Journal of Southeast University (English Edition) Vol. 23, No. 3, pp. 399 – 402 Sept. 2007 ISSN 1003—7985

Fuzzy description logic based on vague sets

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Abstract: To enable the representation and reasoning for fuzzy ontologies with expressive fuzzy knowledge on the semantic web, a new fuzzy extension of description logics called vague ALC which is based on vague sets is presented. The definition of vague set is introduced and then the syntax and semantics of vague ALC are formally defined. The forms of axioms and assertions in the vague ALC knowledge bases are specified. Finally, the tableau algorithm is developed for the reasoning in the vague ALC. The vague ALC based on vague set uses two degrees of membership instead of a single membership degree in the fuzzy sets and is more accurate in representing the imprecision in the degrees of membership. The vague ALC has more expressive power than ALC and can represent fuzzy knowledge and perform reasoning tasks based on them. Therefore, the vague ALC can enable the representation and reasoning for fuzzy ontologies with expressive fuzzy knowledge on the semantic web.

Key words: semantic web; description logic; fuzzy logic; vague sets; tableau algorithm

Description logics (DLs) [1] are the logical foundation of the semantic web, which support knowledge representation and reasoning based on concepts and roles. It should be pointed out, however, that the classical DLs can only deal with crisp knowledge. In the real world, human knowledge and natural language have a great deal of imprecision and vagueness. As a result, fuzzy DLs have been proposed in the literature such as in Refs. [2 – 3] as a way to represent and reason with imprecise and uncertain knowledge. The basic idea of the fuzzy DLs is that, based on Zadeh's fuzzy set theory[4], the fuzzy assertions have expressions of type $\langle \delta, \alpha \rangle$. Here δ is a DL assertion and α is a degree of membership ($\alpha \in [0, 1]$). Note, however, that a single membership degree in the fuzzy sets is inaccurate to represent the imprecision in the membership degrees. In this paper, based on vague sets[5], a fuzzy extension of description logic ALC is presented, called vague ALC. Instead of a crisp degree of membership, two degrees of membership (lower and upper degrees of membership) are used in the vague ALC proposed in this paper. Its syntax, semantics and inference problems are hereby investigated.

1 Vague Sets

Compared with fuzzy sets[4], there are some unique interesting features to vague sets[6] for handling vague data, and, what's more, vague sets have been introduced to deal with imperfect information. Let U be a universe of discourse, where an element of U is denoted by u.

Definition 1 A vague set V in U is characterized by a truth-membership function t_V and a false-membership function f_V. Here $t_V(u)$ is a lower bound on the degree of membership of u derived from the evidence for u, and $f_V(u)$ is a lower bound on the negation of u derived from the evidence against u. Here $t_V(u)$ and $f_V(u)$ are both associated with a real number in the interval $[0, 1]$ with each element in U, where $t_V(u) + f_V(u) \leqslant 1$. Then

$$t_V: U \rightarrow [0, 1] \quad \text{and} \quad f_V: U \rightarrow [0, 1]$$

Suppose that $U = \{u_1, u_2, \ldots, u_n\}$. A vague set V of the universe of discourse U can be represented by

$$V = \{[t_V(u_1), 1 - f_V(u_1)]/u_1, [t_V(u_2), 1 - f_V(u_2)]/u_2, \ldots, [t_V(u_n), 1 - f_V(u_n)]/u_n\}$$

where $t_V(u_i) \leqslant \mu_V(u_i) \leqslant 1 - f_V(u_i)$, $\forall u_i \in U$ and $1 \leqslant i \leqslant n$.

This approach bounds the degree of membership of u to a subinterval $[t_V(u), 1 - f_V(u)]$ of $[0, 1]$. In other words, the exact degree of membership $\mu_V(u)$ of u may be unknown, but is bounded by $t_V(u) \leqslant \mu_V(u) \leqslant 1 - f_V(u)$, where $t_V(u) + f_V(u) \leqslant 1$. The precision of the knowledge about u is characterized by the difference $1 - t_V(u) - f_V(u)$. If the difference is small, the knowledge about u is relatively precise; if it is large, we know correspondingly little. If $t_V(u)$ is equal to $1 - f_V(u)$, the knowledge about u is exact, and the vague

Received 2007-05-18.

Foundation item: Program for New Century Excellent Talents in University (No. NCET-05-0288).

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set reverts back to a fuzzy set. If $t_V(u)$ and $1 - f_V(u)$ are both equal to 1 or 0, depending on whether u does or does not belong to V, the knowledge about u is very exact and the vague set reverts back to an ordinary set. For example, the fuzzy set $\{0.6/u\}$ can be represented as the vague set $\{[0.6, 0.6]/u\}$, while the ordinary set $\{u\}$ can be represented as the vague set $\{[1, 1]/u\}$.

2 Vague ALC

2.1 Syntax and semantics of vague ALC

In fuzzy ALC based on the vague sets (called vague ALC), a concept is interpreted as a vague set. Therefore, concepts and roles become vague. We still have three alphabets of symbols, called primitive vague concepts (denoted by A), primitive vague roles (denoted by R) and vague individuals (denoted by a and b). The vague concepts (denoted by C and D) of the vague ALC are formed out of primitive concepts according to the following abstract syntax:

$C, D \rightarrow \top \mid \bot \mid A \mid C \cap D \mid C \cup D \mid \neg C \mid \forall R.C \mid \exists R.C \mid$

where $\top$, $\bot$, $\cap$, $\cup$, and $\neg$ are used to represent top concept, bottom concept, concept conjunction, concept disjunction, and concept negation, respectively. The symbols $\forall$ and $\exists$ are used to represent universal quantification and existential quantification, respectively.

Now let us focus on the semantics of the vague ALC. The semantics of the vague ALC are provided by a vague interpretation. A vague interpretation I is a pair $I = (\Delta^I, \cdot^I)$. Here Δ^I is a non-empty set of objects and $\cdot^I$ is a vague interpretation function. Vague interpretation function $\cdot^I$ can be defined as follows:

- For an individual $o \in \Delta^I$, $(o)^I = o^I$;
- $A^I: \Delta^I \rightarrow [\alpha, \beta]$, where $\alpha \in [0, 1]$ and $\beta \in [0, 1]$;
- $R^I: \Delta^I \times \Delta^I \rightarrow [\alpha, \beta]$, where $\alpha \in [0, 1]$ and $\beta \in [0, 1]$.

Here α and β are degrees of membership from the truth-membership function t_V and false-membership function f_V in the vague sets, respectively. Let $o, o' \in \Delta^I$, $C^I: \Delta^I \rightarrow [\alpha_C, \beta_C]$, and $D^I: \Delta^I \rightarrow [\alpha_D, \beta_D]$. The semantics of vague ALC concepts and roles are depicted as follows:

- $\top(o) = [1, 1]$;
- $\bot(o) = [0, 0]$;
- $(C \cap D)^I(o) = \min(C^I(o), D^I(o)) = [\min(\alpha_{C(o)}, \alpha_{D(o)}), \min(\beta_{C(o)}, \beta_{D(o)})]$;
- $(C \cup D)^I(o) = \max(C^I(o), D^I(o)) = [\max(\alpha_{C(o)}, \alpha_{D(o)}), \max(\beta_{C(o)}, \beta_{D(o)})]$;
- $(\neg C)^I(o) = [1, 1] - C^I(o) = [\beta_{C(o)}, 1 - \alpha_{C(o)}]$;
- $(\forall R.C)^I(o) = \min_{o' \in I}\{\max\{[1, 1] - R^I(o, o'), C^I(o')\}\} = [\min_{o' \in I}\{\max\{1 - \alpha_{R(o,o')}, \alpha_{C(o')}\}\}, \min_{o' \in I}\{\max\{1 - \beta_{R(o,o')}, \beta_{C(o')}\}\}]$;
- $(\exists R.C)^I(o) = \max_{o' \in I}\{\min\{R^I(o, o'), C^I(o')\}\} = [\max_{o' \in I}\{\min\{\alpha_{R(o,o')}, \alpha_{C(o')}\}\}, \max_{o' \in I}\{\min\{\beta_{R(o,o')}, \beta_{C(o')}\}\}]$.

A vague assertion in ABox is an expression with one of the following types:

- $\langle C(o), [\alpha_C, \beta_C] \rangle$, where o is an instance of C, $\alpha_C \in [0, 1]$ and $\beta_C \in [0, 1]$;
- $\langle R(o, o'), [\alpha_R, \beta_R] \rangle$, where o is related to o' by means of R, $\alpha_R \in [0, 1]$ and $\beta_R \in [0, 1]$.

An interpretation I satisfies a vague assertion $\langle C(o), [\alpha_C, \beta_C] \rangle$ if and only if $\alpha_C \leqslant C^I(o^I) \leqslant \beta_C$. Similarly, I satisfies $\langle R(o, o'), [\alpha_R, \beta_R] \rangle$ if and only if $\alpha_R \leqslant R^I(o^I, o'^I) \leqslant \beta_R$.

A TBox is a finite set of concept axioms. A vague concept axiom is an expression with one of following forms:

- $\langle C \subseteq^* D, [\alpha, \beta] \rangle$, which is called a vague inclusion introduction, the symbol $\subseteq^*$ represents that C is included in D in vague conditions;
- $\langle C \cong D, [\alpha, \beta] \rangle$, which is called a vague equivalence introduction.

As pointed out in Ref. [2], how to deal with general fuzzy concept inclusions still remains an open problem in fuzzy concept languages. So far, a major theoretical and computational limitation of fuzzy DLs has been the inability to handle general concept inclusions[7], which is an important feature of classical DLs. For a vague $C \subseteq^* D$, it is true that $C \subseteq D$ holds with the membership degree, denoted by $[\alpha, \beta]$ ($\alpha \in [0, 1]$ and $\beta \in [0, 1]$). Then an interpretation I satisfies a vague concept axiom $\langle C \subseteq^* D, [\alpha, \beta] \rangle$ if and only if ($\forall o \in \Delta^I$) $(C^I(o) \leqslant D^I(o))$, i.e., $\alpha_C \leqslant \alpha_D$ and $\beta_C \leqslant \beta_D$ for any $o \in \Delta^I$, in which $\alpha = \min_{o \in I}(\alpha_C)$ and $\beta = \min_{o \in I}(\beta_C)$[11]. Similarly, I satisfies $\langle C \cong D, [\alpha, \beta] \rangle$ if and only if $\langle C \subseteq^* D, [\alpha, \beta] \rangle$ and $\langle D \subseteq^* C, [\alpha, \beta] \rangle$.

A vague ALC knowledge base is defined as pair $\Sigma = \langle T, A \rangle$, where T is a vague TBox and A is a vague ABox. An interpretation I satisfies (is a model of) a knowledge base Σ if and only if I satisfies each element in Σ. Given a vague concept axiom or assertion δ, Σ entails δ, written $\Sigma \mid\approx \delta$, if and only if for all models I of Σ, I satisfies δ.

2.2 Inference problems of vague ALC

Let $\Sigma = \langle T, A \rangle$ be a vague ALC knowledge base. Then the inference problems of the vague ALC can also be reduced to vague ABox consistency with regard to T. If ABox has a model, we say it is consistent. We have

- C is α_C-β_C-satisfiable with regard to Σ if and only if $\langle T, \{\alpha_C \leqslant C(o) \leqslant \beta_C\}\rangle$ is satisfiable;
- $\Sigma \mid \approx C \sqsubseteq^* D$ if and only if $\langle T, \{\min_{o \in I}(\alpha_C) \leqslant C(o) \leqslant D(o) \leqslant \min_{o \in I}(\beta_C)\}\rangle$ is satisfiable.

The major inference problems of the vague ALC include concept satisfiability and ABox consistency.

3 Tableau Algorithm for Vague ALC

Many inference tasks can be reduced to the satisfiability of vague ALC concepts. In fact, the satisfiability of concepts is usually checked with tableau algorithms that try to construct a fuzzy tableau for a fuzzy ABox A. In this section, we will design a tableau algorithm for the satisfiability of vague ALC concepts in the style of tableau algorithms in Refs. [2 − 3]. As pointed out in Ref. [2], algorithms that decide consistency of an ABox work on completion-forests rather than on completion-trees. This is because each assertion in an ABox which has lower and upper degrees of membership will be decomposed into four cases according to the tableau rules: $\cup$, $\cap$, $\exists$, $\forall$. This will be illustrated in the following tableau algorithm (see Tab. 1). And because the four cases are very similar to each other, we only give one case.

Such a forest is a collection of trees that correspond to the individuals in the ABox. Nodes in the completion-forest are labeled with a set of triples $\mathscr{L}(o)$ (node triples). More precisely we define $\mathscr{L}(o) = \{\langle * \ \alpha_{C(o)} \bowtie n * \rangle$ or $\langle * \ \beta_{C(o)} \bowtie m * \rangle\}$, where $\alpha_{C(o)}$ is the lower membership degree of an individual o belonging to a concept C; $\beta_{C(o)}$ is the upper membership degree of an individual o belonging to a concept C; $\bowtie = \{=, >, \geqslant, <, \leqslant\}$; $n, m \in [0, 1]$. Furthermore, edges $\langle o, o'\rangle$ are labeled with a set $\mathscr{L}(\langle o, o'\rangle)$ (edge triples) defined as $\mathscr{L}(\langle o, o'\rangle) = \{\langle * \ R_{\langle o, o'\rangle} \bowtie n * \rangle\}$. The algorithm expands the tree either by expanding the set $\mathscr{L}(o)$, or $\mathscr{L}(\langle o, o'\rangle)$, or by adding new leaf nodes.

For the satisfiability of vague ALC concept C_0 w. r. t. an empty TBox to a given degree α_0, β_0, our algorithm starts with an ABox $A = \{\langle C_0(o), [\alpha_0, \beta_0]\rangle\}$, and we first use rule $\bigcirc$ to decompose the assertion to have a result $\mathscr{L}(o) = \{\langle * \ \alpha_{C(o)} = \alpha_0 * \rangle, \langle * \ \beta_{C(o)} = \beta_0 * \rangle\}$, and then use all the other rules.

Tab. 1 Tableau expansion rules

Rule	Description
$\bigcirc$	If $\langle (C)(o), [\alpha_\bigcirc, \beta_\bigcirc]\rangle \in A$, as the definition of vague set, we have $\alpha_{C(o)} = \alpha_\bigcirc, \beta_{C(o)} = \beta_\bigcirc$ and if $\{\langle * \ \alpha_{C(o)} = \alpha_\bigcirc * \rangle, \langle * \ \beta_{C(o)} = \beta_\bigcirc * \rangle\} \not\subseteq \mathscr{L}(o)$; then $\mathscr{L}(o) \to \mathscr{L}(o) \cup \{\langle * \ \alpha_{C(o)} = \alpha_\bigcirc * \rangle, \langle * \ \beta_{C(o)} = \beta_\bigcirc * \rangle\}$.
$\neg$	If $\langle (\neg C)(o), [\alpha_\neg, \beta_\neg]\rangle \in A$, as the vague interpretation I of $\neg$, we have $\beta_{C(o)} = \alpha_\neg$, $1 - \alpha_{C(o)} = \beta_\neg$; then $\alpha_{C(o)} = 1 - \beta_\neg$, $\beta_{C(o)} = \alpha_\neg$; and if $\{\langle * \ \alpha_{C(o)} = 1 - \beta_\neg * \rangle, \langle * \ \beta_{C(o)} = \alpha_\neg * \rangle\} \not\subseteq \mathscr{L}(o)$; then $\mathscr{L}(o) \to \mathscr{L}(o) \cup \{\langle * \alpha_{C(o)} = 1 - \beta_\neg * \rangle, \langle * \ \beta_{C(o)} = \alpha_\neg * \rangle\}$.
$\cup$	If $\langle (C \cup D)(o), [\alpha_\cup, \beta_\cup]\rangle \in A$, as the vague interpretation I of $\cup$, we have $\max(\alpha_{C(o)}, \alpha_{D(o)}) = \alpha_\cup$, $\max(\beta_{C(o)}, \beta_{D(o)}) = \beta_\cup$; then $* * \ \alpha_{C(o)} \leqslant \alpha_\cup, \alpha_{D(o)} = \alpha_\cup, \beta_{C(o)} \leqslant \beta_\cup, \beta_{D(o)} = \beta_\cup$; and if $\{\langle * \ \alpha_{C(o)} \leqslant \alpha_\cup * \rangle, \langle * \ \alpha_{D(o)} = \alpha_\cup * \rangle, \langle * \ \beta_{C(o)} \leqslant \beta_\cup * \rangle, \langle * \ \beta_{D(o)} = \beta_\cup * \rangle\} \not\subseteq \mathscr{L}(o)$; then $\mathscr{L}(o) \to \mathscr{L}(o) \cup \{\langle * \ \alpha_{C(o)} \leqslant \alpha_\cup * \rangle, \langle * \ \alpha_{D(o)} = \alpha_\cup * \rangle, \langle * \ \beta_{C(o)} \leqslant \beta_\cup * \rangle, \langle * \ \beta_{D(o)} = \beta_\cup * \rangle\}$.
$\exists$	If $\langle (\exists R.C)(o), [\alpha_\exists, \beta_\exists]\rangle \in A$, and if o has no R-successor o', as the vague interpretation I of $\exists$: $\max_{o' \in \Delta^I}\{\min\{\alpha_{R(o,o')}, \alpha_{C(o')}\}\} = \alpha_\exists$, $\max_{o' \in \Delta_I}\{\min\{\beta_{R(o,o')}, \beta_{C(o')}\}\} = \beta_\exists$; then $* * \ \alpha_{R(o,o')} \leqslant \alpha_\exists, \beta_{C(o')} \leqslant \beta_\exists$; and if $\{\langle * \ \alpha_{R(o,o')} \leqslant \alpha_\exists * \rangle\} \not\subseteq \mathscr{L}(\langle o, o'\rangle)$ and $\{\langle * \ \beta_{C(o')} \leqslant \beta_\exists * \rangle\} \not\subseteq \mathscr{L}(o')$; then create a new node o', and $\mathscr{L}(\langle o, o'\rangle) \to L(\langle o, o'\rangle) \cup \{\langle * \ \alpha_{R(o,o')} \leqslant \alpha_\exists * \rangle\}$, $\mathscr{L}(o') \to \mathscr{L}(o') \cup \{\langle * \ \beta_{C(o')} \leqslant \beta_\exists * \rangle\}$.

Note: In the above rules $\cup$ and $\exists$, we only list one kind of case, the other three cases can be obtained similarly. Furthermore, the rules $\cap$ and $\forall$ can be obtained similarly according to $\cup$ and $\exists$.

For any individual o, it is said that A contains a clash if $\mathscr{L}(o)$ contains one of the following:

① Two conjugated pairs of triples, for example, $\langle * \ \alpha_{C(o)} \geqslant n \ * \rangle, \langle * \ \beta_{C(o)} \leqslant m \ * \rangle$, and $n > m$;

② One of the triples $\langle * \ \alpha_\perp \geqslant n \ * \rangle, \langle * \alpha_\top \leqslant n \ * \rangle$, with $n > 0$, $n < 1$, $\langle * \ \alpha_\perp > n \ * \rangle, \langle * \ \alpha_\top < n \ * \rangle, \langle * \ \alpha_{C(o)} > 1 \ * \rangle, \langle * \ \alpha_{C(o)} < 0 \ * \rangle$, and $\langle * \ \beta_\perp > n \ * \rangle, \langle * \ \beta_\top < n \ * \rangle, \langle * \beta_{C(o)} > 1 \ * \rangle, \langle * \ \beta_{C(o)} < 0 \ * \rangle$ etc.

③ To all the equations in $\mathscr{L}(o)$, like the form $\langle * \ \alpha_{C(o)} = n_1 * \rangle, \ldots, \langle * \alpha_{C(o)} = n_k * \rangle, \langle * \ \beta_{C(o)} = m_1 * \rangle, \ldots, \langle * \beta_{C(o)} = m_i * \rangle$, if $\max\{n_1 \ldots n_k\} \leqslant \min\{m_1 \ldots m_i\}$.

Otherwise A is clash-free. If none of the tableau expansion rules can be applied to A, then A is complete. While A is clash-free and not complete, apply completion rules to any individual o in A exhaustively. It can yield a complete and clash-free A iff C_0 is α_C-β_C-satisfiable to the given degree α_0, β_0 w. r. t. an empty TBox.

4 Conclusion

In the real world, human knowledge and natural language have a great deal of imprecision and uncertainty. Moreover, the vague set theory is one very important theory for capturing and dealing with vagueness. To this extent, we have presented a fuzzy extension of description logic ALC with vague sets in this paper. We have presented its syntax, semantics and inference problems for the extended language. We also developed an efficient algorithm for reasoning in the vague ALC and analyzed its consistency checking. In the near future, we will analyze the time complexity of the developed algorithm.

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一种基于 vague 集的模糊描述逻辑

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摘要：为实现语义 web 上包含复杂模糊知识模糊本体的表示和推理，提出了一种基于 vague 集的模糊描述逻辑——vague ALC. 首先介绍了 vague 集的定义，然后给出 vague ALC 的语法和语义的形式化描述，并规定 vague ALC 知识库中的公理和断言形式，指出了其推理的基本问题，最后给出了 vague ALC 的检验概念可满足性的 tableau 推理算法. Vague ALC 建立于 vague 集之上，用一个区间来表示一个成员函数的隶属度，解决了 Zadeh 模糊集用单一的数不能准确表示一个成员函数隶属度的问题. Vague ALC 具有比 ALC 更强的表达能力，能够表示复杂的模糊知识并基于它们完成推理任务，因此，vague ALC 可实现语义 web 上包含复杂模糊知识的模糊本体的表示和推理.

关键词：语义 web；描述逻辑；模糊逻辑；vague 集；tableau 算法

中图分类号：TP311