

On S -acts containing no maximal S -subacts

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Abstract: Let S be a semigroup and let A be an S -act. Some necessary and sufficient conditions that S -subacts of A are maximal S -subacts are given. A relation \mathcal{L} which is similar to the Green relation in semigroups is defined. By the relation \mathcal{L} , it is proved that a non-empty set L of A is a maximal S -subact if and only if $A \setminus L$ is a (maximal) \mathcal{L} -class. Finally, the concept of a C -subact is defined, some properties of C -subacts are discussed, and it is proved that A contains no maximal S -subacts if and only if every cyclic S -subact of A is a C -subact. Consequently, the results obtained by Imrich Fabrici that semigroups contain no maximal (left) ideals are the corollary of this paper.

Key words: S -act; maximal subact; C -subact

1 Preliminaries

A representation of a semigroup S by transformation of a set defines an S -act just as a representation of a ring R by endomorphisms of an abelian group defines a R -module. It is well-known that S -acts (also called S -sets, S -systems, S -automata, etc.) play an important role not only in studying properties of monoids, but also in other mathematical areas, such as graph theory and algebraic automata theory^[1-2]. More precisely, a left S -act is a set A equipped with a map $S \times A \rightarrow A$, $(s, a) \rightarrow sa$, such that $(st)a = s(ta)$ for all $a \in A$ and $s, t \in S$, denoted by ${}_S A$. Right S -acts are defined analogously, and in this paper we will often use the term S -act to mean left S -act. An S -subact of a left S -act A is a subset of A that is closed under the S -action. Note that union and intersection of S -subacts are S -subacts. The intersection of all S -subacts of S -act A containing a nonempty subset B of A is the S -subact generated by B , denoted as $L(B)$. The S -subact of S -act A generated by $\{a\}$ ($a \in A$) is denoted as $L(a)$ instead of $L(\{a\})$. An S -act A is called cyclic if $A = L(a)$ for some $a \in A$. An S -subact B of S -act A is called proper if $B \neq A$. A proper S -subact L of A is said to be maximal if there is no proper S -subact B of A such that $L \subset B$. Equivalently, if for any S -subact B of A such that $L \subseteq B$, then $B = A$.

Satyanarayana presented the following problem: Describe semigroups containing no maximal ideals. The

problem was answered by Fabrici^[3]. Let S be a semigroup. It is obvious that a subset I of S is an S -subact of S -act ${}_S S$ if and only if I is a left ideal of S . So it is interesting to know what occurs for S -Act; that is, how to describe S -acts containing no maximal S -subacts. In the paper we will discuss this problem. As an application, some results of Refs. [3 – 4] can be obtained as corollaries of theorem 5 below. Some properties of maximal S -subacts are also given in this paper; other terms are referred to in Ref. [5].

2 Maximal S -Subacts

First we discuss properties of maximal S -subacts.

Lemma 1 Let A be an S -act and B be a non-empty subset of A , then $L(B) = (B \cup SB)$. In particular, for any $a \in A$, $L(a) = (a \cup Sa)$.

Proof Let $D = (B \cup SB)$, then $SD \subseteq D$, so D is an S -subact of A with $B \subseteq D$. If C is an S -subact of S -act A and $B \subseteq C$, then $sb \in C$ for any $b \in B$, $s \in S$, therefore $(B \cup SB) \subseteq C$ and consequently $D \subseteq C$. Hence $L(B) = (B \cup SB)$.

Corollary 1 Let A be an S -act, B and C non-empty subsets of A , then $L(B \cup C) = L(B) \cup L(C)$.

Theorem 1 Let A be an S -act and let B be an S -subact of A . Then the following statements are equivalent:

- ① B is a maximal S -subact of A ;
- ② $L(a) \cup B = A$ for any $a \in A \setminus B$;
- ③ $L(a \cup B) = A$ for any $a \in A \setminus B$.

Proof ① \Rightarrow ② Let B be a maximal S -subact of A and $a \in A \setminus B$, then $L(a) \cup B$ is an S -subact of A and $B \subsetneq L(a) \cup B$. Therefore, $L(a) \cup B = A$ since B is a maximal S -subact of A .

② \Rightarrow ① Suppose that B and C are S -subacts of A

Received 2006-03-16.

Foundation items: The National Natural Science Foundation of China (No. 10626012), Jiangsu Planned Projects for Postdoctoral Research Fund (No. 0502022B).

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such that $B \not\subseteq C$, then there exists $a \in C \setminus B \subseteq A \setminus B$ and so $sa \in C$ for all $s \in S$; therefore, $a \cup Sa \subseteq C$ and consequently $L(a) = (a \cup Sa) \subseteq C$. By hypothesis, we have $A = B \cup L(a) \subseteq C$; that is, $C = A$. Thus B is a maximal S -subact of A .

② \Leftrightarrow ③ From corollary 1 we can obtain the conclusion.

Corollary 2 Let S be a semigroup and let A be an S -act. Then a proper S -subact L of A is maximal if and only if for any $a \in A \setminus L$ we have $A \setminus L = \{a\}$ or $A \setminus L \subseteq Sa$.

Proof \Rightarrow . Suppose that L is a maximal S -subact of A . Then $A = L \cup L(a) = L \cup (a \cup Sa)$ for all $a \in A \setminus L$.

① If there exists $a \in A \setminus L$ such that $Sa \subseteq L$, then $A \setminus L \subseteq \{a\}$.

② If $Sa \not\subseteq L$ for all $a \in A \setminus L$, then $L \subseteq (L \cup Sa)$. Since $L \cup Sa$ is an S -subact of A and L is maximal; therefore, $L \cup Sa = A$ and, consequently, $A \setminus L \subseteq Sa$.

\Leftarrow . Let L be a proper S -subact of A .

① If $A \setminus L = \{a\}$, then $A = L \cup \{a\} \subseteq L \cup (\{a\} \cup Sa) = L \cup L(a) \subseteq A$, by theorem 1, L is maximal.

② If $A \setminus L \subseteq Sa$ for all $a \in A \setminus L$, then $A = L \cup (A \setminus L) \subseteq L \cup Sa \subseteq A$, thus L is maximal by theorem 1.

Let S be a semigroup and let A be an S -act, similarly to the Green's relation \mathcal{L} of a semigroup^[5], we define an equivalence relation \mathcal{L} in S -act A as follows:

$$(a, b) \in \mathcal{L} \Leftrightarrow L(a) = L(b)$$

From lemma 1, we have the following lemma.

Lemma 2 Let S be a semigroup and let A be an S -act. For $a, b \in A$, the following statements are equivalent:

- ① $(a, b) \in \mathcal{L}$;
- ② $S^1 a = S^1 b$;
- ③ There exist $x, y \in S^1$ such that $a = xb, b = ya$.

Denote the \mathcal{L} -class containing a by L^a and assign a partial order relation \leq on the \mathcal{L} -classes as follows:

$$L^a \leq L^b \Leftrightarrow L(a) \subseteq L(b)$$

Definition 1 Let S be a semigroup and let A be an S -act. An S -subact L of A is called a -maximal if L is a maximal S -subact of A with respect to containing none of the element a ($a \in A$).

Remark 1 The a -maximal S -subact of A , if it exists, is the unique a -maximal S -subact of A .

Let L be an S -subact of S -act A and $a \in A$, we denote the intersection of all S -subacts of A containing L and a by L^* , then L^* is an S -subact of A and we have the following lemma.

Lemma 3 Let S be a semigroup and let A be an S -act. An S -subact L is a -maximal if and only if L^* is

a cover of L .

Proof \Rightarrow . Suppose that L is a -maximal and J is an S -subact of A such that $L \subset J \subseteq L^*$, then $a \in J$, therefore $\{a\} \cup L \subseteq J$, and so $L^* \subseteq J$. Consequently $J = L^*$.

\Leftarrow . Let L^* be a cover of L . If K is an S -subact of A which does not contain a such that $L \subset K$, then $L^* \subseteq K$, therefore $a \in L^* \subseteq K$. Impossible. Hence L is a -maximal.

Lemma 4 If L is a -maximal S -subact of A , then the following statements hold:

$$\textcircled{1} A \setminus L = \bigcup_{x \in A \setminus L} L^x;$$

② If $C = \{L^x \mid x \in A \setminus L\}$, then $L^* \setminus L$ is the least of C with respect to the ordering \leq on \mathcal{L} -classes, and any L^y ($y \in L$) is not greater than any element L^x in C .

Proof ① Let $B = \bigcup_{x \in A \setminus L} L^x$. If $b \in A \setminus L$, then $b \in L^b \subseteq B$. That is, $A \setminus L \subseteq B = \bigcup_{x \in A \setminus L} L^x$.

Let $y \notin A \setminus L$, that is, $y \in L$, then $y \notin B$. In fact, if $y \in B$, then $y \in L^x$ for some $x \in A \setminus L$, therefore $L^y \subseteq L^x \subseteq B$, consequently, $y \in A \setminus L$. Impossible. Thus $A \setminus L \supseteq \bigcup_{x \in A \setminus L} L^x$.

② We first show that $L^* \setminus L$ is an \mathcal{L} -class. It is clear that $a \in L^*$. If $x \in L^* \setminus L$, then L is also x -maximal by lemma 3. If $x \notin L(a)$, then $x \notin L(a) \cup L$. Since L is x -maximal and $L(a) \cup L$ is an S -subact of A , we have $L = L(a) \cup L$, and $a \in L$. Impossible. That is $x \in L(a)$ and so $L(x) \subseteq L(a)$. Similarly, we can show that $L(a) \subseteq L(x)$. Hence $L(a) = L(x)$. Moreover, if $y \in A \setminus L^*$, then $L(y) \neq L(a)$. Indeed, if $L(y) = L(a)$, since L^* is an S -subact of A and $L(a) \subseteq L^*$, we have $y \in L(y) = L(a) \subseteq L^*$. A contradiction. Consequently, $L^* \setminus L$ is an \mathcal{L} -class and $L^* \setminus L = L^a$.

Now, consider $y \in A \setminus L^*$. Clearly, $L(a) \subseteq L(y)$, otherwise, we have $a \notin L(y)$, and so $a \notin (L(y) \cup L)$. Since $L(y) \cup L$ is an S -subact of A and L is a -maximal, we would have $L(y) \cup L = L$; therefore, $L(y) \subseteq L$ and consequently $y \in L(y) \subseteq L$. Impossible. Hence $L^* \setminus L$ is the least element in C .

Suppose that $L^a \leq L^y$ for some $y \in L$. Then, we have $a \in L(a) \subseteq L(y) \subseteq L$. But this is clearly impossible.

Theorem 2 Let A be an S -act and let L be its S -subact. Then there exists $a \in A$ such that L is a -maximal S -subact of A if and only if $A \setminus L$ contains the least \mathcal{L} -class among all the \mathcal{L} -classes contained in $A \setminus L$.

Proof \Rightarrow . From lemma 4 we can get the conclusion.

\Leftarrow . Let L^a be the least \mathcal{L} -class in $C = \{L^x \mid x \in A \setminus L\}$, then $a \notin L$. If L is not a -maximal S -subact of A , then, by Zorn's lemma, there exists an a -maximal S -subact K such that $L \subset K$. Let $b \in K \setminus L$. By hypothesis, L

$(a) \subset L(b)$. Thus $a \in L(a) \subseteq L(b) \subseteq K$ which contradicts that K is a -maximal. Therefore L is a -maximal S -subact of A .

Theorem 3 Let A be an S -act. The following statements are equivalent:

- ① L is a maximal S -subact of A ;
- ② $A \setminus L$ is an \mathcal{L} -class of A ;
- ③ $A \setminus L$ is a maximal \mathcal{L} -class of A .

Proof ① \Rightarrow ② Suppose that L is a maximal S -subact of A . Let $a \in A \setminus L$, then L is a -maximal. By lemma 3, L^* is a cover of L . On the other hand, L is a maximal S -subact of A , thus $L^* = A$. By lemma 4, $A \setminus L$ is an \mathcal{L} -class of A .

② \Rightarrow ③ Let $A \setminus L$ be an \mathcal{L} -class of A , then there exists $a \in A \setminus L$ such that $L(x) = L(a)$ for all $x \in A \setminus L$. Suppose that $L^a \leq L^y$ for some $y \in L$. Then, we have $a \in L(a) \subseteq L(y) \subseteq L$. But this is clearly impossible. Hence $A \setminus L$ is a maximal \mathcal{L} -class of A .

③ \Rightarrow ① Let $A \setminus L$ be a maximal \mathcal{L} -class of A . Then L is an S -subact of A . We only prove that $a \in L$ and $s \in S$ imply $sa \in L$. Since $a \in L(a)$ and $L(a)$ is an S -subact of A , we have $sa \in L(a)$ and so $L(sa) \subseteq L(a)$. If $sa \in A \setminus L$, then by hypothesis, $L(sa) = L(a)$. This implies that $a \in A \setminus L$, which is a contradiction. Hence L is an S -subact of A .

Suppose that there exists a proper S -subact K such that $L \subset K$, then one can choose $c \in K \setminus L$, then $c \in A \setminus L$. Since $A \setminus L$ is an S -subact, we have $L(c) \subseteq A \setminus L$. By hypothesis, $L(c) = A \setminus L$. On the other hand, $c \in K$ implies $L(c) \subseteq K$. Thus we have

$$A = (A \setminus L) \cup L = L(c) \cup L \subseteq K$$

Impossible. This means that L is indeed a maximal S -subact of A .

3 On S -Acts Containing No Maximal S -Subacts

Proposition 1 Let A be an S -act. If A contains no maximal S -subacts, then A is a unitary S -act, that is, $A = SA$.

Proof If $A \neq SA$, then $SA \subset A$. Take $a \in A \setminus SA$, we can prove that $L = A \setminus \{a\}$ is a maximal S -subact. Clearly, $A \setminus L = \{a\}$, from corollary 2. We need only show that L is an S -subact. For any $b \in L$, $s \in S$ implies $sb \in L$; otherwise, $a = sb \in SA$. It is contradictory to $a \in A \setminus SA$.

Theorem 4 Let A be an S -act. If $A \neq SA$, then the a -maximal S -subact of A is of the form $A \setminus \{a\}$ for all $a \in A \setminus SA$.

Proof For $a \in A \setminus SA$, we first show that $A \setminus \{a\}$ is an S -subact of A . If $b \in A \setminus \{a\}$ and $s \in S$, then $sb \in A \setminus$

$\{a\}$. In fact, if $a = sb$, then $a = sb \in SA$, this is impossible. Therefore, $A \setminus \{a\}$ is an S -subact of A with $a \notin A \setminus \{a\}$. Moreover, if there exist an S -subact K with $a \notin K$ such that $A \setminus \{a\} \subset K$, then there exist $b \in K$, $b \notin A \setminus \{a\}$. This implies that $a = b$ and so $a \in K$. Impossible. Consequently, $A \setminus \{a\}$ is a a -maximal S -subact of A . According to remark 1, the a -maximal S -subact of A is of the form $A \setminus \{a\}$.

Corollary 3 Let A be an S -act. Then $a \in L(c)$ if and only if $a = c$ for any $a \in A \setminus SA$, $c \in A$. Therefore, $L^a = \{a\}$ for any $a \in A \setminus SA$.

Proof We only show the necessity as the sufficient is obvious. By theorem 4, $A \setminus \{a\}$ is a -maximal S -subact of A . If $a \in L(c)$ then $c \notin SA$. Since if $c \in SA$, then $a \in L(c) \subseteq SA$. If $a \neq c$, then $a \in Sc \subseteq SA$, which is contradictory to $a \in A \setminus SA$.

Definition 2 An S -subact M is called a C -subact of S -act A if $M \subseteq S^1(A \setminus M)$.

For C -subacts of S -act A , we have following properties.

Proposition 2 ① The set of C -subacts of S -act A is a sublattice of the lattice of all S -subacts of A ;

② If S -act A contains maximal S -subacts, then any maximal S -subact of A is not a C -subact.

Proof ① Suppose that M and N are C -subacts of S -act A , then $(M \cup N)$ and $M \cap N$ are S -subacts of A , and $M \subseteq S^1(A \setminus M)$, $N \subseteq S^1(A \setminus N)$. Thus

$$(M \cap N) \subseteq (S^1(A \setminus M)) \cap (S^1(A \setminus N)) \subseteq S^1(A \setminus (M \cap N))$$

$$(M \cup N) \subseteq (S^1(A \setminus M)) \cup (S^1(A \setminus N)) \subseteq S^1(A \setminus (M \cup N))$$

② If M is a maximal S -subact, then $A \setminus M$ is an \mathcal{L} -class of A by theorem 3, thus $A \setminus M = L(A \setminus M) = S^1(A \setminus M)$, therefore M is not a C -subact.

Theorem 5 Let A be an S -act and A contain some proper S -subacts. Then A contains no maximal S -subacts if and only if every cyclic S -subact of A is a C -subact.

Proof \Rightarrow . We first show that $L(a) \subset A$ for all $a \in A$. Suppose that there exists $a \in A$ such that $L(a) = A$.

① If $L^a = A$, then $L(x) = L(a)$ for all $x \in A$. Since A contains some proper S -subacts, say B . For $b \in B$, we have $L(b) \subseteq B$, thus $A = L(b) \subseteq B$. Impossible.

② If $L^a \subset A$, it is clear that L^a is an S -subact of A . If there exists an S -subact K such that $L^a \subseteq K$, then $a \in K$, hence $A = L(a) \subseteq K$ and consequently L^a is a maximal cyclic S -subact of A . However, this contradicts our hypothesis.

From the above two cases, we can obtain $L(a) \subset A$ for all $a \in A$. Thus $A \setminus L^a \neq \emptyset$. For if $A \setminus L^a = \emptyset$, then

$L^a = A$. So we have $x \in L(x) = L(a)$ for $x \in A$, that is, $A \subseteq L(a)$. Impossible. By hypothesis, A contains no maximal S -subacts of A , thus $A \setminus L^a$ is not a maximal S -subact of A . By theorem 3, there exists an \mathcal{L} -class L^b such that $L^a < L^b$, that is, $L(a) \subset L(b)$. This shows that $b \notin L(a)$. Thus $b \in A \setminus L(a)$. Therefore

$$L(a) \subset L(b) = (Sb \cup b) \subseteq (S(A \setminus L(a)) \cup (A \setminus L(a)))$$

\Leftarrow . Let L^a be an \mathcal{L} -class of A . Then $L(a) \neq A$ since A is not a C -subact of A . By hypothesis, $L(a) \subseteq S^1(A \setminus L(a))$. Hence, there exist $b \in A \setminus L(a)$ and $s \in S^1$ such that $a = sb$. Thus, $L(a) \subset L(b)$. Similar to $L(a) \neq A$, we have $L(b) \neq A$. Hence $L(a) \subset L(b) \neq A$, therefore, L^a is not a maximal \mathcal{L} -class of A ; that is, A contains no maximal \mathcal{L} -class of A . Hence, A contains no maximal S -subacts of A by theorem 3.

Recall that S -act A is called local cyclic if, for any $a, b \in A$, there exists $c \in A$ such that $\{a, b\} \subseteq L(c)$.

Theorem 6 If A is a local cyclic S -act. Then A contains no maximal S -subacts.

Proof We claim first that $SA = A$. Suppose that $SA \subset A$ and there exist $a, b \in A - SA$. By hypothesis there exists $c \in A$ such that $\{a, b\} \subseteq Sc \subseteq SA$. This contradicts $\{a, b\} \in A \setminus SA$.

① Suppose $A = L(a)$ for some $a \in A$. If L is a proper S -subact of A , then $a \notin L$; otherwise, $A = L(a)$

$\subseteq L \subseteq A$. Thus, $a \in A \setminus L$. If L is a maximal S -subact of A , then $A \setminus L$ is an \mathcal{L} -class of A by theorem 3. Therefore, $A \subseteq L(a) \subseteq A \setminus L$. This is impossible. Hence A contains no maximal S -subacts.

② Suppose $SA \subsetneq A$ for any $a \in A$. Consider any cyclic S -subact $L(a)$ of A . Choose $b \in A \setminus L(a)$. By hypothesis, there exists $c \in A$ such that $\{a, b\} \subseteq L(c)$. Since $b \notin L(a)$, we have $c \notin L(a)$. From $SA = A$, we have $d \in A \setminus L(a)$ such that $c = Sd$. Hence $L(a) \subseteq L(c) \subseteq Sd \subseteq S(A \setminus L(a))$, proving that $L(a)$ is a C -subact of A . By theorem 5, A contains no maximal S -subacts.

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不包含极大子系的 S -系

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摘要: 设 S 是半群, A 是 S -系. 首先给出了 A 的子系是极大子系的充分必要条件; 其次利用所定义的 \mathcal{L} 关系, 证明了 A 的非空子集 L 是 A 的极大子系的充分必要条件是 $A \setminus L$ 是 A 的 (极大) \mathcal{L} -类; 最后, 定义了 C -子系的概念, 讨论了其性质, 并利用 C -子系刻画了一类 S -系. 证明了 S -系不包含极大子系当且仅当每一个循环子系是 C -子系. 从而 Imrich Fabrici 关于半群不包含极大 (左) 理想的主要结论就是本文的推论.

关键词: S -系; 极大子系; C -子系

中图分类号: O152.7