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Fresh views on some recent developments in the simplex algorithm

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Abstract: First, the main procedures and the distinctive features of the most-obtuse-angle (MOA) row or column pivot rules are introduced for achieving primal or dual feasibility in linear programming. Then, two special auxiliary problems are constructed to prove that each of the rules can be actually considered as a simplex approach for solving the corresponding auxiliary problem. In addition, the nested pricing rule is also reviewed and its geometric interpretation is offered based on the heuristic characterization of an optimal solution.

Key words: linear programming; simplex algorithm; pivot; most-obtuse-angle; nested pricing; large-scale problem

Consider linear programming (LP) problems in the standard form:

$$\begin{aligned} &\min \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ &\text{s. t. } \boldsymbol{A}\boldsymbol{x}=\boldsymbol{b} \qquad \boldsymbol{x}\geqslant \boldsymbol{0} \end{aligned} \tag{1}$$

where $\boldsymbol{A}\in\mathbf{R}^{m\times n}$, $\boldsymbol{c}\in\mathbf{R}^{n}$, and $\boldsymbol{b}\in\mathbf{R}^{m}$ are assumed to be nonzero ($0<m<n$). We assume that rank $(\boldsymbol{A})=m$ and $\boldsymbol{c}$ is not in the range space of $\boldsymbol{A}$.

Since the simplex method for solving LP problems was founded by Dantzig[1] in 1947, this topic has enjoyed considerable interest by researchers in many fields. In 1954, Lemke[2] developed the dual simplex method, so Dantzig's simplex method is also called the primal simplex method.

To create an initial primal (dual) feasible basis to get the primal (dual) simplex algorithm started, one usually solves some sort of "phase 1" problem to produce such a basis. In Refs. [3 - 4], Pan proposed several ratio-test-free rules for phase 1, named the most-obtuse-angle (MOA) pivot rules, for achieving primal (dual) feasibility with neither an auxiliary problem nor an artificial variable. Despite some instances of cycling under such kinds of rules were given[5], their computational performance is very favorable. In Ref. [6], Koberstein and Suhl reported computational results on some major dual phase-1 methods for solving large-scale LP problems, showing the superiority of the MOA rules over others.

On the other hand, great efforts were also made on pivot rules for phase 2 to reduce the number of required iterations. There are basically three types of selection rules: full pricing rules[7-9], finite rules[10-13] and partial pricing rules[14-16]. Recently, Pan reported computational results[17] on a so-called nested pricing rule with large-scale problems, exhibiting its remarkable success over major pivot rules commonly used in practice, such as Dantzig's original rule as well as the steepest-edge rule and the Devex rule.

Why do the rules perform so fast? In this paper, we present some fresh views on these developments. As usual, we give the following notations: e_i denotes the i-th coordinate vector; its dimension is determined by the context. $\|\cdot\|$ refers to the vector two-norm. Let $\boldsymbol{B}$ be the current basis and $\boldsymbol{N}$ the associated nonbasis. Without confusion, we denote both the basic (nonbasic) index set and the basis (nonbasis) by the same notation. For instance, $\boldsymbol{c}_B\in\mathbf{R}^{m}$ is the vector consisting of basic components of $\boldsymbol{c}$, and $\boldsymbol{c}_N\in\mathbf{R}^{n-m}$ consists of its nonbasic components. Assume that $\boldsymbol{B}=\{j_1,j_2,\dots,j_m\}$, where j_i is the index of the i-th basic variable. We denote this by $\overline{\boldsymbol{N}}=\boldsymbol{B}^{-1}\boldsymbol{N}, \bar{\boldsymbol{b}}=\boldsymbol{B}^{-1}\boldsymbol{b}$, and the nonbasic reduced costs may be obtained by a so-called pricing operation:

$$\bar{\boldsymbol{c}}_N=\boldsymbol{c}_N-\boldsymbol{N}^{\mathrm{T}}\boldsymbol{\pi},\quad \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\pi}=\boldsymbol{c}_B$$

1 Fresh Views on the MOA Pivot Rules

Suppose that $\boldsymbol{B}$ is infeasible for both problem (1) and its dual problem; that is, the row index set

$$I=\{i\mid \bar{b}_i<0, i=1,2,\dots,m\}$$

and the column index set

$$J=\{j\mid \bar{c}_j<0, j\in N\}$$

are nonempty. First, we describe the MOA pivot rules for achieving primal and dual feasibility, respectively, in the following.

Pan's proposal[3] for achieving primal feasibility used the row pivot rule in the dual simplex method to determine a leaving index j_r:

$$r=\operatorname{argmin}\{\bar{b}_i\mid i\in I\} \tag{2}$$

and the MOA column pivot rule to determine an entering index q:

$$q=\operatorname{argmin}\{\bar{a}_{rj}\mid j\in J(r)\} \tag{3}$$

where

$$J(r)\triangleq\{j\mid \bar{a}_{rj}<0, j\in N\} \tag{4}$$

Then update the basis correspondingly and complete one iteration step. Such steps are repeated until either set I is empty, implying achievement of primal feasibility, or otherwise I is nonempty but $J(r)$ is empty. In the latter case, it easily follows that problem (1) has no feasible solution. Since there are a finite number of possible bases, the proce-

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dure would terminate in finite iterations if cycling does not occur.

Pan's dual phase-1 approach[4] used Dantzig's column pivot rule to determine an entering index q:

$$q = \text{argmin } \{\bar{c}_j \mid j \in J\} \tag{5}$$

and the MOA row pivot rule to determine a leaving index j_r:

$$r = \text{argmax } \{\bar{a}_{iq} \mid i \in I(q)\} \tag{6}$$

where

$$I(q) \triangleq \{i \mid \bar{a}_{iq} > 0, i = 1, 2, \ldots, m\} \tag{7}$$

Then update the basis correspondingly and repeat the steps. If cycling does not occur, the procedure would terminate at some step at either:

① Set J is empty, implying achievement of dual feasibility; or

② J is nonempty but $I(q)$ is empty, indicating dual infeasibility; that is, problem (1) has no optimal solution.

A distinctive feature of the rules above is the remarkable simplicity. Neither ratio test nor the reduced cost (or the value of the basic variable) is required. Furthermore, it is clear that monotone change in objective values and maintenance of current primal and dual feasibilities cannot be guaranteed.

In the following, we show that each of them can be actually considered as a simplex approach for solving a special phase-1 problem.

Theorem 1 Pan's procedures are equivalent to those of the simplex approach for solving corresponding auxiliary problems.

Proof Apparently, we can obtain a primal feasible basic solution to (1) by solving the following phase-1 problem:

$$\begin{aligned} &\min \mathbf{0}^{\mathrm{T}}\boldsymbol{x} \\ &\text{s. t. } \boldsymbol{Ax} = \boldsymbol{b} \qquad \boldsymbol{x} \geqslant 0 \end{aligned} \tag{8}$$

It is easy to see that any basis of (8) is dually feasible and the associated reduced costs are equal to 0, so we can directly apply the dual simplex method to (8). Let $j_r \in B$ be the current leaving index determined by Eq. (2) and assume that $J(r)$ defined by Eq. (4) is nonempty, then an entering index can be determined by the min-ratio-test, i. e,

$$q = \text{argmin}\left\{\frac{0}{-\bar{a}_{rj}} \mid j \in J(r)\right\}$$

For numerical stabilization, we certainly select q such that

$$q = \text{argmax}\{\,|\bar{a}_{rj}|\, \mid j \in J(r)\}$$

or equivalently

$$q = \text{argmin}\{\bar{a}_{rj} \mid j \in J(r)\}$$

The procedure is just the same as Pan's MOA column pivot rule for achieving primal feasibility.

Similarly, we can also see that Pan's dual phase-1 approach is equivalent to applying the primal simplex method to the homogeneous LP problem (9).

$$\begin{aligned} &\min \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ &\text{s. t. } \boldsymbol{Ax} = \mathbf{0} \qquad \boldsymbol{x} \geqslant \mathbf{0} \end{aligned} \tag{9}$$

The proof is completed.

2 Geometric Interpretation of the Nested Pricing Rule

Recently, Pan presented a successful nested pricing[17-18] in the primal simplex algorithm for solving problem (1). The new rule and the very famous and efficient Devex rule are implemented within MINOS 5.51. In over 80 large-scale problems (consisting of the 48 largest Netlib problems, all of the 16 Kennington problems, and the 17 largest BPMPD problems), the nested pricing rule outperformed the Devex rule with a total time ratio of 5.73. First, we introduce the nested pricing rule in the following.

Nested pricing rule in simplex Let $\boldsymbol{B}$ be the current feasible basis. Let (J_1, J_2) be a partition of $\boldsymbol{N}$.

① If index set $\hat{J}_1 \triangleq \{j \mid \bar{c}_j < 0, j \in J_1\}$ is nonempty, select an entering index q such that

$$q = \text{argmin}\{\bar{c}_j \mid j \in \hat{J}_1\} \tag{10}$$

or else

② If redefined index set $\hat{J}_1 \triangleq \{j \mid \bar{c}_j < 0, j \in J_2\}$ is nonempty, select an entering index q by (10); or else

③ Declare optimality.

If optimality is not declared, the partition of the next N is formed by setting $J_1 = \hat{J}_1 \backslash q$ and then $J_2 = N \backslash J_1$. For the first iteration, J_1 is set to N and J_2 to empty.

We will describe the geometric interpretation of the nested pricing rule as follows. For convenience, we suppose $B = \{1, 2, \ldots, m\}$. The canonical form of (1) corresponding to current basis $\boldsymbol{B}$ is

$$\begin{aligned} &\min \boldsymbol{c}_B^{\mathrm{T}}\bar{\boldsymbol{b}} + \bar{\boldsymbol{c}}_N^{\mathrm{T}}\boldsymbol{x}_N \\ &\text{s. t. } \boldsymbol{x}_B + \bar{\boldsymbol{N}}\boldsymbol{x}_N = \bar{\boldsymbol{b}} \qquad \boldsymbol{x}_B, \boldsymbol{x}_N \geqslant \mathbf{0} \end{aligned} \tag{11}$$

It is equivalent to

$$\begin{aligned} &\min \bar{\boldsymbol{c}}_N^{\mathrm{T}}\boldsymbol{x}_N \\ &\text{s. t. } -\bar{\boldsymbol{N}}\boldsymbol{x}_N \geqslant -\bar{\boldsymbol{b}} \qquad \boldsymbol{x}_N \geqslant \mathbf{0} \end{aligned} \tag{12}$$

The quantity $\bar{c}_j / \|\bar{\boldsymbol{c}}_N\|$ $(j \in N)$ is just the cosine of the angle formed between $\boldsymbol{e}_{j-m}$ (the constraint $x_j \geqslant 0$) and $\bar{\boldsymbol{c}}_N$, and $\bar{c}_j$ is the pivoting index of variable x_j defined in Ref. [12]. According to the plausible characterization of the optimal solution[12], the variable with a lower pivoting index is prior to an entering basis. Dantzig's column rule is to select the minimum one just at any current iteration, not considering the previous iterations. However, the nested pricing rule is to select the minimum one in the set J_1. In fact, each index $j \in J_1$ indicates that the angle formed between $\boldsymbol{e}_{j-m}$ (the constraint $x_j \geqslant 0$) and the reduced gradient always remains obtuse at some previous successive iterations. Therefore, it is reasonable to focus on such indices. This may explain the success of the new rule.

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关于单纯形算法若干进展的新见解

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摘要：首先回顾了采用最钝角行、列主元规则求解线性规画问题的原始、对偶可行解的主要过程，阐述了其与众不同的特性. 然后构造了 2 个特殊的辅助问题，并证明了最钝角行、列主元规则的过程实际上分别等价于采用原始、对偶单纯形算法求解相应的辅助问题. 此外，还对嵌套的 pricing 规则进行了回顾，并基于最优解的启发式特征刻画给出了该规则的一个几何解释.

关键词：线性规划；单纯形算法；主元；最钝角；嵌套的 pricing；大规模问题

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