Journal of Southeast University (English Edition) Vol. 23, No. 1, pp. 148 – 150 Mar. 2007 ISSN 1003—7985

Generalized regular points of a C^1 map between Banach spaces

Shi Ping[1] Ma Jipu[2]

([1]Department of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing 210003, China)
([2]Department of Mathematics, Nanjing University, Nanjing 210093, China)

Abstract: Let f be a C^1 map between two Banach spaces E and F. It has been proved that the concept of generalized regular points of f, which is a generalization of the notion of regular points of f, has some crucial applications in nonlinearity and global analysis. We characterize the generalized regular points of f using the three integer-valued (or infinite) indices $M(x_0)$, $M_c(x_0)$ and $M_r(x_0)$ at $x_0 \in E$ generated by f and by analyzing generalized inverses of bounded linear operators on Banach spaces, that is, if $f'(x_0)$ has a generalized inverse in the Banach space $\mathscr{B}(E, F)$ of all bounded linear operators on E into F and at least one of the indices $M(x_0)$, $M_c(x_0)$ and $M_r(x_0)$ is finite, then x_0 is a generalized regular point of f if and only if the multi-index $(M(x), M_c(x), M_r(x))$ is continuous at x_0.

Key words: Banach space; bounded linear operator; generalized inverse; index; generalized regular point; semi-Fredholm map

There are extensive applications of regular points and regular values of a C^1 mapping f between Banach spaces in nonlinear functional analysis and global analysis. The concept of locally fine points of f was first introduced in Ref. [1], which is a generalization of the notion of regular points of f and has some important applications given in Refs. [2 – 3]. The generalized regular values of f derived from the locally fine points of f extend the contents of regular values of f. The aim of this paper is to characterize the generalized regular points of a C^1 mapping f between two Banach spaces by a new approach.

Let us recall some important concepts and notations. Throughout this paper E and F denote Banach spaces and $\mathscr{B}(E, F)$ is the set of all bounded linear operators on E into F. Let $R(T)$ and $N(T)$ denote the range and null space of $T \in \mathscr{B}(E, F)$, respectively. An operator $T^+ \in \mathscr{B}(F, E)$ is called a generalized inverse of $T \in \mathscr{B}(E, F)$, if $TT^+T = T$ and $T^+TT^+ = T^+$ [4–5]. Let $f: U \subset E \to F$ be a C^1 mapping, where U is open in E. A point $x \in U$ is called a regular point for f if and only if f is a submersion at x, that is, the Fréchet derivative $f'(x)$ is surjective and the null space $N(f'(x))$ splits E. A point $y \in F$ is said to be a regular value of f if and only if the set $f^{-1}(y)$ is empty or only consists of regular points of f [6].

Received 2006-06-12.

Foundation items: The National Natural Science Foundation of China (No. 10271053), the Foundation of Nanjing University of Finance and Economics (No. B0556).

Biography: Shi Ping (1963—), male, associate professor, pshi@eyou.com.

Definition 1 Let $f: U \subset E \to F$ be a C^1 map, where U is open in E. The point $x_0 \in U$ is called a generalized regular point of f, if there exists a generalized inverse $f'(x_0)^+$ of $f'(x_0)$ such that $R(f'(x)) \cap N(f'(x_0)^+) = \{0\}$ near x_0, where $f'(x)$ is the Fréchet derivative of f at $x \in U$.

Definition 2 Let $f: U \subset E \to F$ be a C^1 map, where U is open in E. A point $y \in F$ is called a generalized regular value of f if and only if the preimage $f^{-1}(y)$ is empty or only consists of generalized regular points of f.

For a C^1 mapping f, we give the following three indices.

Definition 3 Let $f: U \subset E \to F$ be a C^1 mapping, where U is an open subset of E. For any $x \in U$, define $M(x) = \dim N(f'(x))$, $M_c(x) = \operatorname{codim} R(f'(x))$ and $M_r(x) = \dim R(f'(x))$.

Clearly, $M(x)$, $M_c(x)$, $M_r(x) \in \mathbf{Z}^{++}$, where $\mathbf{Z}^{++} = \{0, 1, 2, \dots\} \cup \{\infty\}$.

1 Main Results

We first introduce an auxiliary result, which is a solution of the local conjugacy problem suggested by Berger in Refs. [7 – 8] and can be found in Ref. [3].

Lemma 1 Let $f: U \subset E \to F$ be a C^1 mapping with a generalized inverse of $f'(x_0)$, where U is an open subset containing x_0. Then $f(x)$ is conjugate to $f'(x_0)$ near x_0, that is, there exist two neighborhoods U_1 of x_0 and V_1 of $0 \in F$ with two diffeomorphisms u on U_1 and v on V_1 such that $u(x_0) = 0$, $v(0) = f(x_0)$ and $f(x) = v(f'(x_0)u(x))$ for all $x \in U_1$, if and only if x_0 is a generalized regular point of f.

Theorem 1 Suppose that f: $U \subset E \to F$ is a C^1 mapping.

1) If x_0 is a generalized regular point of f, then $M(x), M_c(x)$ and $M_r(x)$ are continuous at x_0.

2) If $f'(x_0)$ has a generalized inverse in $\mathscr{B}(F, E)$ and at least one of the indices $M(x_0)$, $M_c(x_0)$ and $M_r(x_0)$ is finite, then x_0 is a generalized regular point of f if and only if the multi-index $(M(x), M_c(x), M_r(x))$ is continuous at x_0.

Proof 1) Since x_0 is a generalized regular point of f, by lemma 1, $f(x)$ is conjugate to $f'(x_0)$ in the neighborhood $U(x_0)$ of x_0, that is, there exist two diffeomorphisms u: $U(x_0) \to u(U(x_0))$, v: $V(0) \to v(V(0))$ such that $u(x_0) = 0$, $v(0) = f(x_0)$ and

$$f(x) = v(f'(x_0)u(x))$$

for all $x \in U(x_0)$, where $V(0)$ is a neighborhood at 0 of F, whence

$$f'(x) = v'(f'(x_0)u(x))f'(x_0)u'(x)$$

for all $x \in U(x_0)$.

Let $A(x) = v'(f'(x_0)u(x))$, $B(x) = u'(x)$ for all $x \in U(x_0)$. Since u and v are two diffeomorphisms, $A(x) \in \mathscr{B}(F, F)$ and $B(x) \in \mathscr{B}(E, E)$ are invertible for any $x \in U(x_0)$. Then we have that $f'(x) = A(x)f'(x_0)B(x)$, $N(f'(x)) = B(x)^{-1}N(f'(x_0))$ and $R(f'(x)) = A(x)R(f'(x_0))$ for all $x \in U(x_0)$. Hence $\dim N(f'(x)) = \dim N(f'(x_0))$ and $\dim R(f'(x)) = \dim R(f'(x_0))$ for all $x \in U(x_0)$.

Since x_0 is a generalized regular point of f, $f'(x_0)$ has a generalized inverse $f'(x_0)^+$. Furthermore we can easily see that $B(x)^{-1}f'(x_0)^+A(x)^{-1}$ is a generalized inverse of $f'(x)$ for any $x \in U(x_0)$. Hence $R(f'(x))$ is closed for any $x \in U(x_0)$.

Next we will claim that

$$\operatorname{codim} R(f'(x)) = \operatorname{codim} R(f'(x_0))$$

for all $x \in U(x_0)$. In fact, if $\operatorname{codim} R(f'(x_0)) < \infty$, then there exists a finite-dimensional complementary subspace F_1 of $R(f'(x_0))$ in F, that is,

$$F_1 \oplus R(f'(x_0)) = F$$

and

$$\dim F_1 = \operatorname{codim} R(f'(x_0))$$

Therefore

$$A(x)F_1 \oplus A(x)R(f'(x_0)) = F$$

or

$$A(x)F_1 \oplus R(f'(x)) = F$$

Thus

$$\operatorname{codim} R(f'(x)) = \dim A(x)F_1 = \dim F_1 = \operatorname{codim} R(f'(x_0))$$

Suppose that $\operatorname{codim} R(f'(x_0)) = \infty$, but $\operatorname{codim} R(f'(x)) < \infty$ for some $x \in U(x_0)$. By the operator identity $f'(x_0) = A(x)^{-1}f'(x)B(x)^{-1}$ and the above stated proof, we have that $\operatorname{codim} R(f'(x_0)) = \operatorname{codim} R(f'(x)) < \infty$, which is a contradiction. Hence $M(x), M_c(x)$ and $M_r(x)$ are constant in $U(x_0)$, whence $M(x), M_c(x)$ and $M_r(x)$ are continuous at x_0.

2) We only need to prove the sufficiency by 1).

Since $M(x), M_c(x)$ and $M_r(x)$ are continuous at x_0, there exists a neighborhood V at x_0 such that $M(x) = M(x_0)$, $M_c(x) = M_c(x_0)$ and $M_r(x) = M_r(x_0)$ for all $x \in V$. It follows from the continuity of $f'(x)$ at x_0 that there exists a neighborhood V_0 of x_0 such that

$$\|f'(x) - f'(x_0)\| < \|f'(x_0)^+\|^{-1}$$

for all $x \in V_0$, where $f'(x_0)^+$ is a generalized inverse of $f'(x_0)$.

Denote

$$C(x) = I_F + (f'(x) - f'(x_0))f'(x_0)^+$$

and

$$D(x) = I_E + f'(x_0)^+(f'(x) - f'(x_0))$$

for all $x \in V_0$, where I_F and I_E are the identity operators on F and E, respectively. Then $C(x) \in \mathscr{B}(F, F)$ and $D(x) \in \mathscr{B}(E, E)$ are invertible, and we obtain

$$f'(x)f'(x_0)^+f'(x_0) = C(x)f'(x_0)$$

$$f'(x_0)^+f'(x_0)D(x) = f'(x_0)^+f'(x)$$

$$D(x)f'(x_0)^+ = f'(x_0)^+C(x)$$

$$f'(x_0)^+C(x)^{-1} = D(x)^{-1}f'(x_0)^+$$

for all $x \in V_0$. Therefore

$$R(f'(x)) \supseteq R(C(x)f'(x_0))$$

and

$$N(f'(x_0)^+C(x)^{-1}) = N(f'(x_0)^+)$$

for all $x \in V_0$.

We first assume that $M_r(x_0) < \infty$. Then $\dim R(f'(x)) = M_r(x) = M_r(x_0) = \dim R(f'(x_0)) < \infty$ for all $x \in V$, but $\dim R(f'(x)) \geqslant \dim R(C(x)f'(x_0)) = \dim R(f'(x_0))$ for all $x \in V_0$, thus $\dim R(f'(x)) = \dim R(C(x)f'(x_0)) < \infty$ for all $x \in V \cap V_0$, whence $R(f'(x)) = R(C(x)f'(x_0))$ for all $x \in V \cap V_0$. Hence $R(f'(x)) \cap N(f'(x_0)^+) = R(C(x)f'(x_0)) \cap N(f'(x_0)^+C(x)^{-1}) = \{0\}$ for all $x \in V \cap V_0$, that is, x_0 is a generalized regular point of f.

Now assume that $M_r(x_0) = \infty$, either $M(x_0) < \infty$ or $M_c(x_0) < \infty$.

If $M(x_0) < \infty$, then

$$\dim N(f'(x_0)^+f'(x)) = \dim D(x)^{-1}N(f'(x_0)) = \dim N(f'(x_0)) = M(x_0) < \infty$$

for all $x \in V_0$; however, $M(x) = M(x_0)$ for all $x \in V$ and $N(f'(x_0)^+f'(x)) \supseteq N(f'(x))$, thus

$$\dim N(f'(x_0)^+f'(x)) = \dim N(f'(x)) < \infty$$

for all $x \in V \cap V_0$, whence

$$N(f'(x_0)^+f'(x)) = N(f'(x))$$

for all $x \in V \cap V_0$. Thus $R(f'(x)) \cap N(f'(x_0)^+) = \{0\}$ for all $x \in V \cap V_0$. Therefore x_0 is a generalized regular point of f.

If $M_c(x_0) < \infty$, we notice that

$$\dim N(f'(x_0)^+) = \operatorname{codim} R(f'(x_0)) = M_c(x_0) < \infty$$

then there exists a finite-dimensional complementary subspace N^- of $R(f'(x))\cap N(f'(x_0)^+)$ in $N(f'(x_0)^+)$, that is

$$N(f'(x_0)^+) = N^- \oplus [R(f'(x))\cap N(f'(x_0)^+)]$$

whence

$$\begin{aligned} F = R(C(x)f'(x_0)) \oplus N(f'(x_0)^+C(x)^{-1}) = \\ R(C(x)f'(x_0)) \oplus N(f'(x_0)^+) = \\ R(f'(x)) + N(f'(x_0)^+) = R(f'(x)) \oplus N^- \end{aligned}$$

for all $x\in V_0$. Thus

$$\begin{aligned} \operatorname{codim}R(f'(x)) = \dim N^- = \operatorname{codim}R(f'(x_0)) = \\ \dim N(f'(x_0)^+) < \infty \end{aligned}$$

for all $x\in V\cap V_0$. Therefore $R(f'(x))\cap N(f'(x_0)^+) = \{0\}$ for all $x\in V\cap V_0$. Hence x_0 is a generalized regular point of f. This completes the proof.

By theorem 1 and the concept of semi-Fredholm bounded linear operators[9] on Banach spaces in the classical Fredholm operator theory, we have that

Corollary 1 Let f: $U(x_0)\subset E\to F$ be a C^1 mapping, where $U(x_0)$ is an open subset containing x_0 and $f'(x_0)\in\mathscr{B}(E,F)$ be a semi-Fredholm operator with a generalized inverse. Then x_0 is a generalized regular point of f if and only if either $\dim N(f'(x)) = \dim N(f'(x_0)) < \infty$ or $\operatorname{codim}R(f'(x)) = \operatorname{codim}R(f'(x_0)) < \infty$, near x_0.

2 Example

We conclude this paper with a simple example, in which the C^1 mapping f cannot have any regular point and nontrivial regular value, however f has generalized regular points and nontrivial generalized regular values.

Example 1 Let $U = \{x = (x_1,x_2)\in\mathbf{R}^2 \mid x_1^2 - x_2^2 \neq 0\}\subset\mathbf{R}^2$. $f(x)$ is defined by $f(x) = (f_1(x), f_2(x)) : \mathbf{R}^2\to\mathbf{R}^2$, where $f_1(x) = e^{x_1^2-x_2^2}$, $f_2(x) = x_1^2 - x_2^2$, then

$$f'(x) = \begin{pmatrix} 2x_1e^{x_1^2-x_2^2} & -2x_2e^{x_1^2-x_2^2} \\ 2x_1 & -2x_2 \end{pmatrix}$$

f is a C^1 Fredholm mapping, $\dim N(f'(x)) = 1$ and $\operatorname{codim}R(f'(x)) = 1$ for all $x\in U$. Since $f'(x)$ is not surjective, f cannot have regular points and nontrivial regular values. By corollary 1, every $x\in U$ is a generalized regular point of f. Therefore f has nontrivial generalized regular values, e. g., (e^y, y) $(y\neq 0)$ is a nontrivial generalized regular value of f.

References

[1] Ma Jipu. (1.2) inverses of operators between Banach spaces and local conjugacy theorem [J]. *Chinese Annals of Math*: *Series B*, 1999, **20**(1): 57-62.

[2] Ma Jipu. Rank theorems of operators between Banach spaces [J]. *Science in China*: *Series A*, 2000, **43**(1): 1-5.

[3] Ma Jipu. Local conjugacy theorem, rank theorems in advanced calculus and a generalized principle for constructing Banach manifolds [J]. *Science in China*, *Series A*, 2000, **43**(12): 1233-1237.

[4] Nashed M Z. *Generalized inverses and applications* [M]. New York: Academic Press, 1976.

[5] Nashed M Z, Chen X. Convergence of Newton-like methods for singular operator equations using outer inverses [J]. *Numer Math*, 1993, **66**: 235-257.

[6] Zeidler E. *Nonlinear functional analysis and its applications*: Ⅳ *applications to mathematical physics* [M]. New York: Springer-Verlag, 1988.

[7] Berger M S. *Nonlinearity and functional analysis* [M]. New York: Academic Press, 1977.

[8] Shi Ping, Ma Jipu. A note on a problem of M. S. Berger [J]. *Northeast Math J*, 2003, **19**(4): 366-370.

[9] Ma Jipu. Generalized indices of operators in $B(H)$ [J]. *Science in China*: *Series A*, 1997, **40**(12): 1233-1238.

Banach 空间之间 C^1 映射的广义正则点

史 平[1] 马吉溥[2]

([1] 南京财经大学应用数学系,南京 210003)
([2] 南京大学数学系,南京 210093)

摘要:设 f 是 2 个 Banach 空间 E 和 F 之间 C^1 映射. 已经证明 f 的广义正则点概念是 f 的正则点概念的一个推广并且在非线性分析和大范围分析中有非常重要的应用. 用 f 产生的在 $x_0\in E$ 处的 3 个整数(或无穷大)值指标 $M(x_0)$, $M_c(x_0)$ 和 $M_r(x_0)$ 和分析 Banach 空间上有界线性算子的广义逆来刻画 f 的广义正则点,即,如果 $f'(x_0)$ 在从 E 上到 F 的有界线性算子组成的 Banach 空间 $\mathscr{B}(E,F)$ 内有广义逆,且 $M(x_0)$, $M_c(x_0)$ 和 $M_r(x_0)$ 中至少有一个是有限,则 x_0 是 f 的广义正则点的充分必要条件是多重指标$(M(x), M_c(x), M_r(x))$ 在 x_0 点处连续.

关键词:Banach 空间;有界线性算子;广义逆;指标;广义正则点;半 Fredholm 映射
中图分类号:O177.91