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Exponential stability criteria on neural networks with continuously distributed delays

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Abstract: The exponential stability of a class of neural networks with continuously distributed delays is investigated by employing a novel Lyapunov-Krasovskii functional. Through introducing some free-weighting matrices and the equivalent descriptor form, a delay-dependent stability criterion is established for the addressed systems. The condition is expressed in terms of a linear matrix inequality (LMI), and it can be checked by resorting to the LMI in the Matlab toolbox. In addition, the proposed stability criteria do not require the monotonicity of the activation functions and the derivative of a time-varying delay being less than 1, which generalize and improve earlier methods. Finally, numerical examples are given to show the effectiveness of the obtained methods.

Key words: exponential stability; neural networks; free-weighting matrix; continuously distributed delay; linear matrix inequality

Time-delays inevitably exist in neural networks for various reasons, and it can induce chaos instability in the neural networks. Therefore, stability analysis for neural networks with time-delays has been an attractive subject of research in the past years and many good methods have been proposed[1-8]. On the other hand, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths, and hence there is a distribution of propagation delays over a period of time. It is worth noting that, although the signal propagation is sometimes instantaneous and can be modeled with discrete delays, it may also be distributed during a certain time period so that the distributed delays should be incorporated in the model. In other words, it is often the case that the neural network model possesses both discrete and distributed delays[2].

In view of the importance of both discrete and distributed delays in modeling neural networks, the dynamics analysis problem for neural networks with discrete and distributed delays has received much attention[2-8]. In Ref. [2], a two-neuron network model with multiple discrete and distributed delays has been studied. However, its results cannot be directly applied for general neural networks. In Refs. [3 – 8], by employing various methods, various stability criteria are proposed for the neural networks with distributed delays. However, the restriction that the derivative of time-delay is less than 1 is imposed on those stability criteria in Refs. [3 – 5], and in Refs. [6 – 7], the stability criteria are not presented in terms of LMIs, which renders them somewhat difficultly to be checked. To the best of the authors' knowledge, when the stability criteria are presented in LMIs and derivatives of the time-varying delay can take any value, the exponential stability has not been studied for neural networks with time-varying and continuously distributed delays. This remains important and is investigated in this paper.

1 Problem Formulations

Considering the following neural networks with time-varying and continuously distributed delays:

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{A}\boldsymbol{f}_1(\boldsymbol{x}(t)) + \boldsymbol{B}\boldsymbol{f}_2(\boldsymbol{x}(t-\tau(t))) + \boldsymbol{D}\int_{-\infty}^{t}\boldsymbol{K}(t-s)\boldsymbol{f}_3(\boldsymbol{x}(s))\,\mathrm{d}s \tag{1}$$

where $\boldsymbol{x}(\cdot) = \{x_1(\cdot), \dots, x_n(\cdot)\}^{\mathrm{T}} \in \mathbf{R}^n$ is the neuron state vector; $\boldsymbol{f}_i(\cdot) = \{f_{i1}(\cdot), \dots, f_{in}(\cdot)\}^{\mathrm{T}} \in \mathbf{R}^n (i = 1, 2, 3)$ represents neuron activation functions; $\boldsymbol{C} = \mathrm{diag}(c_1, \dots, c_n)$ is a diagonal matrix with $c_i > 0$; $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}$ are the constant matrices with appropriate dimensions; $\boldsymbol{K}(t-s) = \mathrm{diag}(k_1(t-s), \dots, k_n(t-s))$ and the delay kernel $k_j(\cdot)$ is a real-valued non-negative continuous function defined on $[0, \infty)$; and the initial conditions associated with system (1)

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are of the forms: $x_i(t)=\phi_i(t)$, for all $t\in(-\infty,0]$, $i=1,2,\ldots,n$, where $\phi_i(\cdot)$ denotes a real-valued bounded continuous function defined on $(-\infty,0]$.

The following assumptions on system (1) are made throughout this paper:

(H1) $\tau(t)$ denotes the time-varying delay satisfying $0<\tau(t)\leqslant\tau_m$, $\dot{\tau}(t)\leqslant\mu$, in which τ_m,μ are the constants.

(H2) Each activation function $f_{ij}(\cdot)$ in (1) that satisfies $f_{ij}(0)=0(i=1,2,3)$, is bounded and globally Lipschitz with Lipschitz constants $\sigma_j>0,\delta_j>0,\rho_j>0$ such that

$$|f_{1j}(x)-f_{1j}(y)|\leqslant\sigma_j|x-y|,\quad |f_{2j}(x)-f_{2j}(y)|\leqslant\delta_j|x-y|,\quad |f_{3j}(x)-f_{3j}(y)|\leqslant\rho_j|x-y|$$
$$\forall x,y\in\mathbf{R};\ j=1,2,\ldots,n \tag{2}$$

Here, denote $\boldsymbol{\Sigma}_1=\mathrm{diag}(\sigma_1,\ldots,\sigma_n)$, $\boldsymbol{\Sigma}_2=\mathrm{diag}(\delta_1,\ldots,\delta_n)$, $\boldsymbol{\Sigma}_3=\mathrm{diag}(\rho_1,\ldots,\rho_n)$.

(H3) $\int_0^\infty k_j(s)\mathrm{d}s=1$ for all $j=1,2,\ldots,n$.

(H4) There exists a constant number $\lambda>0$ such that

$$\int_0^\infty k_j(\theta)\mathrm{e}^{2\lambda\theta}\theta\mathrm{d}\theta=\pi_j(\lambda)<+\infty,\quad \int_0^\infty k_j(\theta)\mathrm{e}^{2\lambda\theta}\theta^2\mathrm{d}\theta=\gamma_j(\lambda)<+\infty \qquad j=1,2,\ldots,n$$

Denote $\pi(\lambda)=\max\limits_{1\leqslant j\leqslant n}\{\pi_j(\lambda)\}$, $\gamma(\lambda)=\max\limits_{1\leqslant j\leqslant n}\{\gamma_j(\lambda)\}$.

Remark 1 The systems considered in the paper are general enough to cover many results in the literature. First, the description in (H2) is less restrictive than the usual sigmoid functions. Secondly, the derivative of time-varying delay can take any value but it is not necessarily less than 1, which is more meaningful than the ones in Refs. [1-2,4].

It is clear that under assumption (H2), system (1) has the point $\{0,\ldots,0\}^{\mathrm{T}}$ as its equilibrium point. Then, the problem to be addressed is to develop a condition ensuring that system (1) is globally exponentially stable.

2 Main Results

Throughout this paper, the symmetric term in a symmetric matrix is denoted by $*$. Then, the following definition and lemmas are introduced.

Definition 1 The delayed neural networks (1) is said to be globally exponentially stable if there exist scalars $k>0$ and $\gamma>1$ such that $\|\boldsymbol{x}(t)\|\leqslant\gamma\|\boldsymbol{\phi}\|\mathrm{e}^{-kt}$, where $\|\boldsymbol{\phi}\|=\sup\limits_{-\infty\leqslant\theta\leqslant0}\|\boldsymbol{x}(\theta)\|$. Here, $\|\boldsymbol{x}(\theta)\|$ is the Euclidean norm of $\boldsymbol{x}$, and k is called the exponential convergence rate.

Lemma 1[8] Let $\boldsymbol{D}$, $\boldsymbol{S}$ and $\boldsymbol{P}$ be real matrices of appropriate dimensions and $\boldsymbol{P}>0$. Then for any vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with appropriate dimensions, it has $2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{y}\leqslant\boldsymbol{x}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D}\boldsymbol{x}+\boldsymbol{y}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{S}\boldsymbol{y}$.

Lemma 2[1] For any given $\boldsymbol{\alpha}\in\mathbf{R}^m$, $\boldsymbol{\beta}\in\mathbf{R}^n$, $\boldsymbol{N}\in\mathbf{R}^{m\times n}$, $\boldsymbol{X}\in\mathbf{R}^{m\times m}$, $\boldsymbol{Y}\in\mathbf{R}^{m\times n}$, $\boldsymbol{Z}\in\mathbf{R}^{n\times n}$, if $\begin{bmatrix}\boldsymbol{X} & \boldsymbol{Y}\\ \boldsymbol{Y}^{\mathrm{T}} & \boldsymbol{Z}\end{bmatrix}\geqslant\boldsymbol{0}$ holds, the following inequality holds

$$-2\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{N}\boldsymbol{\beta}\leqslant\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{X} & \boldsymbol{Y}-\boldsymbol{N}\\ * & \boldsymbol{Z}\end{bmatrix}\begin{bmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{bmatrix}$$

Lemma 3[8] Given constant matrices $\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{X}_3$, where $\boldsymbol{X}_1^{\mathrm{T}}=\boldsymbol{X}_1$ and $\boldsymbol{0}<\boldsymbol{X}_2^{\mathrm{T}}=\boldsymbol{X}_2$, then $\boldsymbol{X}_1+\boldsymbol{X}_3^{\mathrm{T}}\boldsymbol{X}_2^{-1}\boldsymbol{X}_3<\boldsymbol{0}$ if and only if $\begin{bmatrix}\boldsymbol{X}_1 & \boldsymbol{X}_3^{\mathrm{T}}\\ * & -\boldsymbol{X}_2\end{bmatrix}<\boldsymbol{0}$, or $\begin{bmatrix}-\boldsymbol{X}_2 & \boldsymbol{X}_3\\ * & \boldsymbol{X}_1\end{bmatrix}<\boldsymbol{0}$.

Theorem 1 For any given scalars $\tau_m>0$ and μ, system (1) with (H2) and time-delay satisfying (H1) has one equilibrium point and is globally exponentially stable if there exist matrices $\boldsymbol{P}>\boldsymbol{0}$, $\boldsymbol{Q}>\boldsymbol{0}$, $\boldsymbol{R}>\boldsymbol{0}$, $\boldsymbol{S}>\boldsymbol{0}$, diagonal matrices $\boldsymbol{T}>\boldsymbol{0}$, $\boldsymbol{K}\geqslant\boldsymbol{0}$, $\boldsymbol{U}>\boldsymbol{0}$, $\boldsymbol{V}>\boldsymbol{0}$, $\boldsymbol{W}\geqslant\boldsymbol{0}$, $\boldsymbol{H}\geqslant\boldsymbol{0}$ and some appropriately dimensional matrices $\boldsymbol{M},\boldsymbol{N},\boldsymbol{P}_i(i=1,2)$, $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}$ such that (3) is true

$$\begin{bmatrix}\boldsymbol{X} & \boldsymbol{Y}\\ * & \boldsymbol{Z}\end{bmatrix}=\begin{bmatrix}\boldsymbol{X}_1 & \boldsymbol{X}_2 & \boldsymbol{Y}_1\\ * & \boldsymbol{X}_3 & \boldsymbol{Y}_2\\ * & * & \boldsymbol{Z}\end{bmatrix}\geqslant\boldsymbol{0},\qquad \boldsymbol{\Xi}=\begin{bmatrix}\boldsymbol{\Omega} & \tau_m\boldsymbol{\Pi}\\ * & -\tau_m\boldsymbol{S}\end{bmatrix}<\boldsymbol{0} \tag{3}$$

where $\boldsymbol{\Pi}=\{\boldsymbol{M}^{\mathrm{T}},\boldsymbol{0},\boldsymbol{N}^{\mathrm{T}},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}\}^{\mathrm{T}}$; $\boldsymbol{\Omega}$ is expressed followingly with

$$\boldsymbol{\Omega}=\begin{bmatrix}\boldsymbol{\Omega}_{11} & \boldsymbol{\Omega}_{12} & -\boldsymbol{Y}_1-\boldsymbol{M}+\boldsymbol{N}^{\mathrm{T}} & \boldsymbol{P}_1^{\mathrm{T}}\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{P}_1^{\mathrm{T}}\boldsymbol{B} & \boldsymbol{0} & \boldsymbol{P}_1^{\mathrm{T}}\boldsymbol{D}\\ * & \boldsymbol{\Omega}_{22} & -\boldsymbol{Y}_2 & \boldsymbol{K}^{\mathrm{T}}+\boldsymbol{P}_2^{\mathrm{T}}\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{P}_2^{\mathrm{T}}\boldsymbol{B} & \boldsymbol{0} & \boldsymbol{P}_2^{\mathrm{T}}\boldsymbol{D}\\ * & * & \boldsymbol{\Omega}_{33} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ * & * & * & -\boldsymbol{U} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ * & * & * & * & -\boldsymbol{V}+\boldsymbol{R} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ * & * & * & * & * & -(1-\mu)\boldsymbol{R}-\boldsymbol{W} & \boldsymbol{0} & \boldsymbol{0}\\ * & * & * & * & * & * & -\boldsymbol{H}+\boldsymbol{T} & \boldsymbol{0}\\ * & * & * & * & * & * & * & -\boldsymbol{T}\end{bmatrix}$$

$$\boldsymbol{\Omega}_{12}=\boldsymbol{\Sigma}_1\boldsymbol{K}+\boldsymbol{P}-\boldsymbol{P}_1^{\mathrm{T}}-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}_2+\tau_m\boldsymbol{X}_2+\boldsymbol{Y}_2^{\mathrm{T}}$$

$$\boldsymbol{\Omega}_{11}=-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}_1-\boldsymbol{P}_1^{\mathrm{T}}\boldsymbol{C}+\boldsymbol{Q}+\tau_m\boldsymbol{X}_1+\boldsymbol{M}+\boldsymbol{M}^{\mathrm{T}}+\boldsymbol{Y}_1+\boldsymbol{Y}_1^{\mathrm{T}}+\boldsymbol{\Sigma}_1^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Sigma}_1+\boldsymbol{\Sigma}_2^{\mathrm{T}}\boldsymbol{V}\boldsymbol{\Sigma}_2+\boldsymbol{\Sigma}_3^{\mathrm{T}}\boldsymbol{H}\boldsymbol{\Sigma}_3$$

$$\boldsymbol{\Omega}_{22}=-\boldsymbol{P}_2-\boldsymbol{P}_2^{\mathrm{T}}+\tau_m\boldsymbol{X}_3+\tau_m(\boldsymbol{S}+\boldsymbol{Z})$$

$$\boldsymbol{\Omega}_{33}=-(1-\mu)\boldsymbol{Q}-\boldsymbol{N}-\boldsymbol{N}^{\mathrm{T}}+\boldsymbol{\Sigma}_2^{\mathrm{T}}\boldsymbol{W}\boldsymbol{\Sigma}_2$$

Proof First, similar to Ref. [1], let $\boldsymbol{\Psi}(t)=\boldsymbol{\Psi}(\boldsymbol{x}(t),\boldsymbol{x}(t-\tau(t)))=\mathrm{diag}(\psi_1(x_1(t),x_1(t-\tau(t))),\ldots,\psi_n(x_n(t),x_n(t-\tau(t))))$, where

$$\psi_i(\alpha,\beta)=\begin{cases}\dfrac{f_{2i}(\alpha)-f_{2i}(\beta)}{\alpha-\beta} & \alpha\neq\beta\\ \delta_i & \alpha=\beta\end{cases}$$

Then, by (2), it is easy to see that

$$\boldsymbol{f}_2(\boldsymbol{x}(t))-\boldsymbol{f}_2(\boldsymbol{x}(t-\tau(t)))=\boldsymbol{\Psi}(t)(\boldsymbol{x}(t)-\boldsymbol{x}(t-\tau(t)))\qquad -\boldsymbol{\Sigma}_2\leqslant\boldsymbol{\Psi}(t)\leqslant\boldsymbol{\Sigma}_2 \tag{4}$$

Therefore, system (1) can be rewritten in the following descriptor system as

$$\left.\begin{aligned}\dot{\boldsymbol{x}}(t)&=\boldsymbol{y}(t)\\ \boldsymbol{y}(t)&=-\boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{A}\boldsymbol{f}_1(\boldsymbol{x}(t))+\boldsymbol{B}\boldsymbol{f}_2(\boldsymbol{x}(t))-\boldsymbol{B}\boldsymbol{\Psi}(t)\int_{t-\tau(t)}^{t}\boldsymbol{y}(s)\mathrm{d}s+\boldsymbol{D}\int_{-\infty}^{t}\boldsymbol{K}(t-s)\boldsymbol{f}_3(\boldsymbol{x}(s))\mathrm{d}s\end{aligned}\right\} \tag{5}$$

Construct the Lyapunov-Krasovskii functional:

$$\boldsymbol{V}(\boldsymbol{x}(t))=\boldsymbol{V}_1(\boldsymbol{x}(t))+\boldsymbol{V}_2(\boldsymbol{x}(t))+\boldsymbol{V}_3(\boldsymbol{x}(t)) \tag{6}$$

where

$$\boldsymbol{V}_1(\boldsymbol{x}(t))=\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{x}(t)+2\sum_{j=1}^{n}k_j\int_0^{x_j}(f_{1j}(s)+\sigma_j s)\mathrm{d}s+\int_{t-\tau(t)}^{t}[\boldsymbol{x}^{\mathrm{T}}(s)\boldsymbol{Q}\boldsymbol{x}(s)+\boldsymbol{f}_2^{\mathrm{T}}(\boldsymbol{x}(s))\boldsymbol{R}\boldsymbol{f}_2(\boldsymbol{x}(s))\mathrm{d}s$$

$$\boldsymbol{V}_2(\boldsymbol{x}(t))=\int_{-\tau_m}^{0}\int_{t+\theta}^{t}\boldsymbol{y}^{\mathrm{T}}(s)(\boldsymbol{S}+\boldsymbol{Z})\boldsymbol{y}(s)\mathrm{d}s\mathrm{d}\theta,\quad \boldsymbol{V}_3(\boldsymbol{x}(t))=\sum_{j=1}^{n}t_j\int_0^{\infty}k_j(\theta)\int_{t-\theta}^{t}f_{3j}^2(x_j(s))\mathrm{d}s\mathrm{d}\theta$$

with $\boldsymbol{P}>\boldsymbol{0}$, $\boldsymbol{Q}>\boldsymbol{0}$, $\boldsymbol{R}>\boldsymbol{0}$, $\boldsymbol{S}>\boldsymbol{0}$, $\boldsymbol{Z}\geqslant\boldsymbol{0}$, $\boldsymbol{T}=\mathrm{diag}(t_1,\ldots,t_n)>\boldsymbol{0}$, and $\boldsymbol{K}=\mathrm{diag}(k_1,\ldots,k_n)\geqslant\boldsymbol{0}$. It is clear from (2) that, for any diagonal matrices $\boldsymbol{U}>\boldsymbol{0}$, $\boldsymbol{V}>\boldsymbol{0}$, $\boldsymbol{W}\geqslant\boldsymbol{0}$ and $\boldsymbol{H}\geqslant\boldsymbol{0}$,

$$\begin{aligned}\boldsymbol{0}\leqslant{}&[\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{\Sigma}_1^{\mathrm{T}}\boldsymbol{U}\boldsymbol{\Sigma}_1\boldsymbol{x}(t)-\boldsymbol{f}_1^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{U}\boldsymbol{f}_1(\boldsymbol{x}(t))+\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{\Sigma}_2^{\mathrm{T}}\boldsymbol{V}\boldsymbol{\Sigma}_2\boldsymbol{x}(t)-\boldsymbol{f}_2^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{V}\boldsymbol{f}_2(\boldsymbol{x}(t))+\\ &\boldsymbol{x}^{\mathrm{T}}(t-\tau(t))\boldsymbol{\Sigma}_2^{\mathrm{T}}\boldsymbol{W}\boldsymbol{\Sigma}_2\boldsymbol{x}(t-\tau(t))-\boldsymbol{f}_2^{\mathrm{T}}(\boldsymbol{x}(t-\tau(t)))\boldsymbol{W}\boldsymbol{f}_2(\boldsymbol{x}(t-\tau(t)))+\\ &\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{\Sigma}_3^{\mathrm{T}}\boldsymbol{H}\boldsymbol{\Sigma}_3\boldsymbol{x}(t)-\boldsymbol{f}_3^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{H}\boldsymbol{f}_3(\boldsymbol{x}(t))]\end{aligned} \tag{7}$$

Now, the derivative of $\boldsymbol{V}_1(x(t))$ along the trajectories of system (5) yields

$$\begin{aligned}\dot{\boldsymbol{V}}_1(\boldsymbol{x}(t))\leqslant{}&2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{y}(t)+2[\boldsymbol{f}_1^{\mathrm{T}}(\boldsymbol{x}(t))+\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{\Sigma}_1]\boldsymbol{K}\boldsymbol{y}(t)+[\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{x}(t)+\boldsymbol{f}_2^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{R}\boldsymbol{f}_2(\boldsymbol{x}(t))]-\\ &(1-\mu)[\boldsymbol{x}^{\mathrm{T}}(t-\tau(t))\boldsymbol{Q}\boldsymbol{x}(t-\tau(t))+\boldsymbol{y}_2^{\mathrm{T}}(\boldsymbol{x}(t-\tau(t)))\boldsymbol{R}\boldsymbol{y}_2(\boldsymbol{x}(t-\tau(t)))]\end{aligned} \tag{8}$$

According to (5), it can be deduced that

$$\begin{aligned}2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{y}(t)={}&2\boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{G}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{y}(t)\\ \boldsymbol{0}\end{bmatrix}=2\boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{G}^{\mathrm{T}}\left\{\begin{bmatrix}\boldsymbol{y}(t)\\ -\boldsymbol{y}(t)-\boldsymbol{C}\boldsymbol{x}(t)\end{bmatrix}+\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{A}\end{bmatrix}\boldsymbol{f}_1(\boldsymbol{x}(t))+\right.\\ &\left.\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}\end{bmatrix}\boldsymbol{f}_2(\boldsymbol{x}(t))-\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}\end{bmatrix}\boldsymbol{\Psi}(t)\int_{t-\tau(t)}^{t}\boldsymbol{y}(s)\mathrm{d}s+\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{D}\end{bmatrix}\int_{-\infty}^{t}\boldsymbol{K}(t-s)\boldsymbol{f}_3(\boldsymbol{x}(s))\mathrm{d}s\right\}\end{aligned} \tag{9}$$

where $\boldsymbol{\eta}^{\mathrm{T}}\{t)=\{\boldsymbol{x}^{\mathrm{T}}(t),\ \boldsymbol{y}^{\mathrm{T}}(t)\}$, $\boldsymbol{G}=\begin{bmatrix}\boldsymbol{P} & \boldsymbol{0}\\ \boldsymbol{P}_1 & \boldsymbol{P}_2\end{bmatrix}$ and $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ are the constant matrices of appropriate dimensions. Using lemma 2, it follows from (4) that

$$\begin{aligned}-2\boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{G}^{\mathrm{T}}[\boldsymbol{0}\quad\boldsymbol{B}^{\mathrm{T}}]^{\mathrm{T}}\boldsymbol{\Psi}(t)\int_{t-\tau(t)}^{t}\boldsymbol{y}(s)\mathrm{d}s\leqslant{}&\tau_m\boldsymbol{\eta}^{\mathrm{T}}(t)\boldsymbol{X}\boldsymbol{\eta}(t)+\int_{t-\tau(t)}^{t}\boldsymbol{y}^{\mathrm{T}}(s)\boldsymbol{Z}\boldsymbol{y}(s)\mathrm{d}s+\\ &2\boldsymbol{\eta}^{\mathrm{T}}(t)\{\boldsymbol{Y}(\boldsymbol{x}(t)-\boldsymbol{x}(t-\tau(t)))-\boldsymbol{G}^{\mathrm{T}}[\boldsymbol{0}\quad\boldsymbol{B}^{\mathrm{T}}]^{\mathrm{T}}(\boldsymbol{f}_2(\boldsymbol{x}(t))-\boldsymbol{f}_2(\boldsymbol{x}(t-\tau(t)))\}\end{aligned} \tag{10}$$

Next, together with Eqs. (8) to (10), we can obtain

$$\dot{V}_1(\boldsymbol{x}(t)) \leqslant 2[\boldsymbol{f}_1^{\mathrm{T}}(\boldsymbol{x}(t)) + \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{\Sigma}_1]\boldsymbol{K}\boldsymbol{y}(t) + 2\boldsymbol{\eta}^{\mathrm{T}}(t)\left\{\boldsymbol{G}^{\mathrm{T}}\begin{bmatrix} \boldsymbol{y}(t) \\ -\boldsymbol{y}(t) - \boldsymbol{C}\boldsymbol{x}(t)\end{bmatrix} + \boldsymbol{G}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{A}\end{bmatrix}\boldsymbol{f}_1(\boldsymbol{x}(t)) + \right.$$
$$\left.\boldsymbol{G}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}\end{bmatrix}\boldsymbol{f}_2(\boldsymbol{x}(t-\tau(t))) + \boldsymbol{G}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{D}\end{bmatrix}\int_{-\infty}^{t}\boldsymbol{K}(t-s)\boldsymbol{f}_3(\boldsymbol{x}(s))\mathrm{d}s + \frac{\tau_m}{2}\boldsymbol{X}\boldsymbol{\eta}(t) + \boldsymbol{Y}(\boldsymbol{x}(t) - \boldsymbol{x}(t-\tau(t)))\right\} +$$
$$\int_{t-\tau(t)}^{t}\boldsymbol{y}^{\mathrm{T}}(s)\boldsymbol{Z}\boldsymbol{y}(s)\mathrm{d}s + [\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{f}_2^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{R}\boldsymbol{f}_2(\boldsymbol{x}(t))] - (1-\mu)[\boldsymbol{x}^{\mathrm{T}}(t-\tau(t))\boldsymbol{Q}\boldsymbol{x}(t-\tau(t)) +$$
$$\boldsymbol{f}_2^{\mathrm{T}}(\boldsymbol{x}(t-\tau(t)))\boldsymbol{R}\boldsymbol{f}_2(\boldsymbol{x}(t-\tau(t)))] \tag{11}$$

$$\dot{V}_2(\boldsymbol{x}(t)) \leqslant \tau_m\boldsymbol{y}^{\mathrm{T}}(t)(\boldsymbol{S}+\boldsymbol{Z})\boldsymbol{y}(t) - \int_{t-\tau(t)}^{t}\boldsymbol{y}^{\mathrm{T}}(s)\boldsymbol{S}\boldsymbol{y}(s)\mathrm{d}s - \int_{t-\tau(t)}^{t}\boldsymbol{y}^{\mathrm{T}}(s)\boldsymbol{Z}\boldsymbol{y}(s)\mathrm{d}s \tag{12}$$

$$\dot{V}_3(x(t)) \leqslant \boldsymbol{f}_3^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{T}\boldsymbol{f}_3(\boldsymbol{x}(t)) - \left(\int_{-\infty}^{t}\boldsymbol{K}(t-s)\boldsymbol{f}_3(\boldsymbol{x}(s))\mathrm{d}s\right)^{\mathrm{T}}\boldsymbol{T}\left(\int_{-\infty}^{t}\boldsymbol{K}(t-s)\boldsymbol{f}_3(\boldsymbol{x}(s))\mathrm{d}s\right) \tag{13}$$

The following equality is true for any matrices $\boldsymbol{M},\boldsymbol{N}$ with appropriate dimensions

$$\boldsymbol{0} = 2[\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{M} + \boldsymbol{x}^{\mathrm{T}}(t-\tau(t))\boldsymbol{N}]\left[\boldsymbol{x}(t) - \boldsymbol{x}(t-\tau(t)) - \int_{t-\tau(t)}^{t}\boldsymbol{y}(s)\mathrm{d}s\right] \tag{14}$$

Now, adding the terms on the right of (7), and (11) to (14) to $\dot{V}(\boldsymbol{x}(t))$, it has

$$\dot{V}(\boldsymbol{x}(t)) \leqslant \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t}\boldsymbol{\xi}^{\mathrm{T}}(t,s)\begin{bmatrix}\boldsymbol{\Omega} & -\tau(t)\boldsymbol{\Pi}\\ * & -\tau(t)\boldsymbol{S}\end{bmatrix}\boldsymbol{\xi}(t,s)\mathrm{d}s$$

where $\boldsymbol{\xi}^{\mathrm{T}}(t,s) = \left\{\boldsymbol{\eta}^{\mathrm{T}}(t),\ \boldsymbol{x}^{\mathrm{T}}(t-\tau(t)),\ \boldsymbol{f}_1^{\mathrm{T}}(\boldsymbol{x}(t)),\ \boldsymbol{f}_2^{\mathrm{T}}(\boldsymbol{x}(t)),\ \boldsymbol{f}_2^{\mathrm{T}}(\boldsymbol{x}(t-\tau(t))),\ \boldsymbol{f}_3^{\mathrm{T}}(\boldsymbol{x}(t)),\ \left(\int_{-\infty}^{t}\boldsymbol{K}(t-s)\boldsymbol{f}_3(\boldsymbol{x}(s))\mathrm{d}s\right)^{\mathrm{T}},\ \boldsymbol{y}^{\mathrm{T}}(s)\right\}$,

and $\boldsymbol{\Omega}$ is defined in Eq. (3). Letting $\boldsymbol{\Theta} = \begin{bmatrix}\boldsymbol{\Omega} & -\tau(t)\boldsymbol{\Pi}\\ * & -\tau(t)\boldsymbol{S}\end{bmatrix}$ and by lemma 3, it can be shown that $\boldsymbol{\Theta}<\boldsymbol{0}$ is equivalent to $\boldsymbol{\Omega}+(-\tau(t)\boldsymbol{\Pi})(\tau(t)\boldsymbol{S})^{-1}(-\tau(t)\boldsymbol{\Pi})^{\mathrm{T}}<\boldsymbol{0}$, and $\boldsymbol{\Xi}<\boldsymbol{0}$ in Eq. (3) equals $\boldsymbol{\Omega}+(\tau_m\boldsymbol{\Pi})(\tau_m\boldsymbol{S})^{-1}(\tau_m\boldsymbol{\Pi})^{\mathrm{T}}<\boldsymbol{0}$. Hence, $\boldsymbol{\Xi}<\boldsymbol{0}$ can guarantee $\boldsymbol{\Theta}<\boldsymbol{0}$. Defining $\boldsymbol{\Delta}=\boldsymbol{\Omega}+(\tau_m\boldsymbol{\Pi})(\tau_m\boldsymbol{S})^{-1}(\tau_m\boldsymbol{\Pi})^{\mathrm{T}}$, from lemma 3 and (3), we have $\dot{V}(x(t)) \leqslant \lambda_{\max}(\Delta)\|\boldsymbol{x}(t)\|^2<\boldsymbol{0}$, for all $\boldsymbol{x}(t)\neq\boldsymbol{0}$.

Then let $\bar{V}(\boldsymbol{x}(t)) = \mathrm{e}^{2kt}V(\boldsymbol{x}(t))$, it has

$$\dot{\bar{V}}(\boldsymbol{x}(t)) = 2k\mathrm{e}^{2kt}V(\boldsymbol{x}(t)) + \mathrm{e}^{2kt}\dot{V}(\boldsymbol{x}(t)) \Rightarrow \bar{V}(\boldsymbol{x}(t)) - V(\boldsymbol{x}(0)) \leqslant \int_0^t[2k\mathrm{e}^{2ks}V(\boldsymbol{x}(s)) + \mathrm{e}^{2ks}\lambda_{\max}(\Delta)\,|\boldsymbol{x}(s)|^2]\mathrm{d}s \tag{15}$$

With lemma 1, (H1) to (H4) and defining $\rho = \max_{1\leqslant i\leqslant n}\{\rho_i\}$, it is easy to have

$$V(\boldsymbol{x}(0)) \leqslant \bar{\Lambda}\|\boldsymbol{\phi}\|^2 \tag{16}$$

where $\Lambda = \lambda_{\max}(\boldsymbol{P}) + 4\lambda_{\max}(\boldsymbol{\Sigma}_1\boldsymbol{K}) + \tau_m\lambda_{\max}(\boldsymbol{Q}) + \tau_m\lambda_{\max}(\boldsymbol{\Sigma}_2^{\mathrm{T}}\boldsymbol{R}\boldsymbol{\Sigma}_2) + 2\tau_m{}^2[\lambda_{\max}(\boldsymbol{S}) + \lambda_{\max}(\boldsymbol{Z})][\lambda_{\max}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}) + \lambda_{\max}(\boldsymbol{\Sigma}_1{}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\Sigma}_1) + \lambda_{\max}(\boldsymbol{\Sigma}_2{}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{\Sigma}_2) + \lambda_{\max}(\boldsymbol{\Sigma}_3{}^{\mathrm{T}}\boldsymbol{\Sigma}_3)\lambda_{\max}(\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D})] + \rho^2\pi(0)\lambda_{\max}(\boldsymbol{T})$, and

$$V(\boldsymbol{x}(s)) \leqslant \Theta_0\|\boldsymbol{x}(s)\|^2 + \Theta_1\int_{s-\tau}^{s}\|\boldsymbol{x}(v)\|^2\mathrm{d}v + \sum_{j=1}^{n}t_j\int_0^{\infty}k_j(\theta)\int_{s-\theta}^{s}f_{3j}^2(x_j(v))\mathrm{d}v\mathrm{d}\theta \tag{17}$$

where $\Theta_0 = \lambda_{\max}(\boldsymbol{P}) + 4\lambda_{\max}(\boldsymbol{\Sigma}_1\boldsymbol{K})$, $\Theta_1 = \lambda_{\max}(\boldsymbol{Q}) + \lambda_{\max}(\boldsymbol{\Sigma}_2^{\mathrm{T}}\boldsymbol{R}\boldsymbol{\Sigma}_2) + 2\tau_m\lambda_{\max}(\boldsymbol{S}+\boldsymbol{Z})[\lambda_{\max}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}) + \lambda_{\max}(\boldsymbol{\Sigma}_1^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\Sigma}_1) + \lambda_{\max}(\boldsymbol{\Sigma}_2^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{\Sigma}_2) + \lambda_{\max}(\boldsymbol{\Sigma}_3{}^{\mathrm{T}}\boldsymbol{\Sigma}_3)\lambda_{\max}(\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D})]$.

Then from Eqs. (15) to (17) and (H4), it can de deduced that

$$\bar{V}(\boldsymbol{x}(t)) \leqslant [2k\Theta_0 + \lambda_{\max}(\boldsymbol{\Delta}) + 2k\Theta_1\tau_m\mathrm{e}^{2k\tau_m} + 2k\pi(k)\lambda_{\max}(\boldsymbol{\Sigma}_3{}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{\Sigma}_3)]\int_0^t\mathrm{e}^{2kv}\|\boldsymbol{x}(v)\|^2\mathrm{d}v +$$
$$[2k\Theta_1\tau_m^2\mathrm{e}^{2k\tau_m} + 2k\gamma(k)\lambda_{\max}(\boldsymbol{\Sigma}_3^{\mathrm{T}}\boldsymbol{T}\boldsymbol{\Sigma}_3) + \bar{\Lambda}]\|\boldsymbol{\phi}\|^2$$

Choose a $k_0>0$ such that $0 \geqslant 2k_0\Theta_0 + \lambda_{\max}(\boldsymbol{\Delta}) + 2k_0\Theta_1\tau_m\mathrm{e}^{2k_0\tau_m} + 2k_0\pi(k_0)\lambda_{\max}(\boldsymbol{\Sigma}_3^{\mathrm{T}}\boldsymbol{T}\boldsymbol{\Sigma}_3)$. Then,

$$\bar{V}(\boldsymbol{x}(t)) \leqslant [2k_0\gamma(k_0)\lambda_{\max}(\boldsymbol{\Sigma}_3{}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{\Sigma}_3) + 2k_0\Theta_1\tau_m^2e^{2k_0\tau_m} + \bar{\Lambda}]\|\boldsymbol{\phi}\|^2 = \$\|\boldsymbol{\phi}\|^2$$

Meanwhile, $\bar{V}(\boldsymbol{x}(t)) \geqslant \mathrm{e}^{2k_0t}\lambda_{\min}(\boldsymbol{P})\|\boldsymbol{x}(t)\|^2$, therefore $\|\boldsymbol{x}(t)\| \leqslant \left[\frac{\$}{\lambda_{\min}(\boldsymbol{P})}\right]^{\frac{1}{2}}\|\boldsymbol{\phi}\|\mathrm{e}^{-k_0t}$. Using definition 1, it can be concluded that system (1) is exponentially stable and it completes our proof.

3 Numerical Examples

Example 1 Consider a two-neuron neural network (1):

$$\dot{\boldsymbol{x}}(t) = -\begin{bmatrix}1 & 0\\ 0 & 0.9\end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix}1 & -1.7\\ -1.6 & 1\end{bmatrix}\boldsymbol{f}(\boldsymbol{x}(t)) + \begin{bmatrix}1 & 0.6\\ 0.5 & 0.8\end{bmatrix}\boldsymbol{f}(\boldsymbol{x}(t-\tau(t))) \tag{18}$$

with $f(s) = \{0.2\tanh(s),\ 0.2\tanh(s)\}^{\mathrm{T}}$, $\tau(t) = \sin^2 t + 0.01$. By theorem 1, the method shows system (18) to be globally exponentially stable. However, theorem 1 in Ref. [1] fails to do this.

Example 2 Consider the neural network (1):

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{B}\boldsymbol{f}(\boldsymbol{x}(t-\tau(t))) + \boldsymbol{D}\int_{-\infty}^{t}\boldsymbol{K}(t-s)\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s \tag{19}$$

with $\boldsymbol{C} = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} 1 & -1.7 \\ -1.6 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & 0.6 \\ 0.5 & 0.8 \end{bmatrix}$, $\boldsymbol{D} = \begin{bmatrix} 0.4 & 0.3 \\ 0.1 & 0.2 \end{bmatrix}$, $\tau(t) = \cos^2 t + 0.01$, $\boldsymbol{f}(s) = \{0.2\tanh(s),\ 0.2\tanh(s)\}^{\mathrm{T}}$, $\boldsymbol{K}(t-s) = \mathrm{diag}\{\mathrm{e}^{-(t-s)}, 2\mathrm{e}^{-2(t-s)}\}$. Then by theorem 1 in this paper, it is easy to obtain that (19) is exponentially stable. However, according to theorem 1 in Ref. [10], $\boldsymbol{\Delta} = \boldsymbol{C} - |\boldsymbol{A}|\boldsymbol{\Sigma}_1 - |\boldsymbol{B}|\boldsymbol{\Sigma}_2 - |\boldsymbol{D}|\boldsymbol{\Sigma}_3 = \begin{bmatrix} 0.42 & -0.52 \\ -0.44 & 0.40 \end{bmatrix}$, and its successive principle minors are 0.42 and −0.060 8, respectively. According to definition 1 in Ref. [8], it is not an M-matrix. Thus theorem 1 in Ref. [8] cannot prove its exponential stability.

4 Conclusion

In this paper, through introducing some free-weighting matrices and the equivalent descriptor form of the addressed system, one sufficient condition for the exponential stability of the equilibrium point is derived for a class of neural networks with time-varying and continuously distributed delays. Finally, two numerical examples are used to demonstrate the usefulness of the main results.

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具有连续分布时滞神经网络系统的指数稳定性准则

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摘要:考虑了一类具有时变和连续分布时滞神经网络系统的指数稳定性问题. 通过引入 Lyapunov-Krasovskii 泛函、自由权矩阵和等价的描述系统形式,针对所考虑的系统建立了一个时滞相关的指数稳定性准则. 该准则以线性矩阵不等式的形式给出,能够很容易地用 Matlab 工具箱 LMI 进行检验. 此外,所得到的结论不需要激励函数的单调性且变时滞的导函数只要有上界结论就可以成立,这拓展和发展了现有的一些结论. 最后通过 2 个数值例子说明了所得结论的有效性.

关键词:指数稳定性; 神经网络; 自由权矩阵; 连续分布时滞; 线性矩阵不等式

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