i+\frac{1}{2}} \delta_x \bar{u}_{i+\frac{1}{2}}^k - \frac{h}{2} [\Delta_i u_{i+\frac{1}{2}}^k + a \delta_x \bar{u}_{i+\frac{1}{2}}^k + P(u_{i+\frac{1}{2}}^k, v_{i+\frac{1}{2}}^k)] \quad 0 \leq i \leq M-1, 1 \leq k \leq K-1 \quad (37)$$

It follows from Eqs. (36) and (37) for $1 \leq i \leq M-1, 1 \leq k \leq K-1$ that

$$d_{i+\frac{1}{2}} \delta_x \bar{u}_{i+\frac{1}{2}}^k - \frac{h}{2} [\Delta_i u_{i+\frac{1}{2}}^k + a \delta_x \bar{u}_{i+\frac{1}{2}}^k + P(u_{i+\frac{1}{2}}^k, v_{i+\frac{1}{2}}^k)] = d_{i-\frac{1}{2}} \delta_x \bar{u}_{i-\frac{1}{2}}^k + \frac{h}{2} [\Delta_i u_{i-\frac{1}{2}}^k + a \delta_x \bar{u}_{i-\frac{1}{2}}^k + P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k)] \quad (38)$$

from which we can obtain (27).

Considering Eq. (37) for $i=0$ we know the first equation of (25) is equivalent to

$$\Delta_i u_{i-\frac{1}{2}}^k + a \delta_x \bar{u}_{i-\frac{1}{2}}^k = \frac{2}{h} [d_{i-\frac{1}{2}} \delta_x \bar{u}_{i-\frac{1}{2}}^k - \mu_0 \bar{u}_0^k + \bar{f}^k] - P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k) \quad 1 \leq k \leq K-1$$

Similarly, considering (36) for $i=M$ the second equation of (25) is equivalent to (28).

It is not difficult to verify the equivalence relations. The proof is completed.

We construct the difference scheme (26) to (31) for (1) to (4). We also present an algorithm here to solve the difference scheme. By (30) and (31), we can get $\{u_i^0, u_i^1 \mid 0 \leq i \leq M\}$ and $\{v_{i-\frac{1}{2}}^0, v_{i-\frac{1}{2}}^1 \mid 1 \leq i \leq M\}$ easily. Suppose that we have obtained $\{u_i^{k-1}, u_i^k \mid 0 \leq i \leq M\}$ and $\{v_{i-\frac{1}{2}}^{k-1}, v_{i-\frac{1}{2}}^k \mid 0 \leq i \leq M\}$, then by solving the tridiagonal linear algebraic equations (26) to (28) we can obtain $\{u_i^{k+1} \mid 0 \leq i \leq M\}$, and then get $\{v_{i-\frac{1}{2}}^{k+1} \mid 0 \leq i \leq M\}$ directly by (29).

2 Solvability and Convergence

In this section, we analyze the solvability and convergence of the difference scheme.

Theorem 2 If τ is appropriately small, the difference scheme (26) to (31) is uniquely solvable.

Proof According to theorem 1, we only need to prove that (20) to (25) determines $u^k \equiv \{u_i^k \mid 0 \leq i \leq M\}$ and $v^k \equiv \{v_{i-\frac{1}{2}}^k \mid 1 \leq i \leq M\}$ for $k=0, 1, 2, \dots, K$ uniquely. We can make use of the induction method to prove that.

From (23) and (24), u^0, v^0, u^1 and v^1 are determined uniquely. So we can suppose that we have determined $\{u^k, v^k \mid 1 \leq k \leq l\}$. Consider the homogeneous system for (20) to (25) when $k=l$,

$$\frac{1}{2\tau} u_{i-\frac{1}{2}}^{l+1} = \frac{1}{2} \delta_x w_{i-\frac{1}{2}}^{l+1} - \frac{a}{2d_{i-\frac{1}{2}}} w_{i-\frac{1}{2}}^{l+1} \quad 1 \leq i \leq M \quad (39)$$

$$0 = \frac{1}{2} \delta_x u_{i-\frac{1}{2}}^{l+1} - \frac{1}{2d_{i-\frac{1}{2}}} w_{i-\frac{1}{2}}^{l+1} \quad 1 \leq i \leq M \quad (40)$$

$$\frac{1}{2\tau} v_{i-\frac{1}{2}}^{l+1} = 0 \quad 1 \leq i \leq M \quad (41)$$

$$-w_0^{l+1} + \mu_0 u_0^{l+1} = 0, \quad w_M^{l+1} + \mu_1 u_M^{l+1} = 0 \quad 0 \leq i \leq M \quad (42)$$

Multiplying (39) and (40) by $u_{i-\frac{1}{2}}^{l+1}$ and $w_{i-\frac{1}{2}}^{l+1}$, respectively, and adding the results, we obtain

$$\begin{aligned} \frac{1}{2\tau} (u_{i-\frac{1}{2}}^{l+1})^2 &= \frac{1}{2} (u_{i-\frac{1}{2}}^{l+1} \delta_x w_{i-\frac{1}{2}}^{l+1} + w_{i-\frac{1}{2}}^{l+1} \delta_x u_{i-\frac{1}{2}}^{l+1}) - \frac{1}{2} \frac{a}{d_{i-\frac{1}{2}}} w_{i-\frac{1}{2}}^{l+1} u_{i-\frac{1}{2}}^{l+1} - \frac{1}{2d_{i-\frac{1}{2}}} (w_{i-\frac{1}{2}}^{l+1})^2 \leq \\ &= \frac{1}{2} (u_i^{l+1} w_i^{l+1} - u_{i-1}^{l+1} w_{i-1}^{l+1}) + \frac{1}{2d_{i-\frac{1}{2}}} (w_{i-\frac{1}{2}}^{l+1})^2 + \frac{a^2}{8d_{i-\frac{1}{2}}} (u_{i-\frac{1}{2}}^{l+1})^2 - \frac{1}{2d_{i-\frac{1}{2}}} (w_{i-\frac{1}{2}}^{l+1})^2 = \\ &= \frac{1}{2} (u_i^{l+1} w_i^{l+1} - u_{i-1}^{l+1} w_{i-1}^{l+1}) + \frac{a^2}{8d_{i-\frac{1}{2}}} (u_{i-\frac{1}{2}}^{l+1})^2 \end{aligned} \quad (43)$$

Multiplying the equality above by h , summing up for i from 1 to M and using (42), we obtain

$$\frac{1}{2\tau} \|u^{l+1}\|^2 \leq \frac{1}{2} (w_M^{l+1} u_M^{l+1} - w_0^{l+1} u_0^{l+1}) + \frac{a^2}{8} h \sum_{i=1}^M \frac{1}{d_{i-\frac{1}{2}}} (u_{i-\frac{1}{2}}^{l+1})^2 \leq \frac{a^2}{8} h \sum_{i=1}^M \frac{1}{d_{i-\frac{1}{2}}} (u_{i-\frac{1}{2}}^{l+1})^2 \quad (44)$$

When $\tau < 4c_0/a^2$ we have $\|u^{l+1}\| = 0$, that is $u_{i-\frac{1}{2}}^{l+1} = 0, i = 1, 2, \dots, M$. Combining (39), (40) and (42), we have $u_i^{l+1} = 0, i = 0, 1, 2, \dots, M$. From (41) we can easily know that $v_{i-\frac{1}{2}}^{l+1} = 0$. Thus (20) to (25) determines u^{l+1} and v^{l+1} . The proof is completed.

Let $e_{1,i}^k = U_i^k - u_i^k, e_{2,i}^k = W_i^k - w_i^k, e_{3,i}^k = V_i^k - v_i^k$. Subtracting (20) to (25) from (11) to (16), we obtain the following error equations:

$$\Delta_t e_{1,i-\frac{1}{2}}^k = \delta_x \bar{e}_{2,i-\frac{1}{2}}^k - \frac{a}{d_{i-\frac{1}{2}}} \bar{e}_{2,i-\frac{1}{2}}^k - [P(U_{i-\frac{1}{2}}^k, V_{i-\frac{1}{2}}^k) - P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k)] + \rho_{1,i-\frac{1}{2}}^k \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (45)$$

$$0 = \delta_x \bar{e}_{1,i-\frac{1}{2}}^k - \frac{1}{d_{i-\frac{1}{2}}} \bar{e}_{2,i-\frac{1}{2}}^k + \rho_{2,i-\frac{1}{2}}^k \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (46)$$

$$\Delta_t e_{3,i-\frac{1}{2}}^k = [P(U_{i-\frac{1}{2}}^k, V_{i-\frac{1}{2}}^k) - P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k)] + \rho_{3,i-\frac{1}{2}}^k \quad 1 \leq i \leq M, 1 \leq k \leq K-1 \quad (47)$$

$$e_{1,i}^0 = 0, \quad e_{3,i}^0 = 0 \quad 0 \leq i \leq M \quad (48)$$

$$e_{1,i}^1 = \rho_i, \quad e_{3,i}^1 = \bar{\rho}_i \quad 0 \leq i \leq M \quad (49)$$

$$-\bar{e}_{2,0}^k + \mu_0 \bar{e}_{1,0}^k = 0, \quad \bar{e}_{2,M}^k + \mu_1 \bar{e}_{1,M}^k = 0 \quad 1 \leq k \leq K-1 \quad (50)$$

Theorem 3 If $\tau = O(h^{\frac{1}{4}+\varepsilon})$ for some positive constants ε and h is appropriately small, then there exists the following estimate for difference scheme (26) to (31)

$$\|e_1^k\| \leq c_4(\tau^2 + h^2), \quad \|e_3^k\| \leq c_4(\tau^2 + h^2) \quad k = 1, 2, \dots, K \quad (51)$$

where

$$c_4 = c_3 \left\{ \left[2 + \frac{\frac{3}{2} + c_1}{(2c_2^2 + c_2) + 2\max\left\{1 + \frac{a^2}{2c_0}, 1 + c_2\right\}} \right] e^{[(6c_2^2 + 3c_2) + 6\max\{1 + \frac{a^2}{2c_0}, 1 + c_2\}]\tau} \right\}^{\frac{1}{2}} \quad (52)$$

Proof According to theorem 1, we only need to prove that (51) holds for (20) to (25).

From initial conditions (23), (24) and (48), we have

$$\|e_1^0\| = \|e_3^0\| = 0 \quad (53)$$

From (18) and (49) the estimates for e_1^1 and e_3^1 are

$$\|e_1^1\| \leq c_3(\tau^2 + h^2), \quad \|e_3^1\| \leq c_3(\tau^2 + h^2) \quad (54)$$

So (51) holds for $k=1$. Now we assume that (51) holds for $1 \leq k \leq l$.

According to the induction assumption, and $\tau = O(h^{\frac{1}{4}+\varepsilon})$ and h is appropriately small, we have

$$|e_{1,i-\frac{1}{2}}^l| \leq h^{-\frac{1}{2}} \|e_1^l\| \leq c_4 h^{-\frac{1}{2}} (\tau^2 + h^2) = c_4 (\tau^2 h^{-\frac{1}{2}} + h^{\frac{3}{2}}) \leq \varepsilon_0 \quad 1 \leq i \leq M, 1 \leq k \leq l$$

According to (5), we have

$$|P(U_{i-\frac{1}{2}}^k, V_{i-\frac{1}{2}}^k) - P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k)| \leq c_2 (|U_{i-\frac{1}{2}}^k - u_{i-\frac{1}{2}}^k| + |V_{i-\frac{1}{2}}^k - v_{i-\frac{1}{2}}^k|) \leq c_2 (|e_{1,i-\frac{1}{2}}^k| + |e_{3,i-\frac{1}{2}}^k|) \quad 1 \leq i \leq M, 1 \leq k \leq l \quad (55)$$

Multiplying (45) and (46) by $2\bar{e}_{1,i-\frac{1}{2}}^k$ and $2\bar{e}_{3,i-\frac{1}{2}}^k$, respectively, and adding the results, we obtain

$$\begin{aligned} \frac{1}{2\tau} [(e_{1,i-\frac{1}{2}}^{k+1})^2 - (e_{1,i-\frac{1}{2}}^{k-1})^2] &= \frac{2}{h} (\bar{e}_{1,i}^k \bar{e}_{2,i}^k - \bar{e}_{1,i-1}^k \bar{e}_{2,i-1}^k) - 2 \frac{a}{d_{i-\frac{1}{2}}} \bar{e}_{1,i-\frac{1}{2}}^k \bar{e}_{2,i-\frac{1}{2}}^k + 2\rho_{1,i-\frac{1}{2}}^k \bar{e}_{1,i-\frac{1}{2}}^k - \\ &2[P(U_{i-\frac{1}{2}}^k, V_{i-\frac{1}{2}}^k) - P(u_{i-\frac{1}{2}}^k, v_{i-\frac{1}{2}}^k)] \bar{e}_{1,i-\frac{1}{2}}^k - \frac{2}{d_{i-\frac{1}{2}}} (\bar{e}_{2,i-\frac{1}{2}}^k)^2 + 2\rho_{2,i-\frac{1}{2}}^k \bar{e}_{2,i-\frac{1}{2}}^k \leq \\ &\frac{2}{h} (\bar{e}_{1,i}^k \bar{e}_{2,i}^k - \bar{e}_{1,i-1}^k \bar{e}_{2,i-1}^k) + \frac{1}{d_{i-\frac{1}{2}}} (\bar{e}_{2,i-\frac{1}{2}}^k)^2 + \frac{a^2}{d_{i-\frac{1}{2}}} (\bar{e}_{1,i-\frac{1}{2}}^k)^2 + (\rho_{1,i-\frac{1}{2}}^k)^2 + (\bar{e}_{1,i-\frac{1}{2}}^k)^2 + \\ &2c_2 (|e_{1,i-\frac{1}{2}}^k| + |e_{3,i-\frac{1}{2}}^k|) |\bar{e}_{1,i-\frac{1}{2}}^k| - \frac{2}{d_{i-\frac{1}{2}}} (\bar{e}_{2,i-\frac{1}{2}}^k)^2 + \frac{1}{d_{i-\frac{1}{2}}} (\bar{e}_{2,i-\frac{1}{2}}^k)^2 + d_{i-\frac{1}{2}} (\rho_{2,i-\frac{1}{2}}^k)^2 = \\ &\frac{2}{h} (\bar{e}_{1,i}^k \bar{e}_{2,i}^k - \bar{e}_{1,i-1}^k \bar{e}_{2,i-1}^k) + \left(1 + \frac{a^2}{d_{i-\frac{1}{2}}}\right) (\bar{e}_{1,i-\frac{1}{2}}^k)^2 + 2c_2 (|e_{1,i-\frac{1}{2}}^k| + |e_{3,i-\frac{1}{2}}^k|) |\bar{e}_{1,i-\frac{1}{2}}^k| + \\ &(\rho_{1,i-\frac{1}{2}}^k)^2 + d_{i-\frac{1}{2}} (\rho_{2,i-\frac{1}{2}}^k)^2 \leq \frac{2}{h} (\bar{e}_{1,i}^k \bar{e}_{2,i}^k - \bar{e}_{1,i-1}^k \bar{e}_{2,i-1}^k) + \left(2 + \frac{a^2}{d_{i-\frac{1}{2}}}\right) (\bar{e}_{1,i-\frac{1}{2}}^k)^2 + 2c_2 (|e_{1,i-\frac{1}{2}}^k|^2 + |e_{3,i-\frac{1}{2}}^k|^2) + \\ &(\rho_{1,i-\frac{1}{2}}^k)^2 + d_{i-\frac{1}{2}} (\rho_{2,i-\frac{1}{2}}^k)^2 \leq \frac{2}{h} (\bar{e}_{1,i}^k \bar{e}_{2,i}^k - \bar{e}_{1,i-1}^k \bar{e}_{2,i-1}^k) + \left(2 + \frac{a^2}{d_{i-\frac{1}{2}}}\right) \frac{(e_{1,i-\frac{1}{2}}^{k-1})^2 + (e_{1,i-\frac{1}{2}}^{k+1})^2}{2} + \\ &2c_2^2 (|e_{1,i-\frac{1}{2}}^k|^2 + |e_{3,i-\frac{1}{2}}^k|^2) + (\rho_{1,i-\frac{1}{2}}^k)^2 + d_{i-\frac{1}{2}} (\rho_{2,i-\frac{1}{2}}^k)^2 \quad 1 \leq k \leq l \end{aligned} \quad (56)$$

Multiplying both sides of the above inequality by $2\tau h$, summing up for i from 1 to M , using (19) and (50), we have

$$\left[1 - \left(2 + \frac{a^2}{c_0}\right)\tau\right] \|e_1^{k+1}\|^2 \leq \left[1 + \left(2 + \frac{a^2}{c_0}\right)\tau\right] \|e_1^{k-1}\|^2 + 4c_2^2\tau (\|e_1^k\|^2 + \|e_3^k\|^2) + 2(1 + c_1)c_3^2\tau(\tau^2 + h^2)^2 \quad 1 \leq k \leq l$$

When $(2 + a^2/c_0)\tau < 1/3$, we have

$$\|e_1^{k+1}\|^2 \leq \left[1 + 3\left(2 + \frac{a^2}{c_0}\right)\tau\right] \|e_1^{k-1}\|^2 + 6c_2^2\tau (\|e_1^k\|^2 + \|e_3^k\|^2) + 3(1 + c_1)c_3^2\tau(\tau^2 + h^2)^2 \quad (57)$$

Multiplying (47) by $2(e_{3,i-\frac{1}{2}}^{k-1} + e_{3,i-\frac{1}{2}}^{k+1})$ and applying (55), we have

$$\begin{aligned} \frac{1}{\tau} [(e_{3,i-\frac{1}{2}}^{k+1})^2 - (e_{3,i-\frac{1}{2}}^{k-1})^2] &\leq 2(|e_{3,i-\frac{1}{2}}^{k+1}| + |e_{3,i-\frac{1}{2}}^{k-1}|) [c_2 (|e_{1,i-\frac{1}{2}}^k| + |e_{3,i-\frac{1}{2}}^k|) + |\rho_{3,i-\frac{1}{2}}^k|] \leq \\ &2(1 + c_2) (|e_{3,i-\frac{1}{2}}^{k+1}|^2 + |e_{3,i-\frac{1}{2}}^{k-1}|^2) + 2c_2 (|e_{1,i-\frac{1}{2}}^k|^2 + |e_{3,i-\frac{1}{2}}^k|^2) + (\rho_{3,i-\frac{1}{2}}^k)^2 \end{aligned} \quad (58)$$

which follows

$$\begin{aligned} (e_{3,i-\frac{1}{2}}^{k+1})^2 &\leq (e_{3,i-\frac{1}{2}}^{k-1})^2 + 2(1 + c_2)\tau (|e_{3,i-\frac{1}{2}}^{k+1}|^2 + |e_{3,i-\frac{1}{2}}^{k-1}|^2) + \\ &2c_2\tau (|e_{1,i-\frac{1}{2}}^k|^2 + |e_{3,i-\frac{1}{2}}^k|^2) + \tau(\rho_{3,i-\frac{1}{2}}^k)^2 \quad 1 \leq k \leq l \end{aligned}$$

Applying (19), we obtain

$$[1 - 2(1 + c_2)\tau] (e_{3,i-\frac{1}{2}}^{k+1})^2 \leq [1 + 2(1 + c_2)\tau] (e_{3,i-\frac{1}{2}}^{k-1})^2 + 2c_2\tau (|e_{1,i-\frac{1}{2}}^k|^2 + |e_{3,i-\frac{1}{2}}^k|^2) + c_3^2\tau(\tau^2 + h^2)^2 \quad (59)$$

Multiplying both sides of (59) by h and summing up for i from 1 to M , we have

$$[1 - 2(1 + c_2)\tau] \|e_3^{k+1}\|^2 \leq [1 + 2(1 + c_2)\tau] \|e_3^{k-1}\|^2 + 2c_2\tau (\|e_1^k\|^2 + \|e_3^k\|^2) + c_3^2\tau(\tau^2 + h^2)^2 \quad (60)$$

When $2(1 + c_2)\tau < 1/3$, we have

$$\|e_3^{k+1}\|^2 \leq [1 + 6(1 + c_2)\tau] \|e_3^{k-1}\|^2 + 3c_2\tau (\|e_1^k\|^2 + \|e_3^k\|^2) + \frac{3}{2}c_3^2\tau(\tau^2 + h^2)^2 \quad (61)$$

Adding (57) and (61) together, we obtain

$$\begin{aligned} \|e_1^{k+1}\|^2 + \|e_3^{k+1}\|^2 &\leq \left[1 + \max\left(6 + \frac{3a^2}{c_0}, 6 + 6c_2\right)\tau\right] (\|e_1^{k-1}\|^2 + \|e_3^{k-1}\|^2) + \\ &(6c_2^2 + 3c_2)\tau (\|e_1^k\|^2 + \|e_3^k\|^2) + \left(3c_1 + \frac{9}{2}\right)c_3^2\tau(\tau^2 + h^2)^2 \end{aligned}$$

Let $\tilde{c} = (6c_2^2 + 3c_2) + 6\max\left(1 + \frac{a^2}{2c_0}, 1 + c_2\right)$, then we obtain

$$\|e_1^{k+1}\|^2 + \|e_3^{k+1}\|^2 \leq (1 + \tilde{c}\tau) \max(\|e_1^k\|^2 + \|e_3^k\|^2, \|e_1^{k-1}\|^2 + \|e_3^{k-1}\|^2) + \left(3c_1 + \frac{9}{2}\right)c_3^2\tau(\tau^2 + h^2)^2 \quad 1 \leq k \leq l \quad (62)$$

which follows

$$\max(\|e_1^{k+1}\|^2 + \|e_3^{k+1}\|^2, \|e_1^k\|^2 + \|e_3^k\|^2) \leq (1 + \tilde{c}\tau) \max(\|e_1^k\|^2 + \|e_3^k\|^2, \|e_1^{k-1}\|^2 + \|e_3^{k-1}\|^2) + \left(3c_1 + \frac{9}{2}\right) c_3^2 \tau (\tau^2 + h^2)^2 \quad 1 \leq k \leq l$$

By the Gronwall inequality, using (53) and (54), we can obtain

$$\begin{aligned} \|e_1^{k+1}\|^2 + \|e_3^{k+1}\|^2 &\leq \max(\|e_1^{k+1}\|^2 + \|e_3^{k+1}\|^2, \|e_1^k\|^2 + \|e_3^k\|^2) \leq \\ &e^{\tilde{c}k\tau} \left\{ \max(\|e_1^1\|^2 + \|e_3^1\|^2, \|e_1^0\|^2 + \|e_3^0\|^2) + \frac{1}{c} \left(3c_1 + \frac{9}{2}\right) c_3^2 (\tau^2 + h^2)^2 \right\} \leq \\ &c_3^2 e^{\tilde{c}T} \left[2 + \frac{1}{c} \left(3c_1 + \frac{9}{2}\right) \right] (\tau^2 + h^2)^2 \quad 1 \leq k \leq l \end{aligned}$$

Hence

$$\|e_1^{l+1}\| \leq c_4 (\tau^2 + h^2), \quad \|e_3^{l+1}\| \leq c_4 (\tau^2 + h^2) \quad (63)$$

where c_4 is defined by (52). Based on the induction principle, we can prove that (51) holds for all $0 \leq k \leq K$. The proof is completed.

3 Numerical Example

In this section, we will provide a numerical example to demonstrate the effectiveness of the finite difference scheme (20) to (25). We compute the following problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{10} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1+x}{10} \frac{\partial u}{\partial x} \right) - \frac{1}{20} (2+x) \left(u + e^x - \frac{1}{2+x} v \right) \quad 0 < x < 1, 0 < t \leq 1 \\ \frac{\partial v}{\partial t} &= \frac{1}{20} (2+x) \left(u + e^x - \frac{1}{2+x} v \right) \quad 0 \leq x \leq 1, 0 < t \leq 1 \\ u(x, 0) &= e^x, \quad v(x, 0) = 0 \quad 0 \leq x \leq 1 \\ -\frac{1}{10} \frac{\partial u}{\partial x}(0, t) + u(0, t) &= \frac{9}{10} e^{-\frac{1}{10}t}, \quad \frac{1}{5} \frac{\partial u}{\partial x}(1, t) + 2u(1, t) = \frac{11}{5} e^{1-\frac{1}{10}t} \quad 0 < t \leq 1 \end{aligned}$$

The exact solution of the problem above is

$$u = e^{x-\frac{1}{10}t}, \quad v = (2+x) \left(e^x - e^{x-\frac{1}{10}t} \right)$$

For this problem, we take different divisions of grids. Tab. 1 and Tab. 2 present the maximum norms of errors of u and v at $t=0.7$. Tab. 3 and Tab. 4 present 2-norms of errors of u and v at $t=0.7$. The norms are defined as follows:

$$\begin{aligned} \|u^k\|_\infty &= \max_{0 \leq i \leq M} |u_i^k|, \quad \|u^k\|_A = \left(h \sum_{i=1}^M |u_{i-\frac{1}{2}}^k|^2 \right)^{\frac{1}{2}} \\ \|E(\tau, h)\|_\infty &= \|U^k - u^k\|_\infty, \quad \|F(\tau, h)\|_\infty = \|V^k - v^k\|_\infty \\ \|E(\tau, h)\|_A &= \|U^k - u^k\|_A, \quad \|F(\tau, h)\|_A = \|V^k - v^k\|_A \end{aligned}$$

From the results we can see that the scheme is approximately of the second order.

Tab. 1 Maximum norms of errors of difference solutions u at $t=0.7$

M	K	$\ E(\tau, h)\ _\infty$	$\frac{\ E(\tau, h)\ _\infty}{\ E(\tau/2, h/2)\ _\infty}$
20	40	9.1921×10^{-5}	4.29
40	80	2.1409×10^{-5}	4.10
80	160	5.2249×10^{-6}	3.99
160	320	1.3105×10^{-6}	3.99
320	640	3.2819×10^{-7}	

Tab. 3 2-norms of errors of difference solutions u at $t=0.7$

M	K	$\ E(\tau, h)\ _A$	$\frac{\ E(\tau, h)\ _A}{\ E(\tau/2, h/2)\ _A}$
20	40	6.1773×10^{-5}	3.94
40	80	1.5694×10^{-5}	3.97
80	160	3.9553×10^{-6}	3.98
160	320	9.9280×10^{-7}	3.99
320	640	2.4870×10^{-7}	

Tab. 2 Maximum norms of errors of difference solutions v at $t=0.7$

M	K	$\ E(\tau, h)\ _\infty$	$\frac{\ E(\tau, h)\ _\infty}{\ E(\tau/2, h/2)\ _\infty}$
20	40	9.8177×10^{-5}	3.65
40	80	2.6916×10^{-5}	3.82
80	160	7.0459×10^{-6}	3.91
160	320	1.8025×10^{-6}	3.95
320	640	4.5583×10^{-7}	

Tab. 4 2-norms of errors of difference solutions v at $t=0.7$

M	K	$\ E(\tau, h)\ _A$	$\frac{\ E(\tau, h)\ _A}{\ E(\tau/2, h/2)\ _A}$
20	40	6.0274×10^{-5}	3.89
40	80	1.5507×10^{-5}	3.94
80	160	3.9315×10^{-6}	3.97
160	320	9.8968×10^{-7}	3.99
320	640	2.4827×10^{-7}	

Fig. 1 plots the errors of the difference solutions at $t = 0.7$ with different $M \times K$. Fig. 2 plots the curves $-\log \|U - u\| \sim -\log h$ and $-\log \|V - v\| \sim -\log h$. Their slopes are all about 2.

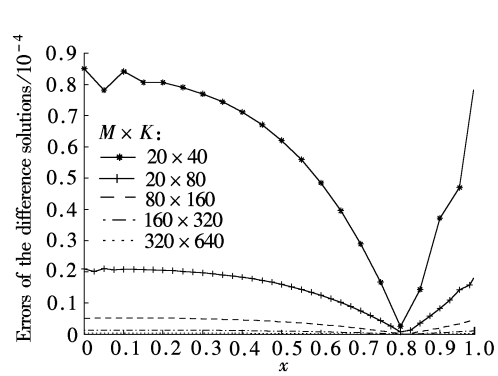


Fig. 1 Difference of analytic and numerical solutions with different grid divisions at $t = 0.7$

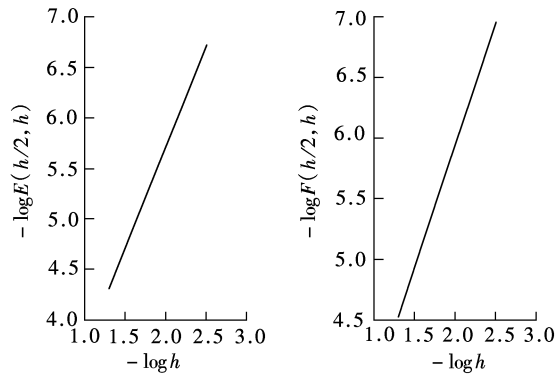


Fig. 2 Curves of the convergence order of u and v

4 Conclusion

In this paper, we study the numerical solution to a model describing process of moisture motion in a wood, which is a system of two nonlinear differential equations. By the method of reduction of order, we present a second-order difference scheme and prove the unique solvability and convergence in L_2 norm. We also present a prior error estimate for the difference scheme. Finally, we give a numerical example and the numerical results are accordant with the theoretical results.

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木材干燥过程中一个非线性模型的二阶收敛差分格式

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摘要: 针对描述木材干燥过程中的一个非线性微分方程模型, 用降阶法对其建立了一个差分格式. 此模型是由一个非线性常微分方程和一个非线性抛物方程组成的耦合微分方程组. 首先引进一个新变量把原问题转化为一阶微分方程组问题, 然后对此一阶微分方程组建立了一个线性化差分格式, 应用能量方法证明了差分格式的可解性、稳定性和收敛性, 并给出了误差估计式. 差分格式关于时间步长和空间步长均为二阶. 在实际计算时, 将引入的新变量分离开, 得到仅含原变量的差分格式, 降低了计算量. 数值计算结果验证了理论结果的可靠性.

关键词: 木材干燥过程; 模型; 非线性微分方程; 差分格式; 降阶法; 稳定性; 收敛性

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