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# Quadratic integrability of solutions based on a class of delayed systems

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**Abstract:** Some properties such as oscillation, stability, existence of periodic solutions and quadratic integrability of solutions based on a class of second order nonlinear delayed systems are analyzed by using the V-function, the Lyapunov functional or the Beuman-Bihari inequality, and some sufficient conditions based on those properties are given. Finally, the conclusions are applied to over-voltage models based on three-phase nonsynchronous closing of switches appearing in the power systems, the results in accord with the background physical meaning are obtained. And all the conditions of the conclusions are easy to validate, so the conclusions have definite theoretical meaning and are easy to apply in practice.

**Key words:** nonlinear delayed system; quadratic integrability; periodic solution; oscillatory solution; over-voltage

In recent years, the solution characteristics of a class of second order nonlinear differential systems have been studied, and a series of pleasing results have been devoted to problems such as oscillation, asymptotic stability and stability, etc. Some results have been obtained by using Gronwall's inequality or its generalizations of this type[1], and some results have been obtained by using the Lyapunov functional[2], analysis[3] or the existence of periodic solutions[4]. But, the problem related to some properties of the second order nonlinear functional differential system, such as oscillation, stability, existence of periodic solutions and so on, has not been discussed in depth.

Let us consider the second order nonlinear functional differential systems as follows:

$$x''(t)+p(t)x'(t)+q_1(t)x(t)+q_2(t)x(t-\tau)+g_1(t,x(t))+g_2(t,x(t-\tau))=f(t) \tag{1}$$

where $\tau\in(0,\infty)$, $f\in C([t_0-\tau,\infty),\mathbf{R})$, $p(t)$, $q_i(t)\in C([t_0-\tau,\infty),\mathbf{R})$, $i=1,2$, and $g_1(t,x)$, $g_2(t,x_\tau)\in C([t_0-\tau,\infty)\times\mathbf{R},\mathbf{R})$, the initial function $\varphi(t)$ is continuous which is defined in $[t_0-\tau,t_0]$; $\mathbf{R}$ denotes a real set, $t$ is the time variable. Some results related to the boundary and the quadratic integrability of solutions of second order nonlinear differential systems with $\tau=0$ and $g_i(\cdot)=0\,(i=1,2)$ can be found in Ref. [5]. In this paper, by using the theoretical analysis, some solution characteristics of Eq. (1) can be obtained; and the V-function, the Lyapunov functional and the Beuman-Bihari inequality are used. Some sufficient conditions are given. Finally, our conclusions are applied to the over-voltage models in the power systems, and some results in accord with the physical meaning are obtained.

First, we present the following assumption and the basic inequality.

**Assumption** Suppose that $p(t)$, $f(t)$ are locally integrable on $[t_0-\tau,\infty)$, $q_i(t)\in C^2([t_0-\tau,\infty),\mathbf{R})$, and $|g_1(t,x(t))|\leqslant h_1(t)|x(t)|$, $|g_2(t,x(t-\tau))|\leqslant h_2(t)|x(t-\tau)|$, where $h_i(t)\in C([t_0-\tau,\infty),\mathbf{R}^+)$, and there exist two nonnegative constants $g_i\,(i=1,2)$ such that $|h_i(t)|\leqslant g_i$.

**Basic inequality** Assume that $a>0$, $b\geqslant 0$, then $-ax^2+bx\leqslant -\dfrac{a}{2}x^2+\dfrac{b^2}{2a}$ for $x\in\mathbf{R}^+$.

Before stating the main results, the following lemmas are needed:

**Lemma 1** (Beuman-Bihari inequality[6]) Assume that $v(t)$, $p(t)$, $g(t)\in([t_0,\infty),\mathbf{R}^+)$, and satisfy the following inequality $v(t)\leqslant v_0+\int_{t_0}^t p(s)v(s)\,\mathrm{d}s+\int_{t_0}^t q(s)v^\gamma(s)\,\mathrm{d}s$, $t\geqslant t_0$, where $v_0\geqslant 0$, $\gamma\in(0,1]$, then

1) $v(t)\leqslant \exp\left(\int_{t_0}^t p(s)\,\mathrm{d}s\right)\left\{v_0^{1-\gamma}+(1-\gamma)\int_{t_0}^t q(u)\exp\left(\int_{t_0}^u(\gamma-1)p(s)\,\mathrm{d}s\right)\mathrm{d}u\right\}^{\frac{1}{1-\gamma}}$ for $\gamma\in(0,1)$ and $t\geqslant t_0$;

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2) $v(t) \leqslant v_0 \exp\left(\int_{t_0}^{t} (p(s) + q(s))\mathrm{d}s\right)$ for $\gamma = 1$ and $t \geqslant t_0$.

**Lemma 2**[6] Consider the functional differential system as follows:

$$x'(t) = F(x_t), \quad x_t = x(t+\theta) \qquad -\gamma \leqslant \theta \leqslant 0; t \geqslant 0 \tag{2}$$

where $F$ is a continuous vector function, and $\forall H_1 < H$, there exists $L(H_1) > 0$ such that $|F(\phi)| \leqslant L(H_1)$ for $\|\phi\| \leqslant H_1$. Assume that $F(0) = 0$, there exists a continuous functional $V(\phi)$ satisfying the Lipschitz condition, and that

1) $W_1(|\phi(0)|) \leqslant V(\phi) \leqslant W_2(\|\phi(\theta)\|)$, where $W_i(i=1,2)$ are the wedge functions;

2) $\dot{V}_{(4)}(\phi) \leqslant 0$;

3) Let $\Lambda = \{\phi: \dot{V}_{(4)}(\phi) = 0\}$, the maximal invariant set $\Omega = \{0\}$ of the set $\Lambda$.

Then the zero solution of (2) is asymptotic stability.

**Lemma 3**[7] Consider the following functional differential system

$$x' = F(t, x_t), \quad x_t = x(t+\theta) \qquad -\tau \leqslant \theta \leqslant 0; t \geqslant 0 \tag{3}$$

and assume that

1) $F(t+T, \varphi) \equiv F(t, \varphi)(T \geqslant \tau > 0)$, $F$ is a continuous vector function, and $\forall M > 0$, then there exists a non-negative constant $L(M) > 0$ such that $|F(t, \varphi)| \leqslant L(M)$ for $\|\varphi\| \leqslant M$;

2) There exists a continuous functional $V(t, \varphi): \mathbf{R} \times C^H \to \mathbf{R}$ satisfying the Lipschitz condition, where $C^H \overset{\text{def}}{=} \{\varphi \in C: |\varphi(0)| \geqslant H\}$;

3) There exist two continuous increasing functions $a(s) > 0, b(s) > 0$ for $s \geqslant H$, and $a(s) \to \infty\ (s \to \infty)$, satisfying $a(|\varphi(0)|) \leqslant V(t, \varphi) \leqslant b(\|\varphi\|)$;

4) There exists a continuous function $w(s) > 0$ for $S \geqslant H$ such that $V'(t, \varphi)_{(5)} \leqslant -w(|\varphi(0)|)$;

5) There exist five constants $H_1, \alpha, \beta, \nu, \gamma$ such that $b(H_1) \leqslant a(\alpha)$, $b(\alpha) \leqslant a(\beta)$, $b(\beta) \leqslant a(\nu)$, $b(\nu) \leqslant a(\gamma)$, where $H_1 > H$ and $\nu L(\gamma) < H_1 - H$.

Then for system (3) there exists a periodic solution $x(t)$ in the periodic $T$.

## 1 Boundary Results without Delay

In this section, we consider the boundary and integrable in $L^2$ of the solutions of Eq. (1) without delay, i. e., considering the case $\tau = 0$ as follows:

$$x''(t) + p(t)x'(t) + q_1(t)x(t) + g(x(t)) = f(t) \tag{4}$$

where $q(t) = q_1(t) + q_2(t)$, $|g(t, x(t))| \leqslant h_1(t)|x|^{\alpha}$, $\alpha \in [0, 1]$. We have the following theorem.

**Theorem 1** Suppose that Eq. (4) satisfies

1) $q_1(t) \in C^2([a, \infty), \mathbf{R}^+)$, $q_2(t)$, $\dfrac{h_1(t)}{\sqrt{q_1(t)}}$ are locally integrable on $[a, \infty)$;

2) There exists a continuous derivable function $F(t)$ satisfying $F(t) > 0$, $F'(t) \geqslant 0$, $Q(t) = \frac{1}{2}(q_1'F + 2pq_1F - F'q_1) > 0$;

3) $F(t)q_2^2(t)Q^{-1}(t)\lambda^{-2}(t)$, $f^2(t)F^2(t)Q^{-1}(t) \in L[a, \infty)$, where $\lambda^2(t) = [Q(q_1F' + Q) + \sqrt{Q^2(q_1F' + Q)^2 + 4q_1q_2^2F^2Q^2}]/(2Q^2)$.

Then there exists $x(t)$ being a solution of Eq. (4) and such solutions bounded on $[a, \infty)$.

**Proof** Suppose that $x(t)$ is a solution of Eq. (4) that is defined in $[a, T](T \leqslant \infty)$.

Let us consider the V-function $V(t) = F(t)(x^2(t) + [x'(t)]^2)$, $t \in [a, T)$, then

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4)} = F'(x^2 + q_1^{-1}x'^2) + F[(-q_1'q_1^{-2} - 2pq_1^{-1})x'^2 - 2q_2q_1^{-1}xx' - 2gq_1^{-1}x' + 2fq_1^{-1}x'] =$$
$$F'x^2 - 2Fq_2q_1^{-1}xx' - 2Qq_1^{-2}x'^2 + 2fFq_1^{-1}x' - 2gFq_1^{-1}x'$$

However, the basic inequality implies that $-2Qq_1^{-1}x'^2 + 2|f|Fq_1^{-1}|x'| \leqslant -Qq_1^{-2}x'^2 + f^2F^2Q^{-1}$. This shows that $2|x||x'| \leqslant \sqrt{q_1}F^{-1}V(t)$, $x^2 \leqslant V(t)F^{-1}$.

Therefore, $2|g|Fq_1^{-1}|x'| \leqslant 2h_1Fq_1^{-1}|x|^{\alpha}|x'| \leqslant h_1q^{-\frac{1}{2}}F^{\frac{1-\alpha}{2}}[V(t)]^{\frac{1+\alpha}{2}}$. This implies that $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4)} \leqslant F'x^2 - 2Fq_2q_1^{-1}xx' - Qq_1^{-2}x'^2 + f^2F^2Q^{-1} + hq^{-\frac{1}{2}}F^{\frac{1-\alpha}{2}}[V(t)]^{\frac{1+\alpha}{2}}$.

Let $E(t)=F'x^2Fq_2q_1^{-1}xx'-Qq_1^{-2}x'^2$, then

$$E(t)=F'x^2-Qq_1^2[\lambda x'+q_1q_2F\lambda^{-1}Q^{-1}x]^2+Qq_1^{-2}(\lambda^2-1)x'^2+q_2^2F^2\lambda^{-2}Q^{-1}x^2\leqslant F'x^2+q_2^2F^2\lambda^{-2}Q^{-1}x^2+Qq_1^{-2}(\lambda^2-1)x'^2$$

This implies $E(t)\leqslant(F'+q_2^2F^2\lambda^{-2}Q^{-1})(x^2+q_1^{-1}x'^2)=(F'F^{-1}+Fq_2^2\lambda^{-2}Q^{-1})V(t)$, i. e., $\left.\frac{dV}{dt}\right|_{(4)}\leqslant(F'F^{-1}+Fq_2^2\lambda^{-2}Q^{-1})V(t)+h_1q_1^{-\frac{1}{2}}F^{\frac{1-\alpha}{2}}V^{\frac{1+\alpha}{2}}+f^2F^2Q^{-1}$, integrating the above inequality from $a$ to $t$, we have $V(t)\leqslant k+\int_a^t R(s)V(s)\mathrm{d}s+\int_a^t h_1q_1^{-\frac{1}{2}}F^{\frac{1-\alpha}{2}}V^{\frac{1+\alpha}{2}}\mathrm{d}s$, where $k=V(a)+\int_a^\infty f^2F^2Q^{-1}\mathrm{d}t>0$ and $R(s)=F'F^{-1}+Fq_2^2\lambda^{-2}Q^{-1}$.

According to lemma 1, these imply the following results:

$$V(t)\leqslant\exp\Big(\int_a^t R(s)\mathrm{d}s\Big)\Big\{k^{\frac{1-\alpha}{2}}+\frac{1-\alpha}{2}\int_a^t h_1q_1^{-\frac{1}{2}}F^{\frac{1-\alpha}{2}}\exp\Big(\frac{\alpha-1}{2}\int_a^s R(u)\mathrm{d}u\Big)\mathrm{d}s\Big\}^{\frac{2}{1-\alpha}}=$$

$$F(t)F^{-1}(a)\exp\Big(\int_a^t Fq_2^2\lambda^{-2}Q^{-1}\mathrm{d}s\Big)\Big\{k^{\frac{1-\alpha}{2}}+\frac{1-\alpha}{2}F(a)^{\frac{1-\alpha}{2}}\int_a^t h_1q_1^{-\frac{1}{2}}\exp\Big(\frac{\alpha-1}{2}\int_a^s Fq_2^2\lambda^{-2}Q^{-1}\mathrm{d}u\Big)\mathrm{d}s\Big\}^{\frac{2}{1-\alpha}}$$

for any $\alpha\in[0,1)$; and $V(t)\leqslant k\exp\Big(\int_a^t(R(s)+h_1q_1^{-\frac{1}{2}})\mathrm{d}s\Big)=kF(t)F^{-1}(a)\exp\Big(\int_a^t(Fq_2^2\lambda^{-2}Q^{-1}+h_1q_1^{-\frac{1}{2}})\mathrm{d}s\Big)$ for $\alpha=1$.

By the conditions, we can know that $V(t)F^{-1}$ is bounded in $[a,T)$, i. e., $x(t)$ is bounded in $[a,T)$. Applying the extended theorem of ordinary differential system solutions, therefore, $x(t)$ exists on $[a,\infty)$ and is bounded. This completes the proof.

**Theorem 2** Assume that Eq. (4) satisfies

1) $q_i(t)>0(i=1,2)$, $p(t)>0$, $q_1'(t)>0$, $t\in[a,\infty)$;

2) There is a $\delta>0$ such that $|q_1'q_1^{-\frac{3}{2}}|\leqslant\delta<4$;

3) $\left|\frac{\mathrm{d}}{\mathrm{d}t}(q_1'q_1^{-\frac{2}{3}})\right|$, $q_1^{-\frac{1}{2}}$, $|q_2|q_1^{-\frac{1}{2}}$, $pq_1'q_1^{-\frac{3}{2}}$, $h_1q_1^{-\frac{1+\alpha}{4}}$, $f^2q_1'q_1^{-\frac{3}{2}}q_2^{-1}$, $f^2q_1^{-\frac{1}{2}}p^{-1}\in L[a,\infty)$.

Then all solutions of Eq. (4) are bounded on $[a,\infty)$ and integrated on $L^2$.

**Proof** Assume that $x(t)$ is a solution of Eq. (4). Let us consider the V-function as follows: $V(t)=\sqrt{q_1}x^2+q_1^{-\frac{1}{2}}x'^2$, then $\left.\frac{dV}{dt}\right|_{(4)}=\frac{1}{2}q'_1q_1^{-\frac{1}{2}}x^2-2q_1^{-\frac{1}{2}}xx'-\Big(\frac{1}{2}q_1^{-\frac{3}{2}}q'_1+2pq_1^{-\frac{1}{2}}\Big)x'+2fq_1^{-\frac{1}{2}}x'-2gq_1^{-\frac{1}{2}}x'$.

The following proof process is similar to that of theorem 1.

## 2 Oscillatory Results

In this section, we consider the oscillation of Eq. (1). Let $y(t)=x'(t)$, then the equivalent system of Eq. (1) is represented as

$$\left.\begin{aligned}&x'(t)=y(t)\\&y'(t)=f(t)-p(t)y(t)-q_1(t)x(t)-q_2(t)x(t-\tau)-g_1(t,x(t))-g_2(t,x(t-\tau))\end{aligned}\right\}\quad(5)$$

We have the following theorems.

**Theorem 3** Assume that

1) $p(t),q_i(t)\in C([t_0-\tau,\infty),\mathbf{R}^+)$, $g_i(t,x)\operatorname{sign}x\geqslant0(i=1,2)$;

2) There exist two continuous functions $F(t)$, $G(t)$ such that $f(t)=F'(t)=G''(t)$, and $F(t)$ is an oscillatory function;

3) $\int_T^\infty[q_1(s)G_\pm(s)+q_2(s)G_\pm(s-\tau)+p(s)F_\pm(s)]\mathrm{d}s=\infty$, where $T>\tau$, and $F_+(t)=\max(0,F(t))$, $F_-(t)=\max(0,-F(t))$.

Then all solutions of Eq. (5) are oscillatory.

**Proof** Assume that $(x(t),y(t))$ is an eventually positive solution of Eq. (5), then there exists $T_1>\tau$ such that $x(t)>0$, $x(t-\tau)>0$, $y(t)>0$, $y(t-\tau)>0$ for any $t>T_1$.

Hence $[y(t)-F(t)]'=-g_1(t,x(t))-g_2(t,x(t-\tau))-q_1(t)x(t)-q_2(t)x(t-\tau)-p(t)y(t)<0$ for any $t>T_1$; i. e., $x(t)$ is increasing and $y(t)-F(t)$ is decreasing. Therefore, two cases may occur for $y(t)-F(t)$:

1) There exists $T_2>T_1$ such that $y(t)-F(t)\leqslant0$ for $t>T_2$. However, $F(t)$ is an oscillatory function; there-

fore, this is impossible.

2) There exists $T_3 > T_1$ such that $y(t) - F(t) > 0$ for $t > T_3$. Then $y(t) > F_+(t)$ and $\lim_{t\to+\infty}[y(t) - F(t)] = k \geqslant 0$, $k$ is a constant. However, $x'(t) = y(t) > F(t) = G'(t)$ for $t > T_3$, i. e., $[x(t) - G(t)]' > 0$. Similar to the above proof, this implies that $x(t) > G_+(t)$. Then, by using the following inequality $-[y(t) - F(t)]' = g_1(t, x(t)) + g_2(t, x(t-\tau)) + q_1(t)x(t) + q_2(t)x(t-\tau) + p(t)y(t) > q_1(t)G_+(t) + q_2(t)G_+(t-\tau) + p(t)F_+(t)$, and integrating the above inequality from $T > T_3 + \tau$ to $t > T$ implies that $y(T) - F(T) - y(t) + F(t) > \int_T^t [q_1(s)G_+(s) + q_2(s)G_+(s-\tau) + p(s)F_+(s)]\,\mathrm{d}s \to +\infty\ (t\to+\infty)$. This is contradictory to $\lim_{t\to+\infty}[y(t) - F(t)] = k \geqslant 0$; i. e., system (1) does not contain any eventually positive solutions, nor any eventually negative solutions for the same reason. This completes our proof.

**Theorem 4** Assume that

1) $p(t), q_i(t) \in C([t_0 - \tau, \infty), \mathbf{R}^+)$, $g_i(t, x)\,\mathrm{sign}x \geqslant 0\ (i = 1, 2)$;

2) There exist two continuous functions $F(t)$, $G(t)$ such that $f(t) = F'(t) = G''(t)$, and there exist two constants $G_i\ (i = 1, 2)$ such that $G_1 \leqslant G(t) \leqslant G_2$, and there exist two sequences $\{t_m\}$, $\{t_n\}$ such that $\lim_{m\to\infty} G(t_m) = G_1$, $\lim_{n\to\infty} G(t_n) = G_2$;

3) The solutions of the following linear systems $z''(t) + q_1(t)z(t) + q_2(t)z(t-\tau) = -q_1(t)G(t) - q_2(t)G(t-\tau) + (q_1(t) + q_2(t))G_i\ (i = 1, 2)$ are oscillatory.

Then all the solutions of Eq. (5) are oscillatory.

**Proof** The proof is similar to that of theorem 3.

## 3 Stability and Existence of Periodic Solutions

In this section, the stability and the existence of the periodic solutions of Eq. (1) are considered. The equivalent system of Eq. (5) can be denoted as

$$\left.\begin{aligned} &x'(t) = y(t) \\ &y'(t) = f(t) - p(t)y(t) - [q_1(t) + q_2(t)]x(t) - g_1(t, x(t)) + g_2(t+\tau, x(t)) + \\ &\qquad \int_{t-\tau}^{t} \Big[\frac{\mathrm{d}(q_2 x)}{\mathrm{d}s} + g_v(s+\tau, x(s))\Big] y(s)\,\mathrm{d}s \end{aligned}\right\} \tag{6}$$

where $g_v(u, v) = \dfrac{\mathrm{d}g_2}{\mathrm{d}v}$. We can conclude some results as follows.

**Theorem 5** Assume that $f(t) \equiv 0$, and

1) $q_i(t) \in C^1([a, \infty), \mathbf{R}^+)$, and there exist two nonnegative constants $q_{0i}\ (i = 1, 2)$ such that $q_i'(t) \leqslant -q_{0i}\ (i = 1, 2)$;

2) $p(t) \in C([a, \infty), \mathbf{R})$, and there exists a nonnegative constant $p$ such that $p(t) \geqslant p$;

3) $g_i\ (i = 1, 2)$ are continuous derivable functions, and there exists a nonnegative constant $g_0$ such that $|g_1(t, x(t)) + g_2(t+\tau, x(t))| \leqslant g_0 |x(t)|$;

4) There exists a nonnegative constant $l$ such that $\left|\dfrac{\mathrm{d}(q_2 x)}{\mathrm{d}t} + g_v(t, x(t))\right| \leqslant l$.

Then the zero solutions of Eq. (6) are stable for $0 \leqslant \tau^2 \leqslant \dfrac{[g_0^2 - p(q_{01} + q_{02})]^2}{4l^2(q_{01} + q_{02})}$ and $p_0(q_{01} + q_{02}) - g_0^2 > 0$.

**Proof** Let us choose the continuous Lyapunov function as follows: $V(x_t, y_t) = [q_1(t) + q_2(t)]x^2 + y^2 + k\int_{-\tau}^{0}\int_{t+s}^{t} y^2(\theta)\,\mathrm{d}s\mathrm{d}\theta$, where $k$ is a parameter that is determined later.

$$\begin{aligned} V'(x_t, y_t)\Big|_{(6)} &= 2y\Big[-p(t)y(t) - g_1(t, x(t)) + g_2(t+\tau, x(t)) + \int_{t-\tau}^{t}\Big[\frac{\mathrm{d}(q_2 x)}{\mathrm{d}s} + g_v(s+\tau, x(s))\Big]y(s)\,\mathrm{d}s\Big] + \\ &\quad k\int_{-\tau}^{0}[y^2(t) - y^2(t+s)]\,\mathrm{d}s + x^2[q_1'(t) + q_2'(t)] \leqslant -2py^2 + 2g_0|xy| - (q_{01} + q_{02})x^2 + 2l\int_{t-\tau}^{t}|y(t)||y(s)|\,\mathrm{d}s + \\ &\quad k\int_{-\tau}^{0}[y^2(t) - y^2(t+s)]\,\mathrm{d}s \leqslant -2py^2 + \frac{{g_0}^2}{(q_{01} + q_{02})}y^2 + k\tau y^2 + \frac{\tau l^2}{k}y^2 \end{aligned}$$

Let us choose the parameter $k$ satisfying

$$\frac{p(q_{01}+q_{02})-g_0^2-\sqrt{[g_0^2-p(q_{01}+q_{02})]^2-4l^2\tau^2(q_{01}+q_{02})^2}}{2\tau(q_{01}+q_{02})}\leqslant k\leqslant$$
$$\frac{p(q_{01}+q_{02})-g_0^2+\sqrt{[g_0^2-p(q_{01}+q_{02})]^2-4l^2\tau^2(q_{01}+q_{02})^2}}{2\tau(q_{01}+q_{02})}$$

Therefore, $V'(x_t,y_t)\mid_{(6)}\leqslant -py^2\leqslant 0$, and Eq. (6) and $V'(x_t,y_t)\Big|_{(6)}=0$ imply that $x=y=0$; i. e., the conditions of lemma 2 are satisfied. This completes our proof.

For the periodic solutions existence of Eq. (6), we have the following results.

**Theorem 6** Assume that

1) $p(t)$ is a continuous function in the period $T$, and there exists a nonnegative constant $p$ such that $p(t)>p$;

2) $q_i(t)\in C^1([a,\infty),\mathbf{R})$, and there exist two nonnegative constants $q_{0i}(i=1,2)$ such that $q_i'(t)\leqslant -q_{0i}(i=1,2)$;

3) $f(t)$ is the continuous function in the period $T$, and there exists a nonnegative constant $f_0$ such that $|f(t)|\leqslant f_0$;

4) $g_1(t,x(t)), g_2(t,x(t-\tau))$ are the continuous derivable functions and satisfy the Lipschitz condition, and there exist two nonnegative constants $l$, $g_0$ such that $|g_1(t,x(t))+g_2(t+\tau,x(t))|\leqslant g_0|x|$, $\left|\frac{\mathrm{d}(q_2x)}{\mathrm{d}t}+g_v(t,x(t-\tau))\right|\leqslant l.$

Then there is a positive constant $\tau_0$ when $0\leqslant\tau\leqslant\tau_0$, for Eq. (6) there exists a periodic solution in the period $T$.

**Proof** Let us choose the continuous Lyapunov function as follows:

$$W(x_t,y_t)=(q_1(t)+q_2(t))x^2+y^2+\frac{l}{2\tau}\int_{-\tau}^{0}\int_{t+s}^{t}y^2(\theta)\,\mathrm{d}s\mathrm{d}\theta+l\int_{-\tau}^{0}\int_{t+s}^{t}|y(\theta)|\,\mathrm{d}s\mathrm{d}\theta$$

The following proof process is similar to that of theorem 5.

## 4 Quadratic Integrability

In this section, we consider the quadratic integrability of solutions of the second order nonlinear functional differential system (1). We have the following results.

**Theorem 7** Assume that

1) There exist two nonnegative constants $g_i(i=1,2)$ such that $|g_1(t,x(t))|\leqslant g_1|x(t)|$, $|g_2(t,x(t-\tau))|\leqslant g_2|x(t-\tau)|$;

2) $q_1(t)=q_{01}(t)+q_{02}(t)$, $q_{01}(t)\in C^1[t_0-\tau,\infty)$, $q_{01}(t)>0$, $q_{02}(t), q_2(t), p(t), f(t)\in C[t_0-\tau,\infty)$;

3) There exists a continuous derivable function $F(t)>0$, $F'(t)\geqslant 0$ and $Q(t)=Fq_{01}^{-2}q_{01}'+2Fpq_{01}^{-1}-F'q_{01}^{-1}>0$, $H(t)=2Q\tau F'+2Qq_{01}^{-2}F^2(g_2+q_2)^2$ such that $\int_{t_0-\tau}^{\infty}\frac{f^2F^2q_{01}^{-2}}{Q}\mathrm{d}t<\infty$, $\int_{t_0-\tau}^{\infty}\frac{F^2q_{01}^{-2}(q_{02}^2+g_1^2)}{Q\lambda^2}\mathrm{d}t<\infty$, $\int_{t_0-\tau}^{\infty}\frac{Fq_{01}^{-2}(g_2+q_2)^2}{\tau}\mathrm{d}t<\infty$, where $\lambda^2(t)=\frac{1+H\tau^{-1}Q^{-2}q_{01}^{-1}+\sqrt{(1+H\tau^{-1}Q^{-1}q_{01}^{-1})^2+16F^2Q^{-2}\tau^{-1}q_{01}^{-3}(q_{02}^2+g_1^2)}}{2}$.

Then the solutions of Eq. (1) satisfy that $|x(t)|=O(1)$, $|x'(t)|=O(\sqrt{q_{01}(t)})$, $t\to\infty$.

**Proof** Let us consider the V-function as follows:

$$V(t)=F(t)(x^2(t)+q_{01}^{-1}(t)(x'(t))^2)+\int_{-\tau}^{0}\int_{t}^{t+s}x^2(\theta-\tau)\,\mathrm{d}\theta\mathrm{d}s$$

then

$$\frac{\mathrm{d}V}{\mathrm{d}t}\Big|_{(1)}=F'(x^2+q_{01}^{-1}x'^2)+F(2xx'+2q_{01}^{-1}x'x''-q_{01}^{-2}q_{01}'x'^2)+\int_{-\tau}^{0}x^2(t+s-\tau)\,\mathrm{d}s-\tau x^2(t-\tau)=$$
$$F'x^2+(F'q_{01}^{-1}-Fq_{01}^{-2}q_{01}')x'^2+2Fxx'+2Fq_{01}^{-1}x'[f-px'-(q_{01}+q_{02})x-q_2x(t-\tau)-$$
$$g_1(t,x(t))-g_2(t,x(t-\tau))]+\int_{-\tau}^{0}x^2(t+s-\tau)\,\mathrm{d}s-\tau x^2(t-\tau)=F'x^2+(F'q_{01}^{-1}-Fq_{01}^{-2}q_{01}'-$$
$$2Fpq_{01}^{-1})x'^2+2fFq_{01}^{-1}x'-2Fq_{01}^{-1}q_2x'x(t-\tau)-2Fq_{01}^{-1}x'g_1(t,x(t))-2Fq_{01}^{-1}x'g_2(t,x(t-\tau))-2Fq_{01}^{-1}q_{02}xx'+$$

$$\int_{-\tau}^{0} x^2(t+s-\tau)\mathrm{d}s - \tau x^2(t-\tau) \leqslant F'x^2 + (F'q_{01}^{-1} - Fq_{01}^{-2}q_{01}' - 2Fpq_{01}^{-1})x'^2 + 2fFq_{01}^{-1}x' - 2Fq_{01}^{-1}q_{02}xx' +$$
$$2Fq_{01}^{-1}g_1 \mid xx' \mid - \tau x^2(t-\tau) + 2q_{01}^{-1}F(g_2+q_2) \mid xx(t-\tau) \mid + \int_{-\tau}^{0} x^2(t+s-\tau)\mathrm{d}s \leqslant$$
$$\left(F' + \frac{q_{01}^{-2}F^2(g_2+q_2)^2}{\tau}\right)x^2 + (F'q_{01}^{-1} - Fq_{01}^{-2}q_{01}' - 2Fpq_{01}^{-1})x'^2 + 2fFq_{01}^{-1}x' - 2Fq_{01}^{-1}q_{02}xx' +$$
$$2Fq_{01}^{-1}g_1 \mid xx' \mid + \int_{-\tau}^{0} x^2(t+s-\tau)\mathrm{d}s$$

By the basic inequality, we have

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)} \leqslant \left(F' + \frac{q_{01}^{-2}F^2(q_2+g_2)^2}{\tau}\right)x^2 - \frac{Q}{2}x'^2 + \frac{2f^2F^2q_{01}^{-2}}{Q} - 2Fq_{01}^{-1}q_{02}xx' + 2Fq_{01}^{-1}g_1 \mid xx' \mid + \int_{-\tau}^{0} x^2(t+s-\tau)\mathrm{d}s$$

Let $E(t) = \left(F' + \frac{q_{01}^{-2}F^2(q_2+g_2)^2}{\tau}\right)x^2 - Fq_{01}^{-1}q_{02}xx' + 2Fq_{01}^{-1}g_1 \mid xx' \mid - \frac{Q}{2}x'^2$, and consider that

$$E(t) = \left(F' + \frac{q_{01}^{-2}F^2(q_2+g_2)^2}{\tau}\right)x^2 - \frac{Q}{2}\left(\lambda x' + \frac{2Fq_{01}^{-1}q_{02}}{\lambda Q}x\right)^2 - \frac{Q}{2}\left(\lambda \mid x' \mid - \frac{2Fq_{01}^{-1}g_1}{\lambda Q} \mid x \mid\right)^2$$
$$\frac{Q}{2}(\lambda^2-1)x'^2 + \frac{4F^2q_{01}^{-2}(q_{02}^2+g_1^2)}{2Q\lambda^2}x^2 \leqslant \left(F' + \frac{q_{01}^{-2}F^2(q_2+g_2)^2}{\tau} + \frac{4F^2q_{01}^{-2}(q_{02}^{\ 2}+g_1^{\ 2})}{2Q\lambda^2}\right)x^2 + \frac{Q}{2}(\lambda^2-1)x'^2$$

where $\lambda(t)$ is a function which is determined as: $F' + \frac{q_{01}^{-2}F^2(q_2+g_2)^2}{\tau} + \frac{4F^2q_{01}^{-2}(q_{02}^{\ 2}+g_1^{\ 2})}{2Q\lambda^2} = \frac{Qq_{01}}{2}(\lambda^2-1)$. Then, we have $\lambda^2(t) = \frac{1 + HQ^{-2}q_{01}^{-1}\tau^{-1} + \sqrt{(1+HQ^{-2}q_{01}^{-1}\tau^{-1})^2 + 16F^2\tau^{-1}Q^{-2}q_{01}^{-3}(q_{02}^{\ 2}+g_1^{\ 2})}}{2} > 1$, this implies that

$$E(t) \leqslant \left(\frac{H}{2\tau Q} + \frac{4F^2q_{01}^{-2}(q_{02}^2+g_1^{\ 2})}{2Q\lambda^2}\right)\left(x^2 + \frac{x'^2}{q_{01}}\right) \leqslant \left(\frac{H}{2\tau QF} + \frac{4Fq_{01}^{-2}(q_{02}^2+g_1^{\ 2})}{2Q\lambda^2}\right)V(t)$$

Therefore, $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)} \leqslant \left(\frac{H}{2\tau QF} + \frac{4Fq_{01}^{-2}(q_{02}^2+g_1^2)}{2Q\lambda^2}\right)V(t) + \frac{2f^2F^2q_{01}^{-2}}{Q} + \int_{-\tau}^{0} x^2(t+s-\tau)\mathrm{d}s.$

Integrating this inequality from $a$ to $t$, we have

$$V(t) \leqslant V(t_0-\tau) + \int_{t_0-\tau}^{t} \frac{2f^2F^2q_{01}^{-2}}{Q}\mathrm{d}s + \int_{t_0-\tau}^{t}\left(\frac{H}{2\tau QF} + \frac{4Fq_{01}^{-2}(q_{02}^2+g_1^2)}{2Q\lambda^2}\right)V(s)\mathrm{d}s + \int_{t_0-\tau}^{t}\int_{-\tau}^{0} x^2(t+s-\tau)\mathrm{d}t\mathrm{d}s \leqslant$$
$$M + \int_{t_0-\tau}^{t}\left(\frac{H}{2\tau QF} + \frac{4Fq_{01}^{-2}(q_{02}^2+g_1^2)}{2Q\lambda^2}\right)V(s)\mathrm{d}s$$

where $M = V(t_0-\tau) + \int_{t_0-\tau}^{\infty} \frac{2f^2F^2q_{01}^{-2}}{Q}\mathrm{d}t + \int_{-\tau}^{0}\mathrm{d}s\int_{t_0-\tau}^{\infty} x^2(t+s-\tau)\mathrm{d}t > 0$ is a constant.

The Gronwall's inequality implies that $V(t) \leqslant M\frac{F(t)}{F(t_0-\tau)}\exp\left\{\int_{t_0-\tau}^{\infty}\left[\frac{Fq_{01}^{-2}(g_2+q_2)^2}{\tau} + \frac{4Fq_{01}^{-2}(q_{02}^2+g_1^2)}{2Q\lambda^2}\right]\mathrm{d}t\right\}$.

This completes our proof by the conditions and the $V$-function.

In the case of $F(t)=1$ or $F(t)=q_{01}(t)$, we have the following results.

**Corollary 1** Assume that

1) There exist two nonnegative constants $g_i$ ($i=1,2$) such that $\mid g_1(t, x(t)) \mid \leqslant g_1 \mid x(t) \mid$, $\mid g_2(t, x(t-\tau)) \mid \leqslant g_2 \mid x(t-\tau) \mid$;

2) Let $Q_1(t) = q_{01}^{-2}q_{01}' + 2pq_{01}^{-1} > 0, H_1(t) = 2Q_1q_{01}^{-2}(g_2+q_2)^2$ satisfy that

$$\int_{t_0-\tau}^{\infty} \frac{f^2q_{01}^{-2}}{Q_1}\mathrm{d}t < \infty, \quad \int_{t_0-\tau}^{\infty} \frac{q_{01}^{-2}(q_{02}^{\ 2}+g_1^{\ 2})}{Q_1\lambda^2}\mathrm{d}t < \infty, \quad \int_{t_0-\tau}^{\infty} \frac{q_{01}^{-2}(g_2+q_2)^2}{\tau}\mathrm{d}t < \infty$$

where $\lambda^2(t) = \frac{1 + H_1\tau^{-1}Q_1^{-2}q_{01}^{-1} + \sqrt{(1+H_1\tau^{-1}Q_1^{-1}q_{01}^{-1})^2 + 4Q_1^{-2}\tau^{-1}q_{01}^{-3}(q_{02}^2+g_1^2)}}{2}$.

Then the solutions of Eq. (1) satisfy that $\mid x(t) \mid = O(1)$, $\mid x'(t) \mid = O(\sqrt{q_{01}(t)})$, $t \to \infty$.

**Corollary 2** Assume that

1) There exists a nonnegative constant $g_0$ such that $\mid g(t, x(t-\tau)) \mid \leqslant g_0 \mid x(t-\tau) \mid$;

2) Let $Q_2(t) = q_{01}^{-1}q_{01}' + 2p - q_{01}'q_{01}^{-1} > 0$, $H_2(t) = 2Q_2\tau q_{01}' + 2Q_2(g_2+q_2)^2$ such that $\int_{t_0-\tau}^{\infty} \frac{f^2}{Q_2}\mathrm{d}t < \infty$,

$\int_{t_0-\tau}^{\infty}\frac{q_{02}^2+g_1^2}{Q_2\lambda^2}\mathrm{d}t<\infty$, $\int_{t_0-\tau}^{\infty}\frac{q_{01}^{-1}(g_2+q_2)^2}{\tau}\mathrm{d}t<\infty$, where $\lambda^2(t)=\frac{1+H_2\tau^{-1}Q_2^{-2}q_{01}^{-1}+\sqrt{(1+H_2\tau^{-1}Q_2^{-1}q_{01}^{-1})^2+4Q^{-2}\tau^{-1}q_{01}^{-1}(q_{02}^{\ 2}+g_1^{\ 2})}}{2}$.

Then the solutions of Eq. (1) satisfy that $|x(t)|=O(1)$, $|x'(t)|=O(\sqrt{q_{01}(t)})$, $t\to\infty$.

**Theorem 8** Assume that

1) There exist two nonnegative constants $g_i\ (i=1,2)$ such that $|g_1(t,x(t))|\leqslant g_1|x(t)|$, $|g_2(t,x(t-\tau))|\leqslant g_2|x(t-\tau)|$;

2) Let $\widetilde{Q}=\frac{1}{2}q_{01}^{-\frac{3}{2}}q_{01}'+2q_{01}^{-\frac{1}{2}}p-2\tau q_{01}^{-1}(g_2+q_2)^2$, and there exists a constant $\delta$ such that $\widetilde{Q}\leqslant\delta<4$, satisfying that $\int_{t_0-\tau}^{\infty}\left(\frac{\widetilde{Q}|f^2|}{4q_{01}}+\frac{2|f^2|}{\widetilde{Q}q_{01}}\right)\mathrm{d}s<\infty$, $\int_{t_0-\tau}^{\infty}q_{01}^{-\frac{1}{2}}\mathrm{d}t<\infty$, and $\int_{t_0-\tau}^{\infty}\left(\left|\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}t}\right|+p\widetilde{Q}+4q_{01}^{-1}q_{02}+2q_{01}^{-1}q_{01}'+2\widetilde{Q}g_1q_{01}^{-\frac{1}{2}}-4\widetilde{Q}q_{01}^{\frac{1}{2}}-2\widetilde{Q}q_{02}q_{01}^{-\frac{1}{2}}\right)\mathrm{d}t<\infty$, for $0<\tau\leqslant\min\left\{\frac{q_{01}}{\widetilde{Q}(g_2+q_2)^2},\frac{q_{01}^{-\frac{1}{2}}q_{01}'+4q_{01}^{\frac{1}{2}}p}{4(g_2+q_2)^2}\right\}$, $2\widetilde{Q}g_1+2q_{01}^{-\frac{1}{2}}q_{01}'\geqslant\widetilde{Q}(q_{01}+2q_{02})$.

Then the solutions $x(t)$ of Eq. (1) satisfy $x(t)\in L^2[t_0-\tau,\infty)\cap L^\infty[t_0-\tau,\infty)$.

**Proof** Let us consider the $V$-function as: $V(t)=\sqrt{q_{01}(t)}(x^2(t)+q_{01}^{-1}(t)(x'(t))^2)+\frac{1}{\tau^2}\int_{-\tau}^{0}\int_{t}^{t+s}x^2(\theta-\tau)\mathrm{d}\theta\mathrm{d}s$, and we can deduce

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)}\leqslant\left(\frac{p\widetilde{Q}}{4}+q_{01}^{-1}q_{02}+\frac{1}{2}q_{01}^{-1}q_{01}'+\frac{\widetilde{Q}g_1q_{01}^{-\frac{1}{2}}}{2}-\frac{q_{01}^{\frac{1}{2}}\widetilde{Q}}{4}-\frac{\widetilde{Q}q_{02}q_{01}^{-\frac{1}{2}}}{2}\right)V(t)-\frac{\widetilde{Q}}{2}(xx')'+\frac{\widetilde{Q}|f^2|}{2q_{01}}+\frac{2|f^2|q_{01}^{-1}}{\widetilde{Q}}+\frac{1}{\tau^2}\int_{-\tau}^{0}x^2(t+s-\tau)\mathrm{d}s$$

Integrating the above inequality, we have

$$V(t)\leqslant K+\int_{t_0-\tau}^{t}\left(\frac{p\widetilde{Q}}{4}+q_{01}^{-1}q_{02}+\frac{1}{2}q_{01}^{-1}q_{01}'+\frac{\widetilde{Q}g_1q_{01}^{-\frac{1}{2}}}{2}-\frac{q_{01}^{\frac{1}{2}}\widetilde{Q}}{4}-\frac{\widetilde{Q}q_{02}q_{01}^{-\frac{1}{2}}}{2}\right)V(s)\mathrm{d}s-\frac{\widetilde{Q}}{2}xx'+\int_{t_0-\tau}^{t}xx'\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\widetilde{Q}}{2}\right)\mathrm{d}s$$

where

$$K=V(t_0-\tau)+\frac{\widetilde{Q}(t_0-\tau)}{2}x(t_0-\tau)x'(t_0-\tau)+\int_{t_0-\tau}^{\infty}\left(\frac{\widetilde{Q}|f^2|}{2q_{01}}+\frac{2|f^2|}{\widetilde{Q}q_{01}}\right)\mathrm{d}s+\frac{1}{\tau^2}\int_{t_0-\tau}^{\infty}\int_{-\tau}^{0}x^2(s+\theta-\tau)\mathrm{d}\theta\mathrm{d}s$$

is a positive constant. The conditions imply that

$$V(t)\leqslant K+\frac{\delta}{4}V(t)+\frac{1}{4}\int_{t_0-\tau}^{t}\left|\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}s}\right|V(s)\mathrm{d}s+\int_{t_0-\tau}^{t}\left(\frac{p\widetilde{Q}}{4}+q_{01}^{-1}q_{02}+\frac{1}{2}q_{01}^{-1}q_{01}'+\frac{\widetilde{Q}g_1q_{01}^{-\frac{1}{2}}}{2}-\frac{\widetilde{Q}q_{01}^{\frac{1}{2}}}{4}-\frac{\widetilde{Q}q_{02}q_{01}^{-\frac{1}{2}}}{2}\right)V(s)\mathrm{d}s$$

i. e.,

$$V(t)\leqslant\frac{4K}{4-\delta}+\frac{1}{4-\delta}\int_{t_0-\tau}^{t}\left(\left|\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}s}\right|+p\widetilde{Q}+4q_{01}^{-1}q_{02}+2q_{01}^{-1}q_{01}'+2\widetilde{Q}g_1q_{01}^{-\frac{1}{2}}-4\widetilde{Q}q_{01}^{\frac{1}{2}}-2\widetilde{Q}q_{02}q_{01}^{-\frac{1}{2}}\right)V(s)\mathrm{d}s$$

Gronwall's inequality implies that

$$V(t)\leqslant\frac{4K}{4-\delta}\exp\left\{\frac{1}{4-\delta}\int_{t_0-\tau}^{t}\left(\left|\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}s}\right|+p\widetilde{Q}+4q_{01}^{-1}q_{02}+2q_{01}^{-1}q_{01}'+2\widetilde{Q}g_1q_{01}^{-\frac{1}{2}}-4\widetilde{Q}q_{01}^{\frac{1}{2}}-2\widetilde{Q}q_{02}q_{01}^{-\frac{1}{2}}\right)\mathrm{d}s\right\}$$

By the conditions, there exists a positive constant $M$ such that $V(t)\leqslant M\ (t\geqslant t_0-\tau)$.

However, $x^2(t)\leqslant q_{01}^{-\frac{1}{2}}V(t)$, then $\int_{t_0-\tau}^{\infty}x^2(t)\mathrm{d}t\leqslant\int_{t_0-\tau}^{\infty}q_{01}^{-\frac{1}{2}}V(t)\mathrm{d}t\leqslant\int_{t_0-\tau}^{\infty}Mq_{01}^{-\frac{1}{2}}\mathrm{d}t<\infty$, i. e., $x(t)\in L^2[t_0-\tau,\infty)$. Considering that $q_{01}'>0$, thus $q_{01}(t)\geqslant q_{01}(t_0-\tau)>0\ (t\geqslant t_0-\tau)$, and $|x(t)|\leqslant\sqrt{V(t)q_{01}^{-\frac{1}{2}}(t)}\leqslant\sqrt{Mq_{01}^{-\frac{1}{2}}(t_0-\tau)}\ (t\geqslant t_0-\tau)$, i. e., $x(t)\in L^\infty[t_0-\tau,\infty)$.

Therefore, $x(t)\in L^2[t_0-\tau,\infty)\cap L^\infty[t_0-\tau,\infty)$. This completes the proof.

## 5 Application to Power Systems

Closing of switches often occurs in power systems. An ideal closing process is often considered that switches of three closed phases are synchronous, some results have been obtained under such circumstances. However, because of the manufacture or other causes, the closing of switches is often nonsynchronous in power systems. For example, closing on phase $B$ and $C$ contractors may occur later than that of $A$, and their times of delay may be $\tau$, or other nonsynchronous conditions. Operational experiences have indicated that switch over-voltage caused by nonsynchronous closing is higher than that caused by synchronous closing. Generally, the former is about 20% to 39% higher than the latter under ordinary circumstances, therefore, the former is of risk. In order to find the reason why the switching over-voltage caused by the former is of higher over-voltage, some laboratory tests and computer-aided analyses have been undertaken[5]. Ferromagnetic resonance over-voltage is a lasting nonlinear resonance phenomenon that appears in tank circuits with the action of iron core inductance saturation magnetization. The resonance formed in circuits is closely related to three phases. It is difficult to study the system's complex mechanism (such as theory, applications, etc). However, as linearizing mathematical models are used, the analysis can be dealt with by the methods of linear superposition of the effects of successive three-phase closing processes (e. g., I. A. Wright has given the unsimplified reason and arranged them in a line to prove it). Germand first set up the three-phase nonlinear mathematical model in power systems in 1975. Even the model is incomprehensive as it is now, but it shows that the discussion of the resonance mechanism of the switching over-voltage must be in high dimensional space. Generally, the nonsynchronous closing of a three-phase process is a time-delay phenomenon in nature. On the other hand, there are some inductive elements with cores in power systems. Due to the operation or other causes cores tend to saturate and inductors with cores may become nonlinear. It is clear that in order to further study the mechanism of over-voltage caused by three-phase nonsynchronous closing of switches, a nonlinear mathematical model considering the time-delay phenomenon must be considered, and then the effects of time-delay on the system are analyzed. According to the above discussion, to study the mechanism of higher over-voltage caused by the nonsynchronous closing of switches in power systems, a nonlinear time-delay mathematical model must be set up.

A kind of typical circuit can be obtained and a closing case is considered. That is, phase A closed, and the other phases $B$ and $C$ closed after time-delays $\tau_1, \tau_2$, respectively, then the dimensionless mathematical model based on the mathematical model of three-phase synchronous closing and considered influences of time-delay are given as[8]

$$\ddot{\phi}_i(t) + (k + a + nb\phi_i^{n-1}(t))\dot{\phi}_i(t) + k(a\phi_i(t) + b\phi_i^n(t)) + W = E\cos\left[t - (i-1)\frac{2}{3}\pi\right] \tag{7}$$

where $i = 1, 2, 3$; $W = \beta a\left[\sum_{i=1}^{3}\phi_i(t) + \sum_{i=1}^{2}\phi_i(t - \omega\tau_i)\right] + \beta b\left[\sum_{i=1}^{3}\phi_i^n(t) + \sum_{i=1}^{2}\phi_i^n(t - \omega\tau_i)\right]$, and $k, a, b, \omega, E, \tau_i$ $(i = 1, 2)$ are all positive constants, and $n \geqslant 3$ is an odd integral number, and $0 < k + a \ll 1, 0 < b \ll 1$. We consider the case $i = 1$, then Eq. (7) becomes

$$\ddot{x}(t) + (k + a + nbx^{n-1}(t))\dot{x}(t) + (k + \beta)(ax(t) + bx^n(t)) + \beta ax(t - \omega\tau) + \beta bx^n(t - \omega\tau) = E\cos t \tag{8}$$

Considering the physical meaning, the solutions $x(t)$ of Eq. (7) are bounded. We rewrite the coefficients of Eq. (8) as

$$\left.\begin{array}{l} p(t) = (k + a + nbx^{n-1}(t)) \geqslant k + a, \quad q_{01}(t) = t^2, \quad q_{02}(t) = (k + \beta)a - t^2, \quad q_2 = \beta a \\ g_1(t, x(t - \omega\tau)) = (k + \beta)bx^n(t), \quad g_2(t, x(t - \omega\tau)) = \beta bx^n(t - \omega\tau), \quad f(t) = E\cos t \end{array}\right\}$$

Then $q_{01}(t) > 0, q_{01}'(t) = 2t > 0 (t \in [0, \infty))$, and there exists a constant $L$ such that $|x(t)| \leqslant L$, satisfying that $|g(t, x(t - \omega\tau))| \leqslant \beta bL^{n-1}|x(t - \omega\tau)|$, and

$$Q_1(t) = 2t^{-3} + 2t^{-2}(k + a + nbx^{n-1}(t)) > 2t^{-3} + 2(k + a)t^{-2} > 0$$

$$\frac{f^2 q_{01}^{-2}}{Q_1} < \frac{E^2\cos^2 t}{2t + 2(k+a)t^2}, \quad \frac{q_{01}^{-2}(g_1{}^2 + q_2{}^2)}{Q_1\lambda_1^2} < \frac{\beta^2 a^2 + (k+\beta)^2 a^2 L^{2n-2}}{2t + 2(k+a)t^2}, \quad \frac{q_{01}^{-2}(g_2 + q_2)^2}{\tau} \leqslant \frac{(\beta bL^{n-1} + \beta a)^2}{\tau t^4}$$

Therefore, all conditions of corollary 1 hold, i. e., the following conclusion satisfies.

**Theorem 9** All the solutions $x(t)$ of Eq. (8) satisfy $|x(t)| = O(1)$, $|x'(t)| = O(t), t \to \infty$; i. e., all solutions $\phi_i(t) (i = 1, 2, 3)$ of the over-voltage model (8) satisfy $|\phi_i(t)| = O(1)$, $|\phi_i'(t)| = O(t), t \to \infty (i = 1, 2, 3)$.

Certainly, choosing different $q_{01}(t)$ can obtain different results.

Let $q_{01}(t) = t^3, t \in [t_0 - \omega\tau, \infty) (t_0 - \omega\tau > 0)$, then $q_{02}(t) = (k + \beta)a - t^3$, and $\widetilde{Q} = \frac{3}{2}t^{-\frac{5}{2}} + 2t^{-\frac{3}{2}}(k + a + nbx^{n-1}) - 2\tau t^{-3}\beta^2(a + bL^{n-1})^2$. Then, it is easy to know that there exists a $t^*$ such that $0 < \widetilde{Q} < 4$ for $t > t^*$. However, $\int_{t_0-\omega\tau}^{\infty} t^{-\frac{3}{2}}\mathrm{d}t < \infty$, hence $\int_{t_0-\omega\tau}^{\infty}\left(\frac{\widetilde{Q}|f^2|}{4q_{01}} + \frac{2|f^2|}{\widetilde{Q}q_{01}}\right)\mathrm{d}t < \int_{t_0-\omega\tau}^{\infty}\left[\frac{1}{8}t^{-\frac{11}{2}} + \frac{1}{2}t^{-\frac{9}{2}}(k + a + nbL^{n-1}) + \frac{t^{-\frac{3}{2}}}{k+a}\right]E^2\cos^2 t\,\mathrm{d}t$ $< \infty$, and $\int_{t_0-\omega\tau}^{\infty}\left(\left|\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}t}\right| + p\widetilde{Q} + 4q_{01}^{-1}q_{02} + 2q_{01}^{-1}q_{01}' + 2\widetilde{Q}g_1 q_{01}^{-\frac{1}{2}} - 4\widetilde{Q}q_{01}^{\frac{1}{2}} - 2\widetilde{Q}q_{02}q_{01}^{-\frac{1}{2}}\right)\mathrm{d}t \leqslant \int_{t_0-\omega\tau}^{\infty}[t^{-\frac{7}{2}} - 4 + 8(k + a +$

$nbL^{n-1})\mathrm{d}t < \int_{t_0-\omega\tau}^{\infty} t^{-\frac{7}{2}}\mathrm{d}t < \infty$, for $k+a+nbL^{n-1} < \frac{1}{2}$. It is easy to see the following inequality holding for the sufficient small time-delay $\tau$, $0<\tau\leqslant\min\left[\frac{q_{01}}{\widetilde{Q}(g_2+q_2)^2}, \frac{q_{01}^{-\frac{1}{2}}q_{01}'+4q_{01}^{\frac{1}{2}}p}{4(g_2+q_2)^2}\right]$, $2\widetilde{Q}g_1+2q_{01}^{-\frac{1}{2}}q_{01}'\geqslant\widetilde{Q}(q_{01}+2q_{02})$. Then we have the following conclusions.

**Theorem 10** All the solutions of Eq. (8) satisfy $x(t)\in L^2[t_0-\tau,\infty)\cap L^\infty[t_0-\tau,\infty)$, i. e., all solutions $\phi_i(t)(i=1,2,3)$ of the over-voltage model (8) satisfy that $\phi_i(t)\in L^2[t_0-\omega\tau,\infty)\cap L^\infty[t_0-\omega\tau,\infty)(i=1,2,3)$.

Similarly, we have

**Theorem 11** There exist oscillatory solutions and periodic solutions for the over-voltage model (7) under some conditions.

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# 基于一类非线性时滞系统解的二次可积性研究


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**摘要:**利用 V 函数、Lyapunov 泛函、Beuman-Bihari 不等式等方法探讨了一类二阶非线性时滞系统的相关性态,诸如振动性、稳定性、周期性及二次可积性等,得到了该类系统相应的充分性存在性条件的有关定量化结论. 最后,将所得结论应用到电力系统中三相非同期合闸的过电压模型中,结果表明所得结论与实际背景的物理意义相吻合,同时结论的所有条件较容易验证,因此该结论具有一定的理论意义并便于在实际中应用.



**关键词:**非线性时滞系统;二次可积性;周期解;振动解;过电压

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