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Frequency domain identification of non-integer order dynamical systems

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Abstract: Two new methods, the generalized Levy method and the weighted iteration method, are presented for identification of non-integer order systems. The first method generalizes the Levy identification method from the integer order systems to the non-integer order systems. Then, the weighted iteration method is presented to overcome the shortcomings of the first method. Results show that the proposed methods have better performance compared with the integer order identification method. For the non-integer order systems, the proposed methods have the better fitting for the system frequency response. For the integer order system, if commensurate order scanning is applied, the proposed methods can also achieve the best integer order model which fits the system frequency response. At the same time, the proposed algorithms are more stable.

Key words: non-integer order dynamical system; non-integer order system identification; generalized Levy method; weighted iteration method

The aim of any system identification is to establish a mathematical model capable of reproducing the system' s physical behavior as faithfully as possible from a series of observations. In the time domain, many mathematical models have been developed such as AR, ARX and ARMAX[1]. In the frequency domain, many methods have been collected in Ref. [2].

Studies on real systems such as thermal[3] and electrochemical[4] systems reveal inherent non-integer (or fractional) differentiation in their behavior. The use of the above integer models is thus inappropriate in identifying these non-integer systems. A new category, called non-integer models, based on the concept of non-integer differentiation, has been developed[5]. Non-integer AR, ARX and ARMAX models in particular can, unlike the integer versions, identify thermal[3] and electrochemical systems[4].

Since the 1970s, many authors have worked in physical modeling of dynamic systems using fractional calculus in order to fit time and/or frequency behaviors[6-8]. Less effort had been made in applying fractional calculus to identification[5,9-11]. In order to identify non-integer order systems, two new methods, the generalized Levy method and the weighted iteration method, are presented. First, the Levy identification method for the integer order systems is extended for the non-integer order systems. Then, the weighted iteration method is presented to overcome the shortcomings of the first method. Results show that our methods obtain better performance compared with the integer order identification method.

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1 Generalized Levy Method

The Levy method is a well-established method for finding the coefficients of an integer transfer function that models a plant with a known frequency behavior[12]. So we extended the Levy method for the case of non-integer orders.

Let us suppose a system G with a known frequency behavior and we want to model it using a commensurate transfer function[11].

$$\tilde{G}(s) = \frac{b_0 + b_1 s^r + b_2 s^{2r} + \ldots + b_m s^{mr}}{1 + a_1 s^r + a_2 s^{2r} + \ldots + a_n s^{nr}} = \frac{\sum_{k=0}^{m} b_k (\mathrm{j}\omega)^{kr}}{1 + \sum_{k=1}^{n} a_k (\mathrm{j}\omega)^{kr}} \tag{1}$$

Here the commensurate order r which can be non-integer is known in advance.

The frequency response of Eq. (1) is given by

$$\tilde{G}(j\omega) = \frac{\sum_{k=0}^{m} b_k (\mathrm{j}\omega)^{kr}}{1 + \sum_{k=1}^{n} a_k (\mathrm{j}\omega)} = \frac{N(\omega)}{D(\omega)} = \frac{\alpha(\omega) + \mathrm{j}\beta(\omega)}{\sigma(\omega) + \mathrm{j}\tau(\omega)} \tag{2}$$

where N and D are complex-valued; α, β, σ and τ are real-valued. The error between the model and the system, for a given frequency ω_i, $i = 1, 2, \ldots, K$, will be

$$\varepsilon(\omega_i) = [R(\omega_i) + \mathrm{j}I(\omega_i)] - \frac{N(\omega_i)}{D(\omega_i)} \tag{3}$$

where $G(j\omega_i) = R(\omega_i) + jI(\omega_i)$.

It might be possible to adjust the parameters of Eq. (1) by minimizing the norm (or the square norm) of the error. But this is a nonlinear optimal problem. In order to simplify this problem, Levy minimized the norm of

$$\sum_{i=1}^{K} \| D(\omega_i)\varepsilon(\omega_i) \|^2 = \sum_{i=1}^{K} \| D(\omega_i)G(j\omega_i) - N(\omega_i) \|^2 \tag{4}$$

In order to alleviate notation, we define

$$J = \sum_{i=1}^{K} \| D(\omega_i)\varepsilon(\omega_i) \|^2 \tag{5}$$

we will have the norm of J

$$J = \sum_{i=1}^{K} \{[R(\omega_i)\sigma(\omega_i) - I(\omega_i)\tau(\omega_i) - \alpha(\omega_i)]^2 + [R(\omega_i)\tau(\omega_i) + I(\omega_i)\sigma(\omega_i) - \beta(\omega_i)]^2\} \tag{6}$$

where

$$\alpha(\omega_i) = \sum_{k=0}^{m} b_k\omega_i^{kr}\cos\left(\frac{kr\pi}{2}\right) \tag{7}$$

$$\beta(\omega_i) = \sum_{k=0}^{m} b_k\omega_i^{kr}\sin\left(\frac{kr\pi}{2}\right) \tag{8}$$

$$\sigma(\omega_i) = 1 + \sum_{k=1}^{n} a_k\omega_i^{kr}\cos\left(\frac{kr\pi}{2}\right) \tag{9}$$

$$\tau(\omega_i) = 1 + \sum_{k=1}^{n} a_k\omega_i^{kr}\sin\left(\frac{kr\pi}{2}\right) \tag{10}$$

Then, we differentiate J with respect to the coefficients a_k and b_k. Then let

$$\frac{\partial J}{\partial b_k} = 0 \tag{11}$$

$$\frac{\partial J}{\partial a_k} = 0 \tag{12}$$

We have

$$\sum_{i=1}^{K} \left\{[R(\omega_i)\sigma(\omega_i) - I(\omega_i)\tau(\omega_i) - \alpha(\omega_i)]\omega_i^{kr}\cos\left(\frac{kr\pi}{2}\right) + [R(\omega_i)\tau(\omega_i) + I(\omega_i)\sigma(\omega_i) - \beta(\omega_i)]\omega_i^{kr}\sin\left(\frac{kr\pi}{2}\right)\right\} = 0 \tag{13}$$

$$\sum_{i=1}^{K} \Big(\sigma(\omega_i)\{[R(\omega_i)]^2 + [I(\omega_i)]^2\}\omega_i^{kr}\cos\left(\frac{kr\pi}{2}\right) + \tau(\omega_i)\{[R(\omega_i)]^2 + [I(\omega_i)]^2\}\omega_i^{kr}\sin\left(\frac{kr\pi}{2}\right) + \alpha(\omega_i)\left\{I(\omega_i)\omega_i^{kr}\sin\left(\frac{kr\pi}{2}\right) - R(\omega_i)\omega_i^{kr}\cos\left(\frac{kr\pi}{2}\right)\right\} + \beta(\omega_i)\left\{-R(\omega_i)\omega_i^{kr}\sin\left(\frac{kr\pi}{2}\right) - I(\omega_i)\omega_i^{kr}\cos\left(\frac{kr\pi}{2}\right)\right\}\Big) = 0 \tag{14}$$

The $(m+1)$ equations given by Eq. (13) and the n equations given by Eq. (14) make up a linear system that may be solved so as to find the coefficients of Eq. (1). According to Eqs. (13) and (14), the linear system on coefficients a and b is

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{bmatrix} = \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{g} \end{bmatrix} \tag{15}$$

where

$$A_{k,l} = \sum_{i=1}^{K} -\omega_i^{(k+l)r}\cos\left(\frac{(k-l)r\pi}{2}\right)$$
$$k = 0, 1, \ldots, m;\ l = 0, 1, \ldots, m \tag{16}$$

$$B_{k,l} = \sum_{i=1}^{K} \left\{R(\omega_i)\omega_i^{(k+l)r}\cos\left(\frac{(k-l)r\pi}{2}\right) + I(\omega_i)\omega_i^{(k+l)r}\sin\left(\frac{(k-l)r\pi}{2}\right)\right\}$$
$$k = 0, 1, \ldots, m;\ l = 1, 2, \ldots, n \tag{17}$$

$$C_{k,l} = \sum_{i=1}^{K} \left\{I(\omega_i)\omega_i^{(k+l)r}\sin\left(\frac{(k-l)r\pi}{2}\right) - R(\omega_i)\omega_i^{(k+l)r}\cos\left(\frac{(k-l)r\pi}{2}\right)\right\}$$
$$k = 1, 2, \ldots, n;\ l = 0, 1, \ldots, m \tag{18}$$

$$D_{k,l} = \sum_{i=1}^{K} \{[R(\omega_i)]^2 + [I(\omega_i)]^2\}\omega_i^{(k+l)r}\cos\left(\frac{(k-l)r\pi}{2}\right)$$
$$k = 1, 2, \ldots, n;\ l = 1, 2, \ldots, n \tag{19}$$

$$e_k = \sum_{i=1}^{K} \left\{-R(\omega_i)\omega_i^{kr}\cos\left(\frac{kr\pi}{2}\right) - I(\omega_i)\omega_i^{kr}\sin\left(\frac{kr\pi}{2}\right)\right\}$$
$$k = 0, 1, \ldots, m \tag{20}$$

$$g_k = -\sum_{i=1}^{K} ([R(\omega_i)]^2 + [I(\omega_i)]^2)\omega_i^{kr}\cos\left(\frac{kr\pi}{2}\right)$$
$$k = 1, 2, \ldots, n \tag{21}$$

2 Weighted Iteration Method

The drawbacks of the Levy method are well known, one of them being that low frequency data has little influence on Eq. (15) and the resulting fit is poor for such frequencies. Using well-chosen weights for decreasing the influence of low frequency data is a means of dealing with this. Here, we select the iteration formula

$$J_L^* = \sum_{i=1}^{K} \left\{\frac{\varepsilon(\omega_i)D(\omega_i)_L}{D(\omega_i)_{L-1}}\right\}^2 \tag{22}$$

where L is the iteration number and D_{L-1} is the denominator found in the previous iteration. In the first iteration this is assumed to be 1 and the result is that according to the Levy method. If convergence exists, subsequent iterations will be J_L^* converging to ε.

Let

$$W_i^L = \frac{1}{|D_{L-1}(\omega_i)|^2} \tag{23}$$

be the weight functions. It depends on coefficients known from the $L-1$ iterations, not the current one, and so derivatives Eqs. (11) and (12) are not affected. The only difference in the method is that matrices and vectors in Eq. (15) will now be given by

$$A_{k,l}^L = \sum_{i=1}^{K} -\omega_i^{(k+l)r}\cos\left(\frac{(k-l)r\pi}{2}\right)W_i^L$$
$$k = 0,1,\dots,m;\ l = 0,1,\dots,m \tag{24}$$

$$B_{k,l}^L = \sum_{i=1}^{K}\left\{R(\omega_i)\omega_i^{(k+l)r}\cos\left(\frac{(k-l)r\pi}{2}\right) + I(\omega_i)\omega_i^{(k+l)r}\sin\left(\frac{(k-l)r\pi}{2}\right)\right\}W_i^L$$
$$k = 0,1,\dots,m;\ l = 1,2,\dots,n \tag{25}$$

$$C_{k,l}^L = \sum_{i=1}^{K}\left\{I(\omega_i)\omega_i^{(k+l)r}\sin\left(\frac{(k-l)r\pi}{2}\right) - R(\omega_i)\omega_i^{(k+l)r}\cos\left(\frac{(k-l)r\pi}{2}\right)\right\}W_i^L$$
$$k = 1,2,\dots,n;\ l = 0,1,\dots,m \tag{26}$$

$$D_{k,l}^L = \sum_{i=1}^{K}\{[R(\omega_i)]^2 + [I(\omega_i)]^2\}\omega_i^{(k+l)r}\cos\left(\frac{(k-l)r\pi}{2}\right)W_i^L$$
$$k = 1,2,\dots,n;\ l = 1,2,\dots,n \tag{27}$$

$$e_k^L = \sum_{i=1}^{K}\left\{-R(\omega_i)\omega_i^{kr}\cos\left(\frac{kr\pi}{2}\right) - I(\omega_i)\omega_i^{kr}\sin\left(\frac{kr\pi}{2}\right)\right\}W_i^L$$
$$k = 0,1,\dots,m \tag{28}$$

$$g_k^L = -\sum_{i=1}^{K}([R(\omega_i)]^2 + [I(\omega_i)]^2)\omega_i^{kr}\cos\left(\frac{kr\pi}{2}\right)W_i^L$$
$$k = 1,2,\dots,n \tag{29}$$

3 Verification of the Method

To illustrate the advantages of these methods in system identification, the proposed procedure is applied to identify systems with known parameters.

First, given a known integer-order system

$$G_1(s) = \frac{3+7s}{1+2s+5s^2+9s^3+s^4} \tag{30}$$

We use the model

$$\hat{G}_1(s) = \frac{b_0+b_1 s}{a_0+a_1 s^r+a_2 s^{2r}+a_3 s^{3r}+s^{4r}} \tag{31}$$

to reconstruct the transfer function with the commensurate order $r = 0.1, 0.2, \dots, 1$, respectively. The relationship between the error index J and the commensurate order r is shown in Fig. 1.

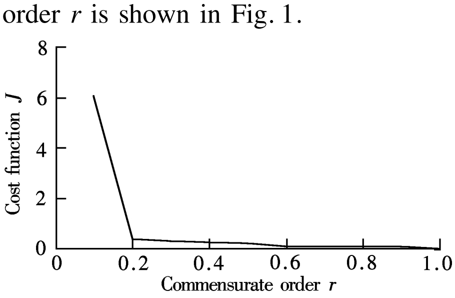

Fig. 1 The relationship between error index J and commensurate order r

The error index J is the minimal, when the commensurate order r equals 1. The minimal error index J is 1.0592×10^{-11}. So $r=1$ should be the best commensurate order, in this case, the result of identification in frequency range from 10^{-3} to 10^3 Hz is

$$G_1(s) = \frac{3+7s}{1+2s+5s^2+9s^3+s^4} \tag{32}$$

Besides this example, we also examined many other examples. All the results show that we can achieve the same degree of fitting when the frequency responses exhibit integer order slopes, provided that an appropriate structure is offered using the non-integer order transfer function model.

Then a known non-integer order system is considered, the transfer function is

$$G_2(s) = \frac{141.1+10\,010s^{0.25}+1.414s^{0.75}+100s^{1.25}}{1+0.1s^{0.25}+0.014\,14s^{0.75}+s^{1.25}} \tag{33}$$

The model

$$\hat{G}_2(s) = \frac{b_0+b_1 s^r+b_2 s^{2r}+b_3 s^{3r}+b_4 s^{4r}+b_5 s^{5r}}{a_0+a_1 s^r+a_2 s^{2r}+a_3 s^{3r}+a_4 s^{4r}+s^{5r}} \tag{34}$$

is used to reconstruct the transfer function with the commensurate order $r=0.25, 0.5, 1$, respectively.

The results of identification which are obtained by the generalized Levy method in frequency ranges from 10^{-5} to 10^4 Hz for $r=0.25, 0.5, 1$ are shown in Fig. 2. Seen from Fig. 2, as the frequency range broadens, the fitting error increases in the low frequency range. This is also the reason why the weighted iteration method is presented.

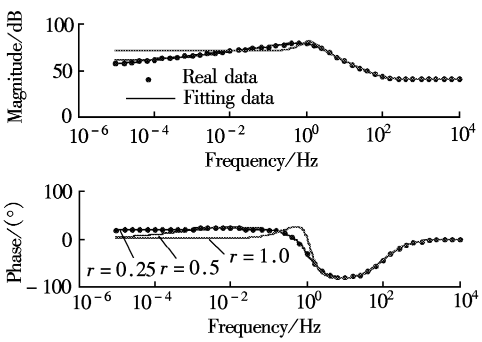

Fig. 2 Real data and results of identification which are obtained by the generalized Levy method

The results of identification which are obtained by the weighted iteration method in frequency ranges from 10^{-5} to 10^4 Hz for $r=0.25, 0.5, 1$ are shown in Fig. 3. Seen from Fig. 2 and Fig. 3, the weighted iteration method can improve on the drawbacks of the generalized Levy method in the low frequency range.

This example also shows that we can achieve better fittings of the frequency response data whose frequency responses exhibit non-integer order slopes using the non-integer order transfer function model.

At the same time we can verify that the algorithm is more stable numerically for $r<1$. In our example

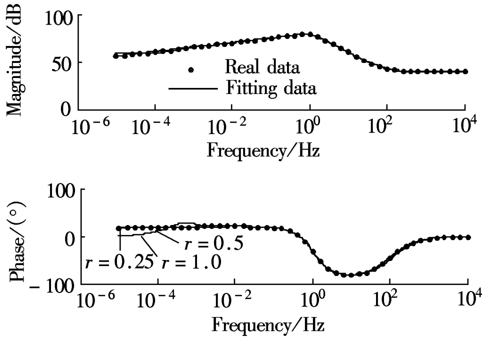

Fig. 3 Real data and results of identification which are obtained by the iterative improvement method

we verify the matrix reciprocal condition number (RCOND). If the matrix is well conditioned, RCOND is near 1.0. If the matrix is badly conditioned, RCOND is near 0. Results are RCOND = 5.1274×10^{-17} for $r = 0.25$, RCOND = 4.1151×10^{-27} for $r = 0.5$, RCOND = 1.4178×10^{-44} for $r = 1$. We find the less the commensurate order r is, the fewer the condition numbers of the matrix is. So the algorithm is more stable numerically.

4 Conclusion

Non-integer order systems are a generalization of the integer systems. These models are more adequate for description of the dynamical systems. In this paper only the identification of the non-integer order systems with commensurate order is presented. How to identify the general non-integer order systems is still an open problem.

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非整数阶动态系统的频域辨识方法

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摘要:提出了2种非整数阶系统辨识方法——广义Levy法和加权迭代法.首先,将辨识整数阶系统的Levy法进行推广得到了适合非整数阶系统辨识的广义Levy法;然后,针对广义Levy法的不足,提出了一种加权迭代法.结果表明:对于非整数阶系统,采用所提出的方法能够得到更好的频域响应拟合;对于整数阶系统,采用该方法,运用阶数扫描仍然能找到拟合其频域响应的最好的整数阶模型;与整数阶系统辨识算法相比,所提出的的系统辨识算法更稳定.

关键词:非整数阶动态系统;非整数阶系统辨识;广义Levy法;加权迭代法

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