Journal of Southeast University (English Edition) Vol. 23, No. 2, pp. 309 – 312 June 2007 ISSN 1003—7985

Extensions of strongly π-regular general rings

Wang Zhou Chen Jianlong

(Department of Mathematics, Southeast University, Nanjing 210096, China)

Abstract: The concept of the strongly π-regular general ring (with or without unity) is introduced and some extensions of strongly π-regular general rings are considered. Two equivalent characterizations on strongly π-regular general rings are provided. It is shown that I is strongly π-regular if and only if, for each $x \in I$, $x^n = x^{n+1}y = zx^{n+1}$ for $n \geqslant 1$ and y, $z \in I$ if and only if every element of I is strongly π-regular. It is also proved that every upper triangular matrix general ring over a strongly π-regular general ring is strongly π-regular and the trivial extension of the strongly π-regular general ring is strongly clean.

Key words: strongly π-regular general ring; strongly clean general ring; upper triangular matrix general ring; trivial extension

An element x in a ring R is called strongly π-regular if there exist $y \in R$, $n \geqslant 1$ such that $x^n = x^{n+1}y$ and $xy = yx$. A ring R is strongly π-regular if all chains of the forms $aR \supseteq a^2R \supseteq \ldots$ terminate, or equivalently, all chains $Ra \supseteq Ra^2 \supseteq \ldots$ terminate. According to Ref. [1], R is strongly π-regular if and only if every element of R is strongly π-regular. In 1999, Nicholson[2] introduced the concept of strongly clean rings (every element is the sum of an idempotent and a unit which commute) and proved that every strongly π-regular ring is strongly clean. For general rings (with or without unity), many people have been interested (see Refs. [3 – 5]). In this paper, we extend the definition of the strongly π-regular ring and the strongly clean ring to general rings, and obtain some interesting results.

Let I be a general ring. An element x of I is strongly π-regular if there exist $n \geqslant 1$ and $y \in I$ such that $x^n = x^{n+1}y$ and $xy = yx$. We call I strongly π-regular if for any $x \in I$ there exist $n \geqslant 1$ and $y \in I$ such that $x^n = x^{n+1}y$. We show that I is strongly π-regular if for every element of I is strongly π-regular. We also prove that the upper triangular matrix general ring over a strongly π-regular general ring is still strongly π-regular. Finally, we consider trivial extensions. It is proved that trivial extensions of strongly π-regular general rings are strongly clean. For the study of strongly clean rings and strongly π-regular rings, we refer to Refs. [6 – 10].

By the term ring we mean an associative ring with unity and by a general ring we mean an associative general ring with or without unity. For clarity, R and S always denote rings, and a general ring is written as I. We denote the Jacobson radical of the general ring I by $J(I)$ and write $T_n(I)$ for the general rings of all $n \times n$ upper triangular matrices over the general ring I. We write the trivial extension of the general ring I as $T(I, I)$ (or as $I[x]/(x^2)$ equivalently).

1 Strongly π-Regular Triangular Matrix General Rings

An element x in a general ring I is called strongly π-regular if there exist $y \in I$ and $n \geqslant 1$ such that $x^n = x^{n+1}y$ and $xy = yx$. Chen et al. [4] introduced a strongly π-regular ideal. We analogously define a strongly π-regular general ring in the case that, for any $x \in I$, there exist $n \geqslant 1$ and $y \in I$ such that $x^n = x^{n+1}y$.

Theorem 1 The following are equivalent for a general ring I:

① I is strongly π-regular.

② For each $x \in I$, $x^n = x^{n+1}y = zx^{n+1}$ for $n \geqslant 1$ and $y, z \in I$.

③ Every element in I is strongly π-regular.

Proof ③⇒①. It is clear.

①⇒②. It is similar to the proof of theorem 2. 3 in Ref. [4].

②⇒③. By ②, we have that $x^n = x^{n+1}y = zx^{n+1}$. Then $x^n = zx^{n+1} = z(zx^{n+1})x = z^2x^{n+2} = \ldots = z^{n+1}x^{2n+1} = z^n(zx^{n+1})x^n = z^n(x^{n+1}y)x^n = z^{n-1}(zx^{n+1})yx^n = z^{n-1}(x^{n+1}y)yx^n = \ldots = x^{n+1}y^{n+1}x^n = x^ny^nx^n$. Set $e = x^ny^n$, then $x^n = ex^n$ and $x^ne = x^nx^ny^n = x^{n-1}(x^{n+1}y)y^{n-1}$

Received 2006-10-08.

Foundation items: The Foundation for Excellent Doctoral Dissertation of Southeast University (No. YBJJ0507), the National Natural Science Foundation of China (No. 10571026), the Natural Science Foundation of Jiangsu Province (No. BK2005207).

Biographies: Wang Zhou (1979—), male, graduate; Chen Jianlong (corresponding author), male, doctor, professor, jlchen@ seu. edu. cn.

$= x^{n-1}x^n y^{n-1} = \ldots = x^{n+1}y = x^n$. Similarly, there exists $f = z^n x^n$, such that $x^n = fx^n = x^n f$. Therefore, $e = x^n y^n = fx^n y^n = fe = z^n x^n e = z^n x^n = f$. Set $b = ey^n e \in eIe$, and $c = ez^n e \in eIe$. Then $e = x^n b = cx^n$. In addition, $bx^n = ebx^n = cx^n bx^n = cex^n = cx^n = x^n b = e$. Set $w = x^{n-1}b = x^{n-1}ebe = x^{2n-1}b^3 x^n \in eIe$, whence $xw = x^n b = e$. Then $w = ew = we$ and $x^{n+1}w = x^{n+1}x^{n-1}b = x^{2n}b = x^n$. Set $g = wx$. Then $g^2 = w(xw)x = wex = wx = g \in eI$, whence $eg = g$. Since $Ix^n = Iex^n = Ibx^{2n} \subseteq Ix^{2n} \subseteq Ix^{n+1} \subseteq Ix^n$, we have $Ie = Ix^n = Ix^{n+1} = Ix^n x = Ix^{n+1}wx \subseteq Iwx = Ig$, whence $e = eg = g = wx$. Therefore, $wx = xw$, as required.

Let I and J be general rings, M an (I, J)-bimodule and

$$T = \begin{bmatrix} I & M \\ 0 & J \end{bmatrix} = \left\{ \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \mid r \in I, s \in J \text{ and } m \in M \right\}$$

In T, define the addition and the multiplication as those in ordinary matrices. Then T is an upper triangular matrix general ring. We will investigate the strongly π-regularity of upper triangular matrix general rings.

Lemma 1 The following are equivalent for a general ring I:

① I is strongly π-regular.

② There exist two ideals A, B of I with $AB = 0$ such that I/A, I/B are both strongly π-regular.

Proof ①⇒②. It is clear.

②⇒①. Let I/A, I/B be strongly π-regular general rings and let $x \in I$. Then there exist some positive integers m, n and $y, z \in I$ such that $x^m = x^{m+1}y \pmod A$, and $x^n = x^{n+1}z \pmod B$. Clearly, we may assume that $x^m = x^{2m+1}y \pmod A$ and $x^m = x^{2m+1}z \pmod B$. Notice that $x^m - x^{2m+1}y \in A$ and $x^m - x^{2m+1}z \in B$, so $(x^m - x^{2m+1}y)(x^m - x^{2m+1}z) \in AB = 0$. This shows that $x^{2m} = x^{2m+1}(yx^m + x^m z - yx^{2m+1}z)$. Therefore, I is strongly π-regular.

Proposition 1 The following are equivalent for a general ring I:

① I is strongly π-regular.

② There exist ideals A_1, A_2, …, A_n of I with $A_1 A_2 \ldots A_n = 0$ such that I/A_1, I/A_2, …, I/A_n are strongly π-regular.

Proof ①⇒②. It is trivial.

②⇒①. The proof will proceed by induction on n. The $n = 1$ case is easily proved. Assume that the result follows whenever $n \leqslant k$. Let $n = k+1$ and $A = A_1 A_2 \ldots A_k$. Then $AA_{k+1} = 0$ and A_1/A, A_2/A, …, A_k/A are all ideals of I/A. From $I/A/A_i/A \cong I/A_i$, we see that $I/A/A_i/A$ is strongly π-regular for all i, $1 \leqslant i \leqslant k$. Clearly, $\prod_{i \leqslant k} A_i/A = 0$ and then I/A is strongly π-regular. Thus I/A and I/A_{k+1} are both strongly π-regular. Since $AA_{k+1} = 0$, by virtue of lemma 1, I is strongly π-regular.

Theorem 2 Let I and J be general rings, M an (I, J)-bimodule and $T = \begin{bmatrix} I & M \\ 0 & J \end{bmatrix}$ an upper triangular matrix general ring. Then I and J are strongly π-regular if and only if T is strongly π-regular.

Proof Suppose that I and J are strongly π-regular. Let $A = \begin{bmatrix} 0 & M \\ 0 & J \end{bmatrix}$, $B = \begin{bmatrix} I & M \\ 0 & 0 \end{bmatrix}$. Then A and B are both ideals of T with $AB = \mathbf{0}$. Notice that $T/A \cong I$ and $T/B \cong J$. Then we know that T/A and T/B are both strongly π-regular. By lemma 1, T is strongly π-regular. Conversely, assume that T is strongly π-regular. Let A and B be ideals of the above. Since $I \cong T/A$ and $J \cong T/B$, it is easily verified that I and J are strongly π-regular.

Corollary 1 Let R and S be rings, M an (R, S)-bimodule and $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ an upper triangular matrix ring. Then R and S are strongly π-regular if and only if T is strongly π-regular.

Proposition 2 Let I be a strongly π-regular general ring. Then the upper triangular matrix general ring $T_n(I)$ is strongly π-regular if and only if I is strongly π-regular.

Proof The proof proceeds by induction on n, being clear for $n = 1$. It follows from theorem 2 for $n = 2$. Assume that the statement is true for $n = k - 1$. Let $J = T_{k-1}(I)$ and $M = \{I, I, \ldots, I\}$. Then $T_k(I) = \begin{bmatrix} I & M \\ 0 & J \end{bmatrix}$. Therefore, by theorem 2 and the induction principle, $T_n(I)$ is strongly π-regular. The converse is clear.

The proof of proposition 2 works for the next result.

Corollary 2 Let R be a strongly π-regular ring. Then the upper triangular matrix ring $T_n(R)$ is strongly π-regular if and only if R is strongly π-regular.

2 Trivial Extensions of Strongly π-Regular General Rings

Let I be a general ring with $p, q \in I$. We write $p * q = p + q + pq$. Let

$$Q(I) = \{q \in I \mid p * q = 0 = q * p \text{ for some } p \in I\}$$

Note that $J(I) \subseteq Q(I)$. Recall that R is strongly clean if every element $a \in R$ can be written as the sum of an idempotent and a unit which commute.

Lemma 2 The following are equivalent for a ring R:

① R is strongly clean.

② For each $a \in R$, $a = e + q$ and $eq = qe$ where e^2

$=e$ and $q\in Q(R)$.

Proof Let $a\in R$. If R is strongly clean, write $a+1=e+u$ and $eu=ue$ where $e^2=e$ and $u\in U(R)$. Then $a=e+q$, $eq=qe$ where $q=u-1$ and $q*p=0=p*q$ with $p=u^{-1}-1$. Conversely, if $a-1=e+q$ and $eq=qe$ where $e^2=e$ and $q\in Q(R)$, then we have $a=e+u$ where $u=q+1\in U(R)$ and $eu=ue$.

An element a in a general ring I is called a strongly clean element if $a=e+q$ and $eq=qe$ where $e^2=e\in I$ and $q\in Q(I)$; and I is called a strongly clean general ring if every element of I is strongly clean. Hence, idempotents, nilpotents and elements of $Q(I)$ are all strongly clean. Clearly, every homomorphic image of a strongly clean general ring is strongly clean, and the direct product $\prod_i K_i$ and the direct sum $\oplus_i K_i$ of general rings are strongly clean if and only if each K_i is strongly clean.

Next we consider trivial extensions of strongly π-regular general rings. Let I be a general ring and M be an (I,I)-bimodule. Write

$$\boldsymbol{T}(I,M)=\left\{\begin{bmatrix} a & m\\ 0 & a\end{bmatrix}\mid a\in I, m\in M\right\}$$

Then $\boldsymbol{T}(I,M)$ is called a trivial extension. In particular, if $M=I$, then we have $\boldsymbol{T}(I,I)\cong I[x]/(x^2)=\{r_0+r_1x\mid r_0,r_1\in I \text{ and } x^2=0\}$.

Theorem 3 Let I be a general ring and let $r=r_0+r_1x\in I[x]/(x^2)$. If r_0 is a strongly π-regular element, then r is strongly clean in $I[x]/(x^2)$.

Proof Suppose that r_0 is a strongly π-regular element. Then there exist $b\in I$ and $m\geqslant 1$ such that $r_0^{\,m}=r_0^{\,m+1}b$ and $r_0b=br_0$. Notice that $r_0^{\,m}=r_0r_0^{\,m}b=r_0(r_0^{\,m+1}b)b=r_0^{\,m+2}b^2=\dots=r_0^{\,2m}b^m=r_0^{\,m}b^mr_0^{\,m}$. Set $y=b^mr_0^{\,m}b^m$. It is easily verified that $r_0^{\,m}=r_0^{\,m}yr_0^{\,m}$ and $y=yr_0^{\,m}y$. Set $e_0=r_0^{\,m}y$, $q_{01}=r_0^{\,m}-r_0^{\,m}y$. Then we have

$$r_0^{\,m}=e_0+q_{01} \tag{1}$$

where $e_0^{\,2}=e_0$, $q_{01}\in Q(I)$ with $p_{01}=y-r_0^{\,m}y$.

We directly calculate that

$$e_0q_{01}=q_{01}e_0=q_{01},\ \ e_0p_{01}=p_{01}e_0=p_{01}$$

Let $n=2m$. Then

$$r_0^{\,n}=r_0^{\,2m}=(e_0+q_{01})^2=e_0+2q_{01}+q_{01}^{\,2}=e_0+q_{02} \tag{2}$$

where $q_{02}=2q_{01}+q_{01}^{\,2}\in Q(I)$ with $p_{02}=2p_{01}+p_{01}^2$.

Furthermore, it is easily checked that $e_0q_{02}=q_{02}e_0=q_{02}$, $e_0p_{02}=p_{02}e_0=p_{02}$.

Next we show that there exist $e^2=e$, $q\in Q(I[x]/(x^2))$ such that

$$r=e+q \quad\text{and}\quad eq=qe$$

First, we need the following claim.

Claim $r_0=e_0+q_0$ is a strongly clean expression of r_0 in I.

Proof Notice that $e_0^{\,2}=e_0$, $e_0r_0=r_0e_0$, so we only need to show $q_0\in Q(I)$. In fact, we may assume m is even and using a virtual 1 for clarity. Let $p_0=(r_0-r_0^{\,2}+r_0^{\,3}-\dots+r_0^{\,m-1})(e_0-1)+r_0^{\,m-1}y-e_0$. Thus we have

$$\begin{aligned}q_0*p_0={}&(r_0-e_0)+(r_0-r_0^{\,2}+r_0^{\,3}-\dots+r_0^{\,m-1})\cdot\\&(e_0-1)+r_0^{\,m-1}y-e_0+(r_0-e_0)[(r_0-r_0^{\,2}+\\&r_0^{\,3}-\dots+r_0^{\,m-1})(e_0-1)+r_0^{\,m-1}y-e_0]=\\&r_0-r_0e_0+(r_0+r_0^{\,m})(e_0-1)=0\end{aligned}$$

Similarly, $p_0*q_0=0$. Therefore, $q_0\in Q(I)$. This completes the claim.

Now let $e=e_0+e_1x$, $q=q_0+(r_1-e_1)x$. Notice that $r=e+q=(e_0+e_1x)+[q_0+(r_1-e_1)x]$ is a strongly clean expression if and only if there exists $e_1\in I$ such that

$$e_0e_1+e_1e_0=e_1 \tag{3}$$

$$e_1r_0-r_0e_1=r_1e_0-e_0r_1 \tag{4}$$

We will use a virtual 1 again for clarity. Set

$$\begin{aligned}e_1={}&(1-e_0)r_1(1+p_{02})r_0^{n-1}+(1+p_{02})r_0^{n-1}r_1\cdot\\&(1-e_0)+\sum_{i=1}^{m-1}(1-e_0)r_0^ir_1[(1+p_{02})r_0^{n-1}]^{i+1}+\\&\sum_{i=1}^{m-1}[(1+p_{02})r_0^{n-1}]^{i+1}r_1r_0^{\,i}(1-e_0)\end{aligned}$$

Next we only need to check that e_1 as above satisfies Eqs. (3) and (4). Notice that the following hold:

$$e_0(1-e_0)r_1=r_1(1-e_0)e_0=0 \tag{5}$$

$$e_0(1-e_0)r_0^ir_1=r_1r_0^i(1-e_0)e_0=0 \tag{6}$$

$$r_0^{\,n-1}e_0=e_0r_0^{\,n-1}=r_0^{\,n-1}\quad\text{(by Eq. (1))} \tag{7}$$

$$(1+p_{02})r_0^{\,n}=e_0\quad\text{(by Eq. (2))} \tag{8}$$

By Eqs. (5), (6) and (7), we clearly see that e_1 satisfies Eq. (3). For Eq. (4) we directly calculate that

$$\begin{aligned}&e_1r_0-r_0e_1=\\&(1-e_0)r_1(1+p_{02})r_0^n+(1+p_{02})r_0^{n-1}r_1(1-e_0)r_0+\\&\sum_{i=1}^{m-1}(1-e_0)r_0^ir_1[(1+p_{02})r_0^{n-1}]^{i+1}r_0+\\&\sum_{i=1}^{m-1}[(1+p_{02})r_0^{n-1}]^{i+1}r_1r_0^{i+1}(1-e_0)-\\&r_0(1-e_0)r_1(1+p_{02})r_0^{\,n-1}-(1+p_{02})r_0^nr_1(1-e_0)-\\&\sum_{i=1}^{m-1}(1-e_0)\ r_0^{\,i+1}r_1[(1+p_{02})r_0^{\,n-1}]^{\,i+1}-\\&\sum_{i=1}^{m-1}r_0[(1+p_{02})r_0^{\,n-1}]^{\,i+1}r_1r_0^{\,i}(1-e_0)=\\&(1-e_0)r_1e_0+(1+p_{02})r_0^{n-1}r_1(1-e_0)r_0+\\&\sum_{i=1}^{m-1}(1-e_0)r_0^ir_1[(1+p_{02})r_0^{n-1}]^ie_0+\\&\sum_{i=2}^{m}[(1+p_{02})r_0^{n-1}]^ir_1r_0^i(1-e_0)-\\&r_0(1-e_0)r_1(1+p_{02})r_0^{n-1}-e_0r_1(1-e_0)-\end{aligned}$$

$$\sum_{i=2}^{m}(1-e_0)r_0{}^i r_1[(1+p_{02})r_0{}^{n-1}]^i -$$
$$\sum_{i=1}^{m-1}e_0[(1+p_{02})r_0{}^{n-1}]^i r_1 r_0{}^i(1-e_0) \text{ (by Eq. (8)) } =$$
$$(1-e_0)r_1e_0 - e_0r_1(1-e_0) +$$
$$\sum_{i=1}^{m-1}(1-e_0)r_0{}^i r_1[(1+p_{02})r_0{}^{n-1}]^i e_0 +$$
$$\sum_{i=1}^{m}[(1+p_{02})r_0{}^{n-1}]^i r_1 r_0{}^i(1-e_0) -$$
$$\sum_{i=1}^{m}(1-e_0)r_0{}^i r_1[(1+p_{02})r_0{}^{n-1}]^i -$$
$$\sum_{i=1}^{m-1}e_0[(1+p_{02})r_0{}^{n-1}]^i r_1 r_0{}^i(1-e_0) =$$
$$(1-e_0)r_1e_0 - e_0r_1(1-e_0) = r_1e_0 - e_0r_1 \text{(by Eqs. (1) and (7))}$$

Thus the proof is completed.

The proof of theorem 3 adapts to proving the following results.

Corollary 3 Let I be a general ring and M be an (I, I)-bimodule. If I is strongly π-regular, then the trivial extension $\boldsymbol{T}(I, M)$ is strongly clean.

Corollary 4 Let R be a ring and let $r = r_0 + r_1x \in R[x]/(x^2)$. If r_0 or $1 - r_0$ is a strongly π-regular element, then r is strongly clean in $R[x]/(x^2)$.

Proof Notice that r is strongly clean in $R[x]/(x^2)$ if and only if so is $1 - r$. The following proof is similar to theorem 3.

Corollary 5 Let R be a ring and M be an (R, R)-bimodule. If R is strongly π-regular, then the trivial extension $\boldsymbol{T}(R, M)$ is strongly clean.

References

[1] Dischinger M F. Sur les anneaux fortement π-réguliers [J]. *C R Acad Sci Paris: Series A*, 1976, **283**: 571 − 573.

[2] Nicholson W K. Strongly clean rings and Fitting's Lemma [J]. *Comm Algebra*, 1999, **27**(8): 3583 − 3592.

[3] Ara P. Extensions of exchange rings [J]. *J Algebra*, 1997, **197**(2): 409 − 423.

[4] Chen H Y, Chen M S. On strongly π-regular ideals [J]. *J Pure Appl* Algebra, 2005, **195**(1): 21 − 32.

[5] Nicholson W K, Zhou Y Q. Clean general rings [J]. *J Algebra*, 2005, **291**(1): 279 − 311.

[6] Ara P. Strongly π-regular rings have stable range one [J]. *Proc Amer Math Soc*, 1996, **124**(11): 3293 − 3298.

[7] Anderson D D, Camillo V P. Commutative rings whose elements are a sum of a unit and an idempotent [J]. *Comm Algebra*, 2002, **30**(7): 3327 − 3336.

[8] Chen J L, Zhou Y Q. Strongly clean power series rings [J]. *Proc Edinburgh Math Soc*, 2007, **50**(1): 73 − 85.

[9] Nicholson W K. Lifting idempotents and exchange rings [J]. *Trans Amer Math Soc*, 1977, **229**: 269 − 278.

[10] Nicholson W K, Zhou Y Q. Rings in which elements are uniquely the sum of an idempotent and a unit [J]. *Glasgow Math J*, 2004, **46**(2): 227 − 236.

强 π-正则一般环的扩张

王 周 陈建龙

(东南大学数学系,南京 210096)

摘要:介绍了强 π-正则一般环(未必有单位元)的概念并考虑了它的一些扩张. 给出了强 π-正则一般环的 2 个等价刻画,即 I 是强 π-正则一般环当且仅当对于每个 $x \in I$,存在 $n \geqslant 1$ 以及 $y, z \in I$,使得 $x^n = x^{n+1}y = zx^{n+1}$ 当且仅当 I 中的每个元都是强 π-正则的. 还考虑了强 π-正则一般环上的上三角矩阵一般环和平凡扩张,证明了强 π-正则一般环上的上三角矩阵一般环仍是强 π-正则的并且其平凡扩张是强 clean 的.

关键词:强 π-正则一般环;强 clean 一般环;上三角矩阵一般环;平凡扩张

中图分类号:O153.3