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# Exchange general rings

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**Abstract:** The exchange rings without unity, first introduced by Ara, are further investigated. Some new characterizations and properties of exchange general rings are given. For example, a general ring $I$ is exchange if and only if for any left ideal $L$ of $I$ and $\bar{a} = \bar{a}^2 \in I/L$, there exists $w \in$ r. ureg$(I)$ such that $\bar{w} = \bar{a}$; $E(R, I)$ (the ideal extension of a ring $R$ by its ideal $I$) is an exchange ring if and only if $R$ and $I$ are both exchange. Furthermore, it is presented that if $I$ is a two-sided ideal of a unital ring $R$ and $I$ is an exchange general ring, then every central element of $I$ is a clean element in $I$.

**Key words:** exchange ring; clean element; stable range one; ideal extension

Throughout this paper, let $R$ be an associative ring with identity. In this case, $R$ is also called a unital ring. All the modules are unitary and all the ideals are always two-sided. $U(R)$, r. $U(R)$ and l. $U(R)$ denote the set of units, right units and left units of $R$, respectively. An element $a$ in $R$ is said to be regular, unit-regular or right unit-regular, if $a = axa$ for some $x \in R$, $x \in U(R)$ or $x \in$ r. $U(R)$, respectively. Write reg$(A)$, ureg$(A)$ or r. ureg$(A)$ for the set of regular, unit-regular or right unit regular elements of subset $A$ in $R$, respectively. $l_R(\cdot)$ and $r_R(\cdot)$ will be the left and right annihilators in $R$.

Following Crawley and Jónsson[1], a right $R$-module $M_R$ is said to have the exchange property if for every module $A_R$ and any two decompositions $A_R = M'_R \oplus N_R = \oplus_{i \in I} A_i$ with $M_R \cong M_R$, there exist submodules $A_i' \subseteq A_i$ such that $A_R = M'_R \oplus (\oplus_{i \in I} A'_i)$. $M_R$ is said to have the finite exchange property if the above condition is satisfied whenever the index set $I$ is finite. In Ref. [2], Warfield called a ring $R$ an exchange ring if the right regular module $R_R$ has the finite exchange property and proved that the definition is left-right symmetric. The structure of exchange rings has been investigated by many authors recently. It is well known that regular rings, $\pi$-regular rings, strongly $\pi$-regular rings, semiperfect rings, left or right continuous rings, clean rings and unit $C^*$-algebras of real rank zero are all exchange rings. In Ref. [3], Ara defined and explored the notion of an exchange ring without unity which is called exchange general ring throughout this paper. Ara showed that $I$ being an exchange general ring does not depend on the particular unital ring in which $I$ is embedded as an ideal. Some new characterizations and properties of this kind of rings are given.

If $S$ is a unital ring and $I$ is a general ring such that $I = {}_SI_S$ is a bimodule, the ideal extension of $S$ by $I$ is defined to be the additive group $E(S; I) = S \oplus I$ with multiplication $(r, a)(s, b) = (rs, rb + as + ab)$. This is an associative ring if and only if the condition $s(ab) = (sa)b$, $a(sb) = (as)b$ and $(ab)s = a(bs)$ are satisfied for all $s \in S$ and $a, b \in I$. In this paper, all the ideal extensions will always be associative rings. It is shown that if $I$ is a two-sided ideal of ring $R$ and $I$ is an exchange general ring then every central element of $I$ is a clean element in $I$.

**Lemma 1**[3] Let $I$ be a ring without unit and let $R$ be a ring containing $I$ as an ideal. Then the following conditions are equivalent for an element $a \in I$:

① There exists $e^2 = e \in I$ such that $e - a \in R(a - a^2)$;

② There exist $e^2 = e \in Ia$ and $c \in R$ such that $(1 - e) - c(1 - a) \in J(R)$;

③ There exists $e^2 = e \in Ia$ such that $R = Ie + R(1 - a)$;

④ There exists $e^2 = e \in Ia$ such that $1 - e \in R(1 - a)$;

⑤ There exist $r, s \in I$ and $e^2 = e \in I$ such that $e = ra = s + a - sa$.

A ring without unity is called a left exchange general ring if every element satisfies any of the conditions in lemma 1. The right exchange general ring is defined analogously.

**Remark** Ara[3] showed that a general ring $I$ is

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left exchange if and only if it is right exchange. It is called an exchange general ring when it is either left or right exchange.

**Lemma 2** Let $I$ be an exchange ideal of a ring $R$. Then for any element $a \in I$, there exists an idempotent $e \in I$ such that $l_R(a) \subseteq R(1-e)$ and $l_R(1-a) \subseteq Re$.

**Proof** Since $I$ is an exchange ideal, for any $a \in I$, there exists an idempotent $e \in aI$ and $1-e \in (1-a)R$. So we have $l_R(a) \subseteq l_R(e) = R(1-e)$ and $l_R(1-a) \subseteq l_R(1-e) = Re$.

**Proposition 1** Let $I$ be an ideal of a ring $R$. Suppose that for any $a \in I$, $aR$ is a right annihilator in $R$. Then $I$ is an exchange ideal if and only if for any $a \in I$, there exists an idempotent $e \in I$ such that $l_R(a) \subseteq R(1-e)$ and $l_R(1-a) \subseteq Re$.

**Proof** By lemma 2, it suffices to show the sufficiency. Let $a \in I$ and there exists an idempotent $e \in I$ such that $l_R(a) \subseteq R(1-e)$ and $l_R(1-a) \subseteq Re$. Then $aR = r_R(l_R(a)) \supseteq r_R(1-e) = eR$ and $(1-a)R = r_R(l_R(1-a)) \supseteq r_R(e) = (1-e)R$. Hence $e = at$, and so $e = e^2 = ate \in aI$ and $(1-e) \in (1-a)R$. By lemma 1, the proof is complete.

**Corollary 1** Let $R$ be a left $P$-injective ring. Then $R$ is exchange if and only if for any $a \in R$, there exists an idempotent $e \in R$ such that $l_R(a) \subseteq Re$ and $l_R(1-a) \subseteq R(1-e)$.

**Proof** Since $R$ is a left $P$-injective ring i. e., $aR = r_R(l_R(a))$ for each $a \in R$. Thus the result follows.

In Ref. [4], Nicholson and Zhou showed that a general ring $I$ is exchange if and only if idempotents lift modulo every one sided ideal of $I$. Here, we give another characterization of it. The argument for the following results of this section is similar to that used in Ref. [5].

**Lemma 3** Let $I$ be an ideal of a ring $R$. Then for any element $a \in I$, the following conditions are equivalent to those in lemma 1.

① There exists $w \in$ r. ureg$(I)$ such that $w - a \in R(a-a^2)$;

② There exist $w \in$ r. ureg$(Ia)$ and $c \in R$ such that $(1-w) - c(1-a) \in J(R)$;

③ There exists $w \in$ r. ureg$(Ia)$ such that $R = Iw + R(1-a)$;

④ There exists $w \in$ r. ureg$(Ia)$ such that $1 - w \in R(1-a)$.

**Proof** A quick check shows that the conditions in lemma 1 imply ① and ④ here. We now show that the conditions in this lemma are equivalent and they can deduce one condition in lemma 1.

Given $w \in$ r. ureg$(I)$, then following from theorem 1.2.3 of Ref. [5], there exist $u \in$ l. $U(R)$ and $e = e^2 \in R$ such that $w = ue$. Given $vu = 1$, $v \in R$, then $e = vw$.

①⇒② Set $w - a = r(a-a^2)$ for some $r \in R$, hence $w = a + r(a-a^2) \in Ra$ and so $1 - w = (1-ra)(1-a)$. Thus $e = vw \in Ra \subseteq I$ and $e = e^2 \in Ia$, therefore, $w \in Ia$, as asserted.

②⇒③ By hypothesis, it is not hard to check that $R = Rw + R(1-a)$. And $Rw = Rue = Re = (Re)e = (Rvw)(vw) \subseteq Iw$. Thus, $R = Iw + R(1-a)$.

③⇒④ Suppose that $R = Iw + R(1-a)$ for some $w \in$ r. ureg$(Ia)$. Then $R = Iue + R(1-a) = Ie + R(1-a)$, $e = vw \in vIa \subseteq Ia$, this guarantees that ③ of lemma 1. Therefore, there exists some idempotent $e \in Ia$ such that $1 - e \in R(1-a)$.

④⇒① Assume that $w \in$ r. ureg$(Ia)$ with $1 - w \in R(1-a)$, thus $w - a = w(1-a) - (1-w)a \in R(a-a^2)$.

From lemma 3 in this paper and theorem 3 in Ref. [4], we have the following theorem.

**Theorem 1** A general ring $I$ is exchange if and only if for any left ideal $L$ of $I$ and $\bar{a} = \bar{a}^2 \in I/L$, there exists $w \in$ r. ureg$(I)$ such that $\bar{w} = \bar{a}$.

Recall that, in Ref. [6], an element $a$ in a ring $R$ is said to have stable range one (written $sr(a) = 1$) if, for any $b \in R$, $Ra + Rb = R$ implies that $a + xb \in U(R)$ for some $x \in R$.

**Proposition 2** Let $I$ be an ideal of a ring $R$. Then the following conditions are equivalent:

① $I$ is an exchange general ring and $sr(a) = 1$ for any $a \in I$;

② For any $a \in I$ and $x \in R$, there exists $w \in$ ureg$(I)$ such that $w - a \in R(a - axa)$;

③ For any $a \in I$ and $x \in R$, there exists $w \in$ ureg$(Ia)$ such that $1 - wx \in R(1-ax)$;

④ For any $a \in I$ and $x \in R$, there exists $b \in R$ such that $abxa \in$ ureg$(Ia)$ with $1 - bxa \in R(1-xa)$;

⑤ For any $a \in I$ and $x \in R$, there exists $b \in R$ such that $abxa \in$ ureg$(Ia)$ with $abxa - a \in R(a - axa)$.

**Proof** ①⇒② Following from lemma 1, for any $a \in I$ and $x \in R$, there exists some idempotent $e \in I$ such that $e - xa \in Rxa(1-xa)$. So we can set $e - xa = rxa(1-xa)$ for some $r \in R$, and then $1 = (1-rxa)(1-xa) + e$ with $e \in Ra$. Hence, $R = Ra + R(1-xa)$, $a + z(1-xa) = u \in U(R)$ for some $z \in R$. Therefore, $ue = [a + z(1-xa)][xa + rxa(1-xa)] = axa + [z + zr + (1-zx)ar]xa \cdot (1-xa)$. Set $c = z + zr + (1-zx)ar$, $w = ue \in$ ureg$(I)$, hence $w - a = axa - a + cxa(1-xa)$

$=(cx-1)(a-axa)$, and so $w-a\in R(a-axa)$.

②⇒① By supposition, for any $a\in I$, there exists $u\in \mathrm{ureg}(I)\subseteq \mathrm{r.\,ureg}(I)$ such that $w-a\in R(a-a^2)$. Then following from lemma 3①, it is easy to check that $I$ is an exchange general ring. By lemma 1 ④, there exists $e=e^2\in Ixa$ such that $1-e\in R(1-xa)$. Then $ae=(ae)e^2=(ae)bx(ae)$ for some $b\in I$, and by ②, there exists $w\in \mathrm{ureg}(I)$ such that $w-ae\in R(ae-(ae)bx(ae))=0$. Hence, $ae=w\in \mathrm{ureg}(I)$. Write $ae=aeuae$, $u\in U(R)$, then we can easily show that $e(uae)=e(euae)=bxa(euae)=bx(aeuae)=ae=(bxa)e=e^2=e$, and so by lemma 1.2.2 of Ref. [5] $(1-e)+uae=v\in U(R)$, i. e., $ua+(1-ua)(1-e)=v\in U(R)$. Hence, $a+[u^{-1}(1-ua)c](1-xa)=u^{-1}v\in U(R)$ for some $c\in R$, and so $sr(a)=1$.

②⇒③ Given $w=ue\in \mathrm{ureg}(I)$ for some $u\in U(R)$, $e=e^2\in R$, with $w-a=r(a-axa)$ for some $r\in R$. Thus $w=a+r(a-axa)\in Ra$, and so $e=e^2=(u^{-1}wu^{-1})w\in Ia$. Hence, $w=ue\in Ia$. Moreover, $1-wx=1-ax-r(a-axa)x=(1-rax)(1-ax)\in R(1-ax)$.

③⇒② By hypothesis, there exists $w\in \mathrm{ureg}(Ia)\subseteq \mathrm{ureg}(I)$ such that $w-a=w(1-xa)-(1-wx)a\in R(a-axa)$.

①, ②⇒④, ⑤ For any $a\in I$ and $x\in R$, there exists $e=e^2\in I$ such that $e-xa\in Rxa(1-xa)$. Write $e=xa+rxa(1-xa)$ and $b=1+r(1-xa)$ for some $r\in R$, $b\in R$, then $e=bxa$ and $1-e\in R(1-xa)$. In a similar manner to that proof of ②⇒①, $abxa=ae\in \mathrm{ureg}(I)$. Hence $ae\in \mathrm{ureg}(Ia)$ and $1-e\in R(1-xa)$. Therefore, ④ is proved. And because $abxa-a=a[1+r(1-xa)]xa-a=(arx-1)(a-axa)\in R(a-axa)$, ⑤ is followed.

④⇒③ Set $w=abxa\in \mathrm{ureg}(Ia)$ and $1-bxa=r(1-xa)$ for some $b, r\in R$, then $1-wx=(1-ax)+a(1-bxa)x=(1+arx)(1-ax)\in R(1-ax)$.

⑤⇒② is clear.

**Corollary 2** Let exchange general ring $I$ be an ideal of ring $R$. Then the following conditions are equivalent:

① For any $a\in I$, $sr(a)=1$;

② $\mathrm{reg}(I)=\mathrm{ureg}(I)$;

③ For any $a\in I$, $x\in R$ and $e=e^2\in Ixa$, $ae\in \mathrm{ureg}(I)$;

④ For any $a\in I$, $x\in R$, there exists $e=e^2\in Ixa$ with $1-e\in R(1-xa)$, such that $ae\in \mathrm{ureg}(I)$;

⑤ For any $a\in I$ and $x\in R$, there exists $b\in Ixa$ with $1-b\in R(1-xa)$ such that $ab\in \mathrm{ureg}(I)$.

**Proof** ②⇒③ For any $e=e^2\in Ixa$, set $e=txa$ for some $t\in I$. Thus $ae=aetxae$, so $ae\in \mathrm{ureg}(I)$ by ②.

③⇒④ Since $I$ is an exchange general ring, there exists $e=e^2\in Ixa$ such that $1-e\in R(1-xa)$. From ③, $ae\in \mathrm{ureg}(I)$.

④⇒⑤ is obvious.

⑤⇒① ⑤ implies proposition 2⑤ and so it implies ①.

**Proposition 3** Let $E(R, I)$ be the ideal extension of $R$ by its ideal $I$. $E(R, I)$ is an exchange ring if and only if $R$ and $I$ are all exchange.

**Proof** Let $K=\{(0,a)\mid a\in I\}$, then $K$ is a two-sided ideal of $E(R, I)$, and $E(R, I)/K\cong R$. Assume that $E(R, I)$ is an exchange ring, then $I\cong K$ and $R\cong E(R, I)/K$ are both exchange rings[3].

Conversely, suppose that $R$ and $I$ are both exchange rings, then $K\cong I$ and $E(R, I)/K\cong R$ are both exchange. And for any $a^2=a\in R$, $(a,0)^2=(a,0)\in E(R, I)$. For any $(x,a)^2-(x,a)\in K$, there exists $(x,0)^2=(x,0)\in E(R, I)$ such that $(x,0)-(x,a)=(0,a)\in K$ i. e., in $E(R, I)$, idempotents can be lifted modulo $K$. According to Ref. [3], $E(R, I)$ is an exchange ring.

In Ref. [4], Nicholson and Zhou introduced the notion of clean element in a general ring $I$. Now, we give a result which generalizes that in Ref. [7]. If $I$ is a general ring and $p, q\in I$, we write $p*q=p+q+pq$. Then $(I, *)$ is a monoid with unity 0, and we denote the group of units of $(I, *)$ by $Q=Q(I)=\{q\in I\mid \exists p\in I, p*q=0=q*p\}$. With this in mind, an element $a$ in a general ring $I$ is called a clean element if $a=e+q$ where $e^2=e$ and $q\in Q=Q(I)$; and $I$ is called a clean general ring if every element is clean.

**Theorem 2** Let $I$ be an exchange general ring. Then every central element of $I$ is a clean element in $I$.

**Proof** Let $S=E(Z, I)$, which is the ideal extension of $Z$ by $I$. Since $I$ is exchange, then $(0,x)$ is an exchange element in $S$ for any $x\in I$. And it is clear that $(0,x)$ is in the center of $S$ if $x$ is in the center of $I$, hence $(1,0)-(0,x)=(1,-x)$ is in the center of $S$. Set $K=\{(0,x)\mid x\in I\}$, then $K$ is a two-sided ideal of $S$ and $K\cong I$ is an exchange general ring. For any $(0,x)\in S$, there exists $e^2=e\in K(0,x)$, hence $e=(0,r)(0,x)=(0,x)(0,r)$, and so $1-e=(1,-rx)=(1,-xr)=b(1,-x)=(1,-x)b$ for some $b\in S$ by lemma 1④. Set $a=(0,r)$, we can assume that $a=ea=ae$, $b=(1-e)b=b(1-e)$. So $ab=ba=0$ and hence $-a+ea-eb=-a(1-e)-eb=0$. Therefore, $[(0,x)-(1-e)](a-b)=(0,x)a+((1,-x)b-b+(-a+ea-eb)+b=e+(1-e)-b+b=1$. Similarly, it can be shown that $(a-b)[(0,x)-(1-e)]=1$. Hence $(0,x)-(1-e)$ is an invertible element and $(0,x)=(0,x)-(1-e)+(1-e)$. This implies that $(0,x)$ is clean in $S$. By the proof of ③⇒① of proposition 7 in Ref. [4], $x$ is a clean element in $I$.

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# Exchange 一般环


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**摘要：**进一步研究了由 Ara 首次引入并研究的没有单位元的 exchange 环. 给出了它的一些新的等价刻画和性质. 例如：一个一般环 $I$ 是 exchange 的当且仅当对它的任意理想 $L$ 以及 $\bar{a}=\bar{a}^2\in I/L$，存在 $w\in$ r. ureg($I$) 使得 $\bar{w}=\bar{a}$; $E(R,I)$（环 $R$ 通过它的理想 $I$ 生成的理想扩张）是一个 exchange 环当且仅当 $R$ 和 $I$ 都是 exchange 环. 还证明了如果环 $R$ 的双边理想 $I$ 是一个 exchange 一般环，则 $I$ 的每一个中心元素都是 $I$ 中一个 clean 元素.



**关键词：**exchange 环；clean 元素；稳定度 1；理想扩张

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