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Making the category of entwined modules into a braided monoidal category

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Abstract: The question of how the category of entwined modules can be made into a braided monoidal category is studied. First, the sufficient and necessary conditions making the category into a monoidal category are obtained by using the fact that if (A, C, ψ) is an entwining structure, then $A \otimes C$ can be made into an entwined module. The conditions are that the algebra and coalgebra in question are both bialgebras with some extra compatibility relations. Then given a monodial category of entwined modules, the braiding is constructed by means of a twisted convolution invertible map Q, and the conditions making the category form into a braided monoidal category are obtained similarly. Finally, the construction is applied to the category of Doi-Hopf modules and (α, β)-Yetter-Drinfeld modules as examples.

Key words: Doi-Hopf module; entwined module; braided monoidal category

In 1998, Brzezinski and Majid[1] introduced the notion of entwined modules, which contained the Long modules, Yetter-Drinfeld modules and Doi-Koppinen modules etc. So it is very important to study entwined modules. The category of Yetter-Drinfeld modules is a braided monoidal category[2]. Caenepeel et al. [3] studied how the category of Doi-Hopf modules can be made into a braided monoidal category. The general category of the entwined modules is not a braided monoidal category. In this paper, we discuss the following question: how to make the category of entwined modules into a braided monoidal category. Our results are formulated for right-right entwining structures. As an application, we study the category of Doi-Hopf modules and the (α, β)-Yetter-Drinfeld modules.

Throughout this paper, we assume that k is a commutative ring. If a tensor product is written without an index, then it is assumed to be taken over k, that is $\otimes = \otimes_k$.

1 Making the Category of Entwined Modules into a Braided Monoidal Category

Definition 1 A right-right entwining structure is a triple (A, C, ψ), where A is an algebra and C is a coalgebra with a linear map $\psi: C\otimes A \to A\otimes C$, $c\otimes a \mapsto \psi(c\otimes a) = a_\psi \otimes c^\psi$ satisfying the following conditions:

$$(ab)_\psi \otimes c^\psi = a_\psi b_\phi \otimes c^{\psi\phi} \tag{1}$$

$$1_{A\psi} \otimes c^\psi = 1_A \otimes c \tag{2}$$

$$a_\psi \otimes \Delta(c^\psi) = a_{\psi\phi} \otimes c_1^\phi \otimes c_2^\psi \tag{3}$$

$$\varepsilon(c^\psi) a_\psi = \varepsilon(c) a \tag{4}$$

Over an entwining structure (A, C, ψ), a right-right entwined module M is both a right C-comodule and a right A-module such that

$$\rho(ma) = m_{(0)} \psi(m_{(1)} \otimes a) = m_{(0)} a_\psi \otimes m_{(1)}^\psi$$

for any $a \in A$ and $m \in M$.

The category of right-right entwined modules and A-linear C-colinear homomorphisms is denoted by $M_A^C(\psi)$. Entwined modules were introduced by Brzezinski et al. [1] as a generalization of Doi-Koppinen modules in Refs. [4 – 5]. We now want to make $M_A^C(\psi)$ into a monoidal category.

Definition 2 We call (A, C, ψ) a monoidal entwining datum if (A, C, ψ) is a right-right entwining structure and A and C are bialgebras, with the additional compatibility relations

$$a_{1\psi} \otimes a_{2\phi} \otimes c^\psi c'^\phi = \Delta(a_\psi) \otimes (cc')^\psi \tag{5}$$

$$\varepsilon_A(a) 1_C = \varepsilon_A(a_\psi) 1_C^\psi \tag{6}$$

for all $a \in A$ and $c, c' \in C$.

Proposition 3 Let (A, C, ψ) be a monoidal entwining structure over a commutative ring k. Then the tensor product of two entwined modules M and N is again an entwined module. The structure maps are given by

$$\rho_{M\otimes N}(m\otimes n) = m_{(0)} \otimes n_{(0)} \otimes m_{(1)} n_{(1)}$$

$$(m\otimes n) a = ma_1 \otimes na_2$$

The category $P = M_A^C(\psi)$ is a monoidal category.

Proof We show that $M\otimes N$ is an entwined module. For all $a \in A$, $m \in M$ and $n \in N$, we have

$$\begin{aligned}\rho_{M\otimes N}((m\otimes n)a) = \\ &(ma_1)_{(0)} \otimes (na_2)_{(0)} \otimes (ma_1)_{(1)} (na_2)_{(1)} = \\ &m_{(0)} a_{1\psi} \otimes n_{(0)} a_{2\phi} \otimes m_{(1)}{}^\psi n_{(1)}{}^\phi = \\ &m_{(0)} (a_\psi)_1 \otimes n_{(0)} (a_\psi)_2 \otimes (m_{(1)} n_{(1)})^\psi = \\ &(m_{(0)} \otimes n_{(0)}) a_\psi \otimes (m_{(1)} n_{(1)})^\psi = \\ &(m\otimes n)_{(0)} \psi((m\otimes n)_{(1)} \otimes a)\end{aligned}$$

Thus $M\otimes N \in M_A^C(\psi)$, as needed.

It is clear that the tensor product defines a functor $P\otimes P \to P$. It follows from Eq. (6) that the trivial A-action and C-coaction given by $xa = \varepsilon_A(a)x$ and $\rho(x) = x\otimes 1_C$ make k into an object of P. It is clear that k is a unit object of P. Let U, V and W be entwined modules. The isomorphisms

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$$a_{UVW}: (U\otimes V)\otimes W\to U\otimes(V\otimes W):\\ (u\otimes v)\otimes w\mapsto u\otimes(v\otimes w)$$

$$r_U: U\otimes k\to U: u\otimes t\mapsto ut$$

$$l_U: k\otimes U\to U: t\otimes u\mapsto tu$$

obviously satisfy the pentagon axiom and the triangle axiom[6]. Hence P is a monoidal category.

Let (A, C, ψ) be a monoidal entwining datum. We know that a braiding on $M_A^C(\psi)$ is a natural family of isomorphisms

$$t_{M,N}: M\otimes N\to N\otimes M$$

in $M_A^C(\psi)$ such that the following equations hold, for all $M, N, P\in M_A^C(\psi)$,

$$(I_N\otimes t_{M,P})\circ t_{M,N\otimes P}=t_{M,N}\otimes I_P \tag{7}$$

$$(t_{M,P}\otimes I_N)\circ t_{M\otimes N,P}=I_M\otimes t_{N,P} \tag{8}$$

Consider a map Q: $C\otimes C\to A\otimes A$ with twisted convolution inverse R. The last condition means that

$$Q^2(c_2\otimes d_2)_\psi R^1(d_1^\phi\otimes c_1^\psi)\otimes Q^1(c_2\otimes d_2)_\phi R^2(d_1^\phi\otimes c_1^\psi)=\\ \eta_A(\varepsilon_C(c))\otimes\eta_A(\varepsilon_C(d)) \tag{9}$$

for all $c, d\in C$. Here we write $Q(c\otimes d)=Q^1(c\otimes d)\otimes Q^2(c\otimes d)\in A\otimes A$. Consider two entwined modules M and N. We define $t_{M,N}: M\otimes N\to N\otimes M$ by

$$t_{M,N}(m\otimes n)=(n_{(0)}\otimes m_{(0)})Q(n_{(1)}\otimes m_{(1)}) \tag{10}$$

for all $m\in M$ and $n\in N$. It follows from Eq. (9) that $t_{M,N}$ is bijective. Our aim is to give necessary and sufficient conditions on Q such that $t_{M,N}$ defines a braiding on the category of entwined modules.

Remark If (A, C, ψ) is an entwining structure, then $A\otimes C$ can be made into an entwined module as follows: The right A-module action and right C-comodule action are given by

$$(b\otimes c)a=ba\otimes c,\ \rho_{A\otimes C}(b\otimes c)=b_\psi\otimes c_1\otimes c_2^\psi$$

respectively, for all $a, b\in A$, and $c\in C$.

Proof $A\otimes C$ is a right C-comodule, and the right A-module can be obtained straightforwardly. In fact,

$$(\rho_{A\otimes C}\otimes I)\rho_{A\otimes C}(b\otimes c)=\rho_{A\otimes C}(b_\psi\otimes c_1)\otimes c_2{}^\psi=\\ b_{\psi\phi}\otimes c_1\otimes c_2^\phi\otimes c_3^\psi$$

$$(I\otimes\Delta)\rho_{A\otimes C}(b\otimes c)=b_\psi\otimes c_1\otimes\Delta(c_2^\psi)=\\ b_{\psi\phi}\otimes c_1\otimes c_2^\phi\otimes c_3^\psi$$

The compatibility can be proved as follows:

$$\rho_{A\otimes C}((b\otimes c)a)=(ba)_\psi\otimes c_1\otimes c_2^\psi=b_\psi a_\phi\otimes c_1\otimes c_2{}^{\psi\phi}=\\ (b_\psi\otimes c_1)a_\phi\otimes c_2^{\psi\phi}=(b_\psi\otimes c_1)\phi(c_2{}^\psi\otimes a)=\\ (b\otimes c)_{(0)}\psi((b\otimes c)_{(1)}\otimes a)$$

Lemma 4 With the above notations, the map $t_{M,N}$ is A-linear for all entwined modules M and N if and only if

$$(a_{2\psi}\otimes a_{1\phi})Q(d^\psi\otimes c^\phi)=Q(d\otimes c)\Delta_A(a) \tag{11}$$

for all $a\in A$ and $c, d\in C$.

Proof Suppose that $t_{A\otimes C,A\otimes C}$ is A-linear. It turns out to be convenient to consider $t_{A\otimes C,A\otimes C}$ as the map

$$t: A\otimes A\otimes C\otimes C\to A\otimes A\otimes C\otimes C$$

$$a\otimes b\otimes c\otimes d\mapsto(b_\psi\otimes a_\phi)Q(d_2^\psi\otimes c_2^\phi)\otimes d_1\otimes c_1$$

Because t is A-linear, we obtain

$$t((1\otimes 1)a\otimes c\otimes d)=t(a_1\otimes a_2\otimes c\otimes d)=\\ (a_{2\psi}\otimes a_{1\phi})Q(d_2^\psi\otimes c_2^\phi)\otimes d_1\otimes c_1$$

$$t(1\otimes 1\otimes c\otimes d)a=Q(d_2\otimes c_2)\cdot\Delta_A(a)\otimes d_1\otimes c_1$$

for all $a\in A$ and $c, d\in C$. We apply $I_A\otimes I_A\otimes\varepsilon_C\otimes\varepsilon_C$ to both sides of $(a_{2\psi}\otimes a_{1\phi})Q(d_2^\psi\otimes c_2^\phi)\otimes d_1\otimes c_1=Q(d_2\otimes c_2)\cdot\Delta_A(a)\otimes d_1\otimes c_1$ and we prove that Eq. (11) holds.

Conversely, suppose that Eq. (11) holds, and consider two entwined modules M and N. We have for all $a\in A$, $m\in M$ and $n\in N$ that

$$t_{M,N}((m\otimes n)a)=t_{M,N}(ma_1\otimes na_2)=\\ ((na_2)_{(0)}\otimes(ma_1)_{(0)})Q((na_2)_{(1)}\otimes(ma_1)_{(1)})=\\ (n_{(0)}a_{2\psi}\otimes m_{(0)}a_{1\phi})Q(n_{(1)}^\psi\otimes m_{(1)}^\phi)=\\ (n_{(0)}\otimes m_{(0)})Q(n_{(1)}\otimes m_{(1)})\Delta_A(a)=t_{M,N}(m\otimes n)a$$

and it follows that $t_{M,N}$ is A-linear.

By similar computations, we can obtain the following lemmas.

Lemma 5 With the above notations, the map $t_{M,N}$ is C-colinear for all entwined modules M and N if and only if

$$Q^1(d_2\otimes c_2)_\psi\otimes Q^2(d_2\otimes c_2)_\phi\otimes d_1^\psi c_1^\phi=Q(d_1\otimes c_1)\otimes c_2d_2 \tag{12}$$

for all $c, d\in C$.

Lemma 6 With the above notations, Eq. (7) holds for all $c, d, e\in C$ if and only if

$$(\Delta_A\otimes I_A)Q(de\otimes c)=Q^1(d\otimes c_2)\otimes\\ (1\otimes Q^2(d\otimes c_2)_\psi)q(e\otimes c_1^\psi) \tag{13}$$

Here we write $Q(c\otimes d)=q(c\otimes d)$.

Lemma 7 With the above notations, Eq. (8) holds for all $c, d, e\in C$ if and only if

$$(I_A\otimes\Delta_A)Q(e\otimes cd)=(Q^1(e_2\otimes d)_\psi\otimes 1)q(e_1{}^\psi\otimes c)\otimes\\ Q^2(e_2\otimes d) \tag{14}$$

We summarize our results as follows.

Theorem 8 Let (A, C, ψ) be a monoidal entwining datum, and Q: $C\otimes C\to A\otimes A$ a twisted convolution invertible map. Then the family of maps

$$t_{M,N}: M\otimes N\to N\otimes M,\ m\otimes n\mapsto(n_{(0)}\otimes m_{(0)})Q(n_{(1)}\otimes m_{(1)})$$

defines a braiding on the category of entwined modules $M_A^C(\psi)$ if and only if Q satisfies Eqs. (11) to (14).

2 Examples

2.1 Doi-Hopf modules

Let H be a Hopf algebra with a bijective antipode, (A, ρ^A) is a right H-comodule algebra, and (C, ψ^C) a right H-

module coalgebra. Then (A, C, ψ) forms a ψ-entwining structure,

$$\psi: C\otimes A\to A\otimes C,\quad c\otimes a\mapsto a_{(0)}\otimes ca_{(1)}$$

In particular, for all $M\in M_A^C(\psi)$, we have

$$\rho_M(ma)=m_{(0)}a_{(0)}\otimes m_{(1)}a_{(1)}$$

M is a right-right Doi-Hopf module. If ψ satisfies Eqs. (5), (6), and (11) to (14), we can obtain the conditions making the category of Doi-Hopf modules into a braided monoidal category in Ref. [3].

2.2 (α,β)-Yetter-Drinfeld modules

Let H be a Hopf algebra with a bijective antipode. Define $\psi: H\otimes H\to H\otimes H$, $g\otimes h\mapsto h_2\otimes \alpha S(h_1)g\beta(h_3)$, where $\alpha,\beta\in \mathrm{Aut}_{\mathrm{Hopf}}(H)$ means that α,β are Hopf algebra automorphisms. If $M\in M_H^H(\psi)$, then

$$\rho_M(mh)=m_{(0)}h_2\otimes\alpha S(h_1)m_{(1)}\beta(h_3)$$

this means that M is the right-right (α,β)-Yetter-Drinfeld module[7].

Now we make the category of (α,β)-Yetter-Drinfeld modules into a braided monoidal category for some (α,β) by applying the results of the category of entwined modules.

First, we do a calculation. Compute $\varepsilon_A(a)1_C=\varepsilon_A(a_\psi)1_C^\psi$ as $\varepsilon(a_\psi)1^\psi=\alpha S(a_1)\beta(a_2)=\varepsilon(a)1_H$, so we can obtain $S(a_1)\alpha^{-1}\beta(a_2)=\varepsilon(a)1_H$. Because the involution inverse of S is unique, we obtain that Eq. (6) is satisfied if and only if $\alpha=\beta$.

If $\alpha=\beta$, we prove that Eq. (5) holds as follows:

$$a_{1\psi}\otimes a_{2\phi}\otimes c^\psi c'^\phi=a_2\otimes a_5\otimes\alpha S(a_1)c\alpha(a_3)\alpha S(a_4)c'\alpha(a_6)=\\ a_2\otimes a_3\otimes\alpha S(a_1)cc'\alpha(a_4)$$

and

$$\Delta_H(a_\psi)\otimes(cc')^\psi=a_2\otimes a_3\otimes\alpha S(a_1)cc'\alpha(a_4)$$

So the category is a monoidal category if and only if $\alpha=\beta$.

Next, we define the braidings as follows:

$$t_{M,N}: M\otimes N\to N\otimes M,\quad m\otimes n\mapsto n_{(0)}\otimes m\alpha^{-1}(n_{(1)})$$

The corresponding map Q is as follows:

$$Q: H\otimes H\to H\otimes H,\ n\otimes k\mapsto\eta(\varepsilon(k))\otimes\alpha^{-1}(h)$$

Now we compute

$$(a_{2\psi}\otimes a_{1\phi})Q(d^\psi\otimes c^\phi)=\varepsilon(c)a_1\otimes\alpha^{-1}(d)\alpha^{-1}\beta(a_2)$$
$$Q(d\otimes c)\Delta_A(a)=\varepsilon(c)a_1\otimes\alpha^{-1}(d)a_2$$

So the equation $(a_{2\psi}\otimes a_{1\phi})Q(d^\psi\otimes c^\phi)=Q(d\otimes c)\Delta_A(a)$ holds if and only if $\alpha=\beta$. Similarly we can prove that the condition $\alpha=\beta$ is also the condition making Eq. (12) hold.

Finally, we check that Eqs. (13) and (14) are satisfied straightforwardly.

From the above computation, we can obtain the following theorem.

Theorem 9 The category of (α,β)-Yetter-Drinfeld modules is a braided monoidal category if and only if $\alpha=\beta$.

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构造 entwined 模范畴成为辫子张量范畴

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摘要:研究了如何使 entwined 模范畴成为辫子张量范畴. 首先,利用如果(A,C,ψ)是一个 entwining 结构,那么 $A\otimes C$ 形成 entwined 模的结论可以得到 entwined 模范畴成为张量范畴的充要条件. 条件是要求问题中的代数和余代数都必须为双代数而且满足某些相容条件. 然后,在给定的张量 entwined 模范畴上,通过一个扭曲卷积可逆映射 Q 定义了辫子,并且由类似的方法得到使 entwined 模范畴构成辫子张量范畴的充分必要条件. 最后,作为示例将得到的结果应用到 Doi-Hopf 模和(α,β)-Yetter-Drinfeld 模范畴中.

关键词: Doi-Hopf 模;entwined 模;辫子张量范畴

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