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$L(s, t)$ edge spans of trees and product of two paths

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Abstract: $L(s, t)$-labeling is a variation of graph coloring which is motivated by a special kind of the channel assignment problem. Let s and t be any two nonnegative integers. An $L(s, t)$-labeling of a graph G is an assignment of integers to the vertices of G such that adjacent vertices receive integers which differ by at least s, and vertices that are at distance of two receive integers which differ by at least t. Given an $L(s, t)$-labeling f of a graph G, the $L(s, t)$ edge span of f, $\beta_{st}(G, f) = \max\{|f(u) - f(v)|: (u, v) \in E(G)\}$ is defined. The $L(s, t)$ edge span of G, $\beta_{st}(G)$, is $\min \beta_{st}(G, f)$, where the minimum runs over all $L(s, t)$-labelings f of G. Let T be any tree with a maximum degree of $\Delta \geqslant 2$. It is proved that if $2s \geqslant t \geqslant 0$, then $\beta_{st}(T) = (\lceil \Delta/2 \rceil - 1)t + s$; if $0 \leqslant 2s < t$ and Δ is even, then $\beta_{st}(T) = \lceil (\Delta - 1)t/2 \rceil$; and if $0 \leqslant 2s < t$ and Δ is odd, then $\beta_{st}(T) = (\Delta - 1)t/2 + s$. Thus, the $L(s, t)$ edge spans of the Cartesian product of two paths and of the square lattice are completely determined.

Key words: $L(s, t)$-labeling; $L(s, t)$ edge span; tree; Cartesian product; square lattice

$L(2, 1)$-labeling was first introduced by Griggs and Yeh[1] as a variation of the channel assignment problem in the radio system formulated as a graph coloring problem by Hale[2]. Suppose that a number of transmitters or stations are given. We ought to assign a channel to each of the given transmitters or stations. In order to avoid interference, "close" transmitters must receive different channels, and channels for "very close" transmitters are at least two apart. We can construct an interference graph for this problem. The transmitters are represented by the vertices of a graph, two vertices are "very close" if they are adjacent in the graph and "close" if they are of distance two in the graph. And so a feasible channel assignment is corresponding to an $L(2, 1)$-labeling of the interference graph, which is a special case of $L(s, t)$-labeling of the graph defined below.

Let s and t be any two nonnegative integers. Given a graph $G = (V(G), E(G))$, an $L(s, t)$-labeling of G is a function f from $V(G)$ to integers such that $|f(u) - f(v)| \geqslant s$ if u and v are adjacent and $|f(u) - f(v)| \geqslant t$ if u and v are at distance 2. The integers assigned to vertices are called labels. The span of f is the difference between the largest and the smallest labels used by f. The minimum span over all $L(s, t)$-labelings of G, denoted by $\lambda_{st}(G)$, is called the $L(s, t)$-labeling number of G. The $L(s, t)$-labeling numbers of graphs have been investigated in many papers, see Refs. [3 – 5]. For surveys on this topic, we refer readers to Refs. [6 – 7].

Let G be a graph. Suppose that f is an $L(s, t)$-labeling of G. We define the $L(s, t)$ edge span of f, denoted by $\beta_{st}(G, f)$, to be the value $\max\{|f(x) - f(y)|: xy \in E(G)\}$. The $L(s, t)$ edge span of G, denoted by $\beta_{st}(G)$, is $\min \beta_{st}(G, f)$, where the minimum runs over all $L(s, t)$-labelings f of G. If $s = t = 0$, then $\beta_{st}(G) = 0$ for any graph G. So we assume throughout this paper that at least one of s and t is positive. Obviously, $\beta_{st}(G) \geqslant s$ for any graph G with at least one edge. Also it is easy to see that $\beta_{st}(G) \leqslant \lambda_{st}(G)$ for any graph G. The equality may hold for some graphs. For example, $\beta_{st}(G) = \lambda_{st}(G) = (|V(G)| - 1)s$ for all complete graphs G. For many graphs G, $\beta_{st}(G)$ will be much less than $\lambda_{st}(G)$. Note that if H is an induced subgraph of G, then the restriction of any $L(s, t)$-labeling of a graph G on its subgraph H is an $L(s, t)$-labeling of H. Therefore, if H is an induced subgraph of G, then $\beta_{st}(H) \leqslant \beta_{st}(G)$. It is worth pointing out that this may not be true when s is less than t and H is not an induced subgraph of G.

The $L(2, 1)$ edge spans of graphs were first introduced by Yeh in Ref. [8]. The author determined the $L(2, 1)$ edge spans of cycles, trees, complete k-partite graphs and investigated the $L(2, 1)$ edge spans of triangular lattice and square lattice. Feng and Song studied the $L(d, 1)$ edge spans of several classes of graphs[9].

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$L(s,t)$ edge spans of graphs have not been investigated before. In this paper, we study the $L(s,t)$ edge spans of graphs. Let T be any tree with a maximum degree of $\Delta \geqslant 2$. We show that, if $2s \geqslant t \geqslant 0$, then $\beta_{st}(T) = (\lceil \Delta/2 \rceil - 1)t + s$; if $0 \leqslant 2s < t$ and Δ is even, then $\beta_{st}(T) = \lceil (\Delta-1)t/2 \rceil$; and if $0 \leqslant 2s < t$ and Δ is odd, then $\beta_{st}(T) = (\Delta-1)t/2 + s$. We also completely determine the $L(s,t)$ edge spans of the Cartesian products of two paths and of the square lattice. Our results generalize the results in Refs. [8-9].

Note that the labels used by an $L(s,t)$-labeling may be negative or nonnegative. And s may be less than or greater than or equal to t. The graphs we considered in this paper are simple and finite except that the square lattice is an infinite graph.

1 $L(s,t)$ Edge Span of a Tree

Let T be any tree with maximum degree Δ. If $\Delta = 1$ then T is the complete graph K_2 and we clearly have $\beta_{st}(K_2) = s$. Thus we assume that $\Delta \geqslant 2$. Let f be an $L(s,t)$-labeling of a graph G. For any subset S of $V(G)$, we denote by $f(S)$ the label set $\{f(v) \mid v \in S\}$.

Theorem 1 Let T be a tree with the maximum degree $\Delta \geqslant 2$, then

$$\beta_{st}(T) = \begin{cases} \left(\lceil \dfrac{\Delta}{2} \rceil - 1\right)t + s & s \geqslant \dfrac{t}{2}, t \geqslant 0 \\ \lceil \dfrac{(\Delta-1)t}{2} \rceil & 0 \leqslant s < \dfrac{t}{2}, \Delta \text{ is even} \\ \dfrac{\Delta-1}{2}t + s & 0 \leqslant s < \dfrac{t}{2}, \Delta \text{ is odd} \end{cases}$$

Proof Since each tree with maximum degree Δ is a subgraph of some tree with degree set $\{1, \Delta\}$, w. l. o. g., we may assume that each vertex of T has degree Δ or 1. We split the proof of the theorem into three cases.

Case 1 $2s \geqslant t$.

Let $m = (\lceil \Delta/2 \rceil - 1)t + s$. We prove $\beta_{st}(T) = m$ by giving an $L(s,t)$-labeling of T with edge span m and by showing that $\beta_{st}(K_{1,\Delta}) \geqslant m$. For any integer y (negative or nonnegative), we define

$$L_y = \{y - s - (\lceil \Delta/2 \rceil - 1 - i)t \mid i \in \mathbf{Z} \text{ and } 0 \leqslant i \leqslant \lceil \Delta/2 \rceil - 1\} \cup \{y + s + (i - \lceil \Delta/2 \rceil - 1)t \mid i \in \mathbf{Z} \text{ and } \lceil \Delta/2 \rceil + 1 \leqslant i \leqslant \Delta\}$$

We define an $L(s,t)$-labeling f of T as follows:

1) Choose a vertex v_0 of T of degree Δ. Let $f(v_0) = m$ and $f(N(v_0)) = L_m$.

2) If v is a vertex of maximum degree such that v and exactly one of its neighbors u are labeled, then give the other unlabeled $\Delta - 1$ neighbors of v the label set $L_{f(v)} \backslash \{f(u)\}$. (It is easy to check that $f(u) \in L_{f(v)}$.) If no such vertex exists, then all vertices of T are labeled and the labeling is completed.

It is not difficult to verify that f is an $L(s,t)$-labeling of T and the $L(s,t)$ edge span of f is m. Thus, $\beta_{st}(T) \leqslant m$.

We now show that $\beta_{st}(K_{1,\Delta}) \geqslant m$. Let v_0 be the root of $K_{1,\Delta}$, and let $N(v_0) = \{v_1, v_2, \ldots, v_\Delta\}$. Suppose to the contrary that $\beta_{st}(K_{1,\Delta}) < m$. Let f be an $L(s,t)$-labeling of T with $\beta_{st}(G,f) < m$. Then we shall obtain contradictions. Suppose the minimum label used by f is 0 and the maximum label is M. If $f(v_0) = 0$, then $\beta_{st}(G,f) \geqslant \max\{f(v_i): i = 1, 2, \ldots, \Delta\} \geqslant s + (\Delta - 1)t > m$: a contradiction. Thus, $f(v_0) \neq 0$. Symmetrically, $f(v_0) \neq M$. W. l. o. g., we may assume that $f(v_1) = 0$, then $f(v_0) \geqslant s$. Let $f(v_0) = s + x$ with $x \geqslant 0$. Since $\beta_{st}(G, f) < m$, we have $f(v_0) = s + x < m = (\lceil \Delta/2 \rceil - 1)t + s$ and so $x < (\lceil \Delta/2 \rceil - 1)t$. Therefore, at most $\lceil \Delta/2 \rceil - 1$ neighbors of v_0 have labels in $[0, x]$. It follows that at least $\lfloor \Delta/2 \rfloor + 1$ vertices have labels in $[2s + x, M]$. This implies that $M - (2s + x) \geqslant \lfloor \Delta/2 \rfloor t$. Then $\beta_{st}(G,f) \geqslant M - (s + x) \geqslant s + \lfloor \Delta/2 \rfloor t \geqslant s + (\lceil \Delta/2 \rceil - 1)t = m$, thus contradicting the assumption that $\beta_{st}(G,f) < m$.

Case 2 $2s < t$ and $\Delta(\geqslant 3)$ is odd.

We first show that $\beta_{st}(T) \leqslant (\Delta - 1)t/2 + s$. It suffices to give an $L(s,t)$-labeling of T with its $L(s,t)$ edge span $(\Delta-1)t/2 + s$. For any integer y, we define

$$L_y = \{y - s - it \mid i = 0, 1, 2, \ldots, (\Delta-1)/2\} \cup \{y - s + it \mid i = 1, 2, \ldots, (\Delta-1)/2\}$$

$$L'_y = \{y + s - it \mid i = 0, 1, 2, \ldots, (\Delta-1)/2\} \cup \{y + s + it \mid i = 1, 2, \ldots, (\Delta-1)/2\}$$

Clearly $|L_y| = |L'_y| = \Delta$. We define an $L(s,t)$-labeling f of T as follows:

1) Choose a vertex v_0 in $V(T)$ of degree Δ. Let $f(v_0) = (\Delta-1)t/2 + s$, and $f(N(v_0)) = L_{f(v_0)}$.

2) Find a vertex v of the maximum degree such that v and exactly one of its neighbor vertices u are labeled. If $d(v, v_0)$ is odd, then give $N(v) \backslash \{u\}$ the label set $L'_{f(v)} \backslash \{f(u)\}$. (It is easy to see that $f(u) \in L'_{f(v)}$.) If $d(v, v_0)$ is even, then give $N(v) \backslash \{u\}$ the label set $L_{f(v)} \backslash \{f(u)\}$. (Also it is easy to see that $f(u) \in L_{f(v)}$.) If no such vertex exists, then all vertices of T are labeled and the labeling is completed.

It is not difficult to verify that f is an $L(s,t)$-labeling and its $L(s,t)$ edge span is $(\Delta-1)t/2 + s$. Therefore, $\beta_{st}(T) \leqslant (\Delta-1)t/2 + s$.

We now show that $\beta_{st}(K_{1,\Delta}) \geqslant (\Delta-1)t/2 + s$. Let the root of $K_{1,\Delta}$ be v_0 and let $N(v_0) = \{v_1, v_2, \ldots, v_\Delta\}$. Suppose to the contrary that $\beta_{st}(K_{1,\Delta}) < (\Delta-1)t/2 + s$.

Let f be an $L(s, t)$-labeling with its $L(s, t)$ edge span less than $(\Delta - 1)t/2 + s$. We shall derive contradictions. Suppose the minimum label used by f is 0 and the maximum label is M. If $f(v_0) = 0$, then $\beta_{st}(G,f) = M \geqslant s + (\Delta - 1)t > (\Delta - 1)t/2 + s$: a contradiction. Thus, $f(v_0) \neq 0$. Symmetrically, $f(v_0) \neq M$. W. l. o. g., assume $f(v_1) = 0$. Then $f(v_0) \geqslant s$. Suppose $f(v_0) = s + x$ $(x \geqslant 0)$. Since we assume $\beta_{st}(G,f) < (\Delta - 1)t/2 + s$, $s + x < (\Delta - 1)t/2 + s$ and so $x < (\Delta - 1)t/2$. Then there are at most $(\Delta - 1)/2$ neighbor vertices of v_0 with labels in $[0, x]$. It follows that there are at least $(\Delta + 1)/2$ neighbors of v_0 with labels in $[2s + x, M]$. This implies that $M - (2s + x) \geqslant (\Delta - 1)t/2$; that is, $M - (s + x) \geqslant (\Delta - 1)t/2 + s$, contradicting the assumption that $\beta_{st}(G,f) < (\Delta - 1)t/2 + s$.

Case 3 $2s < t$ and Δ is even.

We first show that $\beta_{st}(T) \leqslant \lceil (\Delta - 1)t/2 \rceil$. Let $m_1 = \lfloor (\Delta - 1)t/2 \rfloor$ and $m_2 = \lceil (\Delta - 1)t/2 \rceil$. For any integer y, define $L_y = \{y - m_1 + it \mid i = 0, 1, \ldots, \Delta - 1\}$ and $L'_y = \{y - m_2 + it \mid i = 0, 1, \ldots, \Delta - 1\}$. Note that it is not difficult to verify $|y - (y - m_k + it)| \geqslant s$ for $k = 1, 2$ and $i = 0, 1, \ldots, \Delta - 1$. Using a similar process as in the proof of case 2, we can obtain an $L(s, t)$-labeling of T with its $L(s, t)$ edge span m_2. Thus, $\beta_{st}(T) \leqslant m_2$. We also can prove $\beta_{st}(K_{1,\Delta}) \geqslant m_2$ by using the similar arguments as in the proof of case 2. This completes the proof of theorem 1.

2 $L(s,t)$ Edge Span of the Product of Two Paths

We now turn to the Cartesian product of two paths and the square lattice. Let P_n be the path on n vertices. Given two positive integers m and n, the Cartesian product of two paths P_m and P_n, denoted by $P_m \times P_n$, has vertex set $\{(i,j) \mid 0 \leqslant i \leqslant m - 1, 0 \leqslant j \leqslant n - 1\}$. Two vertices (i_1, j_1) and (i_2, j_2) are adjacent if $|i_1 - i_2| + |j_1 - j_2| = 1$. The square lattice, denoted by $\square$, has vertex set $\mathbf{Z}^2$ and any two vertices are adjacent if the Euclidean distance between them is 1.

Theorem 2 For any two integers m and n with $m, n \geqslant 3$,

$$\beta_{st}(P_m \times P_n) = \beta_{st}(\square) = \begin{cases} s + t & \text{if } 2s \geqslant t \\ \left\lceil \dfrac{3t}{2} \right\rceil & \text{if } 2s < t \end{cases}$$

and $\beta_{st}(P_2 \times P_n) = s + t$ for $n \geqslant 2$.

Proof Let $G = P_m \times P_n$. Suppose $m, n \geqslant 2$. Let f be any $L(s, t)$-labeling of G. Clearly G contains a 4-cycle, say, $C = v_0 v_1 v_2 v_3$. W. l. o. g., suppose $f(v_0) = \min\{f(v_i) \mid i = 0, 1, 2, 3\}$. Then $\max\{f(v_1), f(v_3)\} \geqslant f(v_0) + s + t$. It follows that $\beta_{st}(G, f) \geqslant s + t$. Hence, $\beta_{st}(G) \geqslant s + t$.

We now show that if $2s \geqslant t$ then $\beta_{st}(G) \leqslant s + t$. Let $f: V(G) \to \mathbf{Z}$ be defined as $f(i,j) = si + (s + t)j$, where $f(i,j)$ stands for $f((i,j))$. Suppose (i_1, j_1) and (i_2, j_2) are any two vertices of G. If $d((i_1, j_1), (i_2, j_2)) = 1$, then $|i_1 - i_2| = 1$ and $j_1 = j_2$, or $i_1 = i_2$ and $|j_1 - j_2| = 1$. In both cases, we have $|f(i_1, j_1) - f(i_2, j_2)| \geqslant s$. If $d((i_1, j_1), (i_2, j_2)) = 2$, then $|i_1 - i_2| = 1 = |j_1 - j_2|$, or $|i_1 - i_2| = 2$ and $j_1 = j_2$, or $i_1 = i_2$ and $|j_1 - j_2| = 2$. In each case, we have $|f(i_1, j_1) - f(i_2, j_2)| \geqslant t$. So f is an $L(s, t)$-labeling of G. Also, it is easy to see that the $L(s, t)$ edge span of f is $s + t$. This implies that $\beta_{st}(G) = s + t$ if $2s \geqslant t$.

We now turn to the case $2s < t$. If $m, n \geqslant 3$, then $K_{1,4}$ is an induced subgraph of G. It follows from theorem 1 that $\beta_{st}(G) \geqslant \beta_{st}(K_{1,4}) = \lceil 3t/2 \rceil$. On the other hand, let $f: V(G) \to \mathbf{Z}$ be defined as $f(i,j) = \lceil t/2 \rceil i + \lceil 3t/2 \rceil j$. It is not difficult to see that f is an $L(s, t)$-labeling of G and its $L(s, t)$ edge span is $\lceil 3t/2 \rceil$. Hence, $\beta_{st}(G) = \lceil 3t/2 \rceil$ if $2s < t$ and $m, n \geqslant 3$. For the case $m = 2$, let $f: V(G) \to \mathbf{Z}$ be defined as $f(0,j) = jt$ if j is even and $f(0,j) = jt + s$ if j is odd; $f(1,j) = jt + s$ if j is even and $f(1,j) = jt$ if j is odd. It is easy to check that if $2s < t$, then f is an $L(s, t)$-labeling of G and its $L(s, t)$ edge span is $s + t$. Hence, $\beta_{st}(P_2 \times P_n) = s + t$ for $n \geqslant 2$.

Since $P_m \times P_n$ is a subgraph of $\square$, $\beta_{st}(\square) \geqslant \beta_{st}(P_m \times P_n)$. On the other hand, it is easy to see that $L(s, t)$-labelings of $P_m \times P_n$ for $m, n \geqslant 3$ defined above can be extended to $L(s, t)$-labelings of $\square$. Thus $\beta_{st}(\square) = \beta_{st}(P_m \times P_n)$.

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树和路乘积图的 $L(s,t)$ 边跨度

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摘要: 图的 $L(s,t)$-标号的概念来自频道分配问题. 设 s 和 t 是 2 个非负整数. 图 G 的一个 $L(s,t)$-标号是一个从 G 的顶点集到整数集的映射,满足:①任意 2 个相邻顶点对应的整数相差至少为 s;②任意 2 个距离为 2 的顶点对应的整数相差至少为 t. 给定图 G 的一个 $L(s,t)$-标号 f,f 的 $L(s,t)$ 边跨度定义为 $\max\{|f(u)-f(v)|:(u,v)\in E(G)\}$,记为 $\beta_{st}(G,f)$. 图 G 的 $L(s,t)$ 边跨度定义为 $\min\{\beta_{st}(G,f):f$ 取遍图 G 的所有 $L(s,t)$-标号$\}$,记为 $\beta_{st}(G)$. 设 T 是一棵最大度为 $\Delta(\geqslant 2)$ 的树. 证明了:若 $2s\geqslant t\geqslant 0$,则 $\beta_{st}(T)=(\lceil \Delta/2 \rceil -1)t+s$;若 $0\leqslant 2s<t$ 且 Δ 为偶数,则 $\beta_{st}(T)=\lceil (\Delta-1)t/2 \rceil$;若 $0\leqslant 2s<t$ 且 Δ 为奇数,则 $\beta_{st}(T)=(\Delta-1)t/2+s$. 同时完全确定了 2 条路的笛卡儿乘积图和正四边形格图的 $L(s,t)$ 边跨度.

关键词: $L(s,t)$-标号;$L(s,t)$ 边跨度;树;笛卡儿乘积图;正四边形格图

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