

# Comparison of two kinds of approximate proximal point algorithms for monotone variational inequalities

Tao Min

(School of Applied Mathematics and Physics, Nanjing University of Posts and Telecommunications, Nanjing 210046, China)

**Abstract:** This paper proposes two kinds of approximate proximal point algorithms (APPA) for monotone variational inequalities, both of which can be viewed as two extended versions of Solodov and Svaiter's APPA in the paper "Error bounds for proximal point subproblems and associated inexact proximal point algorithms" published in 2000. They are both prediction-correction methods which use the same inexactness restriction; the only difference is that they use different search directions in the correction steps. This paper also chooses an optimal step size in the two versions of the APPA to improve the profit at each iteration. Analysis also shows that the two APPAs are globally convergent under appropriate assumptions, and we can expect algorithm 2 to get more progress in every iteration than algorithm 1. Numerical experiments indicate that algorithm 2 is more efficient than algorithm 1 with the same correction step size.

**Key words:** monotone variational inequality; approximate proximate point algorithm; inexactness criterion

A classical variational inequality problem, denoted by  $VI(\Omega, F)$ , is to find a vector  $x^* \in \Omega$ , such that

$$VI(\Omega, F) \quad (x - x^*)^T F(x^*) \geq 0 \quad \forall x \in \Omega \quad (1)$$

where  $\Omega \in \mathbf{R}^n$  is a nonempty closed convex subset of  $\mathbf{R}^n$  and  $F$  is a continuous mapping from  $\mathbf{R}^n$  into itself.

$VI(\Omega, F)$  problems include nonlinear complementarity problems (when  $\Omega = \mathbf{R}_+^n$ ) and a system of nonlinear equations (when  $\Omega \subset \mathbf{R}^n$ ), and thus have many important applications in economics, operations research and nonlinear analysis, and they have been studied by many researchers<sup>[1]</sup>. For any  $\beta > 0$ , it is well known that<sup>[2]</sup>

$$u^* \text{ is a solution of } VI(\Omega, F) \Leftrightarrow u^* = P_\Omega[u^* - \beta F(u^*)] \quad (2)$$

where  $P_\Omega(\cdot)$  denotes the projection on  $\Omega$ . Denote

$$e(u, \beta) := u - P_\Omega[u - \beta F(u)] \quad (3)$$

A well known method for solving monotone variational inequalities is the proximal point algorithm (PPA)<sup>[3-4]</sup>. For given  $u^k \in \Omega$  and  $\beta_k > 0$ , the new iterate  $u^{k+1}$  of the exact version of the PPA is taken by  $u^{k+1} = u_*^{k+1}$ , where  $u_*^{k+1}$  is the exact solution of the following variational inequality:

$$(PPA) \quad u \in \Omega, (u' - u)^T F_k(u) \geq 0 \quad \forall u' \in \Omega \quad (4)$$

with

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**Biography:** Tao Min (1979—), female, master, lecturer, taominnju@yahoo.com.

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$$F_k(u) = (u - u^k) + \beta_k F(u) \quad (5)$$

An equivalent recursion form of the exact PPA is

$$u^{k+1} = P_\Omega[u^{k+1} - F_k(u^{k+1})] \quad (6)$$

The ideal form Eq. (6) of the method is often impracticable since in many cases solving problems exactly is either impossible or too expensive. In order to overcome such obstacles, some APPA methods have been proposed in Refs. [3, 5]. In this paper, we pay attention to compare two kinds of them. Given  $u^k \in \Omega$  and  $\beta_k > 0$ , let  $v_k \in \Omega$  be an approximate solution in the sense that

$$v^k \approx P_\Omega[v^k - F_k(v^k)] \quad (7)$$

and define

$$\tilde{v}^k := P_\Omega[v^k - F_k(v^k)] \quad (8)$$

We denote

$$\zeta^k := v^k - \tilde{v}^k \quad (9)$$

$$\Delta(u) := 2(u - P_\Omega[u - F_k(u)])^T F_k(u) - \|u - P_\Omega[u - F_k(u)]\|^2 \quad (10)$$

According to Eqs. (8) and (9), we have

$$\Delta(v^k) = 2(\zeta^k)^T F_k(v^k) - \|\zeta^k\|^2 \quad (11)$$

The general updated forms of algorithms 1 and 2 are

$$u^{k+1}(\alpha, v^k) := u_1^{k+1}(\alpha, v^k) = P_\Omega[u^k - \alpha(u^k - v^k + \zeta^k)] \quad (12)$$

$$u^{k+1}(\alpha, v^k) := u_2^{k+1}(\alpha, v^k) = P_\Omega[u^k - \alpha\beta_k F(v^k)] \quad (13)$$

respectively. We note that such updated forms are indirectly based on  $v^k$  and consider the following inexactness restriction:

$$\Delta(v^k) \leq v \|u^k - v^k\|^2 \quad v < 1 \quad (14)$$

First, we remark that in the case  $\alpha \equiv 1$ , it is clear that algorithm 1 reduces to algorithm 2, which was presented by Solodov and Svaiter<sup>[5]</sup>. If we adopt an optimal step length along this direction, we can achieve more profit at each iteration.

Our interest in this paper, however, is only to compare the efficiencies of algorithms 1 and 2. For any solution point  $u^* \in \Omega^*$  ( $\Omega^*$  denotes the solution set of  $VI(\Omega, F)$ , which is nonempty), let

$$\Theta_1(\alpha, v^k) := \|u^k - u^*\|^2 - \|u_1^{k+1}(\alpha, v^k) - u^*\|^2$$

$$\Theta_2(\alpha, v^k) := \|u^k - u^*\|^2 - \|u_2^{k+1}(\alpha, v^k) - u^*\|^2$$

We will prove that for two suitable introduced amounts  $\Phi_1$  and  $\Phi_2$ ,

$$\Theta_1(\alpha, \mathbf{v}^k) \geq \Phi_1(\alpha, \mathbf{v}^k) := \Phi(\alpha) + \|\mathbf{u}^k - \alpha(\mathbf{u}^k - \bar{\mathbf{v}}^k) - \mathbf{u}_1^{k+1}(\alpha, \mathbf{v}^k)\|^2 \quad (15)$$

$$\Theta_2(\alpha, \mathbf{v}^k) \geq \Phi_2(\alpha, \mathbf{v}^k) := \Phi(\alpha) + \|\mathbf{u}^k - \alpha(\mathbf{u}^k - \bar{\mathbf{v}}^k) - \mathbf{u}_2^{k+1}(\alpha, \mathbf{v}^k)\|^2 \quad (16)$$

$$\Phi_2(\alpha, \mathbf{v}^k) \geq \Phi_1(\alpha, \mathbf{v}^k) + \|\mathbf{u}_1^{k+1}(\alpha, \mathbf{v}^k) - \mathbf{u}_2^{k+1}(\alpha, \mathbf{v}^k)\|^2 \quad (17)$$

where

$$\Phi(\alpha) = 2\alpha\{\|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2 - \beta_k(\boldsymbol{\zeta}^k)^T F(\mathbf{v}^k)\} - \alpha^2\|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2$$

Moreover, it will be shown by an example that both the inequalities (15) and (16) are tight. Inequality (17) shows that algorithm 2 can achieve more profit at each iteration for given  $\mathbf{u}^k$ ,  $\mathbf{v}^k$  and  $\alpha$ .

Throughout this paper, we assume that  $\{\beta_k\} \subset [\beta, +\infty)$  and  $\beta > 0$ . The operator  $F$  is monotone and continuous on  $\Omega$ , and the solutions set of  $\text{VI}(\Omega, F)$ , denoted by  $\Omega^*$ , is not a singleton, and it is nonempty. We use  $\mathbf{u}^*$  to denote any point in  $\Omega^*$ .

## 1 Preliminaries

Let  $\Omega \subset \mathbf{R}^n$  be a nonempty closed convex set. For given  $\mathbf{w} \in \mathbf{R}^n$ , the projection mapping under the Euclidean norm ( $\|\cdot\|$ ), denoted by  $P_\Omega(\mathbf{w})$ , is defined as

$$P_\Omega(\mathbf{w}) = \arg \min\{\|\mathbf{w} - \mathbf{u}\| \mid \mathbf{u} \in \Omega\}$$

It follows from this definition that

$$(\mathbf{v} - P_\Omega(\mathbf{v}))^T (P_\Omega(\mathbf{v}) - \mathbf{w}) \geq 0 \quad \forall \mathbf{v} \in \mathbf{R}^n, \forall \mathbf{w} \in \Omega \quad (18)$$

Consequently, following from inequality (18), we have

$$\|P_\Omega(\mathbf{v}) - P_\Omega(\mathbf{w})\| \leq \|\mathbf{v} - \mathbf{w}\| \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{R}^n \quad (19)$$

$$\|P_\Omega(\mathbf{v}) - \mathbf{u}\| \leq \|\mathbf{v} - \mathbf{w}\| \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{R}^n \quad (20)$$

Note that  $e(\mathbf{u}, \beta)$  is a continuous function of  $\mathbf{u}$  because the projection mapping is non-expansive. The next lemma states that  $\|e(\mathbf{u}, \beta)\|$  is a non-decreasing function for  $\beta > 0$ .

**Lemma 1**<sup>[6]</sup> For all  $\mathbf{u} \in \mathbf{R}^n$  and  $\beta \geq \beta > 0$ , it holds that

$$\|e(\mathbf{u}, \bar{\beta})\| \geq \|e(\mathbf{u}, \beta)\| \quad (21)$$

**Lemma 2**<sup>[7]</sup> Let  $\beta > 0$  and  $\{\mathbf{u}^k\}$  be a bounded sequence and  $\lim_{k \rightarrow \infty} e(\mathbf{u}^k, \beta) = 0$ , then  $\{\mathbf{u}^k\}$  has a subsequence  $\{\mathbf{u}^{k_j}\}$  which converges to some  $\mathbf{u}^\infty$  which is a solution point of  $\text{VI}(\Omega, F)$ .

## 2 Preliminary Analysis for Algorithm 1 and Algorithm 2

In this section, we set up some preliminary results for algorithm 1 and algorithm 2.

**Theorem 1**<sup>[7]</sup> Given  $\mathbf{u}^k \in \Omega$  and  $\beta_k > 0$ , let  $\mathbf{v}^k \in \Omega$  be an approximate solution of inequality (4) in the sense of inequality (7) and the new iterate  $\mathbf{u}^{k+1}(\alpha, \mathbf{v}^k)$  be given by the general forms of algorithm 1 (Eq. (12)). Then for any  $\alpha > 0$  we have

$$\Theta_1(\alpha, \mathbf{v}^k) \geq \Phi_1(\alpha, \mathbf{v}^k) \quad (22)$$

where

$$\Theta_1(\alpha, \mathbf{v}^k) := \|\mathbf{u}^k - \mathbf{u}^*\|^2 - \|\mathbf{u}_1^{k+1}(\alpha, \mathbf{v}^k) - \mathbf{u}^*\|^2 \quad (23)$$

$$\Phi_1(\alpha, \mathbf{v}^k) := \Phi(\alpha) + \|\mathbf{u}^k - \alpha(\mathbf{u}^k - \bar{\mathbf{v}}^k) - \mathbf{u}_1^{k+1}(\alpha, \mathbf{v}^k)\|^2 \quad (24)$$

$$\Phi(\alpha) = 2\alpha\{\|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2 - \beta_k(\boldsymbol{\zeta}^k)^T F(\mathbf{v}^k)\} - \alpha^2\|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2 \quad (25)$$

$\bar{\mathbf{v}}^k$  and  $\boldsymbol{\zeta}^k$  are defined in Eqs. (8) and (9), respectively.

**Theorem 2**<sup>[7]</sup> Given  $\mathbf{u}^k \in \Omega$  and  $\beta_k > 0$ , let  $\mathbf{v}^k \in \Omega$  be an approximate solution of inequality (4) in the sense of inequality (7) and the new iterate  $\mathbf{u}^{k+1}(\alpha, \mathbf{v}^k)$  be given by the general forms of algorithm 2 (Eq. (13)). Then for any  $\alpha > 0$  we have

$$\Theta_2(\alpha, \mathbf{v}^k) \geq \Phi_2(\alpha, \mathbf{v}^k) \quad (26)$$

where

$$\Theta_2(\alpha, \mathbf{v}^k) := \|\mathbf{u}^k - \mathbf{u}^*\|^2 - \|\mathbf{u}_2^{k+1}(\alpha, \mathbf{v}^k) - \mathbf{u}^*\|^2 \quad (27)$$

$$\Phi_2(\alpha, \mathbf{v}^k) := \Phi(\alpha) + \|\mathbf{u}^k - \alpha(\mathbf{u}^k - \bar{\mathbf{v}}^k) - \mathbf{u}_2^{k+1}(\alpha, \mathbf{v}^k)\|^2 \quad (28)$$

$\Phi(\alpha)$  is defined in Eq. (25).

## 3 Main Theoretical Result

We are now at the stage to prove the convergence of algorithm 1 and algorithm 2. First, using  $\mathbf{u}^k - \bar{\mathbf{v}}^k = (\mathbf{u}^k - \mathbf{v}^k) + \boldsymbol{\zeta}^k$  (see Eq. (9)) and by some regrouping, we obtain

$$2(\|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2 - \beta_k(\boldsymbol{\zeta}^k)^T F(\mathbf{v}^k)) = (\|\mathbf{u}^k - \mathbf{v}^k\|^2 + \|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2 - \Delta(\mathbf{v}^k)) \quad (29)$$

Substituting Eq. (29) into Eq. (25), we have

$$\Phi(\alpha) = \alpha\{(\|\mathbf{u}^k - \mathbf{v}^k\|^2 + \|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2) - \Delta(\mathbf{v}^k)\} - \alpha^2\|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2$$

Since  $\Phi(\alpha)$  is a quadratic function of  $\alpha$ , it reaches its maximum at

$$\alpha_k^* = \frac{(\|\mathbf{u}^k - \mathbf{v}^k\|^2 + \|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2) - \Delta(\mathbf{v}^k)}{2\|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2} \quad (30)$$

with

$$\Phi(\alpha_k^*) = \frac{1}{2}\alpha_k^* ((\|\mathbf{u}^k - \mathbf{v}^k\|^2 + \|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2) - \Delta(\mathbf{v}^k)) \quad (31)$$

Under inexactness restriction (14), it follows from Eqs. (30) and (31) that

$$\alpha_k^* \geq \frac{1-\nu}{2}$$

$$\Phi(\alpha_k^*) \geq \frac{(1-\nu)^2}{4} (\|\mathbf{u}^k - \mathbf{v}^k\|^2 + \|\mathbf{u}^k - \bar{\mathbf{v}}^k\|^2)$$

For fast convergence, we propose a relaxation factor  $\gamma_k \in [\gamma_L, \gamma_U] \subset [1, 2)$  and set the step-size  $\alpha_k$  in Eqs. (12) and (13) by

$$\alpha_k = \gamma_k \alpha_k^* \quad (32)$$

By simple manipulations, we obtain

$$\begin{aligned} \Phi(\gamma_k \alpha_k^*) &= 2\gamma_k \alpha_k^* (\|u^k - \tilde{v}^k\|^2 - \beta_k (\zeta^k)^T F(v^k)) - \\ &(\gamma_k^2 \alpha_k^*) (\alpha_k^* \|u^k - \tilde{v}^k\|^2) = (2\gamma_k \alpha_k^* - \gamma_k^2 \alpha_k^*) \cdot \\ &(\|u^k - \tilde{v}^k\|^2 - \beta_k (\zeta^k)^T F(v^k)) = \gamma_k (2 - \gamma_k) \Phi(\alpha_k^*) \end{aligned} \quad (33)$$

It follows from theorems 1 and 2 that

$$\begin{aligned} \|u^{k+1} - u^*\|^2 &\leq \|u^k - u^*\|^2 - \frac{\gamma_k (2 - \gamma_k) (1 - \nu)^2}{4} \cdot \\ &(\|u^k - v^k\|^2 + \|u^k - \tilde{v}^k\|^2) \end{aligned} \quad (34)$$

**Theorem 3** Given  $u^k \in \Omega$  and  $\beta_k \geq \beta > 0$ , let  $v^k \in \Omega$  be an approximate solution of inequality (4) in the sense of inequality (7). If the inexactness criterion (14) holds, then the sequence  $\{u^k\}$  generated by algorithms 1 and 2 converges to some  $u^\infty$  which is a solution point of  $VI(\Omega, F)$ .

**Proof** It follows from inequality (34) that  $\{u^k\}$  is bounded,

$$\lim_{k \rightarrow \infty} \|u^k - v^k\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|u^k - \tilde{v}^k\| = 0 \quad (35)$$

Moreover, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \|\zeta^k\| &= \lim_{k \rightarrow \infty} \|(u^k - \tilde{v}^k) - (u^k - v^k)\| \leq \\ \lim_{k \rightarrow \infty} \|u^k - \tilde{v}^k\| + \lim_{k \rightarrow \infty} \|u^k - v^k\| &= 0 \end{aligned} \quad (36)$$

Using lemma 1,

$$\begin{aligned} \|e(v^k, \beta)\| &\leq \|v^k - P_\Omega[v^k - \beta_k F(v^k)]\| = \\ \|\zeta^k + \tilde{v}^k - P_\Omega[v^k - \beta_k F(v^k)]\| &\leq \|\zeta^k\| + \\ \|P_\Omega[u^k - \beta_k F(v^k)] - P_\Omega[v^k - \beta_k F(v^k)]\| &\leq \\ \|\zeta^k\| + \|u^k - v^k\| \end{aligned} \quad (37)$$

It follows from Eqs. (35) to (37) and the continuousness of  $e(u, \beta)$  that

$$\lim_{k \rightarrow \infty} e(u^k, \beta) = 0$$

Then it follows from lemma 2 that we have a subsequence  $\{u^{k_j}\} \subset \{u^k\}$  which converges to some  $u^\infty$ , and  $u^\infty$  is a solution point of  $VI(\Omega, F)$ . Since inequality (34) is true for all solution points of  $VI(\Omega, F)$ , we have

$$\|u^{k+1} - u^\infty\| \leq \|u^k - u^\infty\| \quad \forall k \geq 0$$

and it follows that the sequence  $\{u^k\}$  generated by algorithms 1 and 2 converges to  $u^\infty$ .

In general, inequality (22) in theorem 1 (inequality (26) in theorem 2) is tight. This can be seen from the following example. Let us consider a  $VI(\Omega, F)$  with

$$\Omega = \mathbf{R}^2, \quad F(u) = Mu, \quad M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This variational inequality is monotone and has a unique solution  $u^* = \mathbf{0}$ . Note that

$$M^2 = -I, \quad M^T M = I$$

For any  $u^k \in \Omega$ ,  $\beta \in (0, 1)$ , let  $v^k = \tilde{v}^k$ , obviously it satisfies inexactness restriction (14), then we have

$$v^k = \tilde{v}^k = u^k - \beta_k M v^k \quad (38)$$

so

$$u^k = v^k + \beta_k M v^k \quad (39)$$

Using  $(v^k)^T M v^k = \mathbf{0}$ ,  $\|M v^k\| = \|v^k\|$  and Eq. (39), we obtain

$$\|u^k - \alpha \beta_k M v^k\|^2 = \|u^k\|^2 - 2\alpha \beta_k^2 \|v^k\|^2 + \alpha^2 \beta_k^2 \|v^k\|^2 \quad (40)$$

When the problem is solved by algorithm 1 (Eq. (12)), we have

$$\begin{aligned} \Theta_1(\alpha, v^k) &= \|u^k\|^2 - \|u_1^{k+1}(\alpha, v^k)\|^2 = \|u^k\|^2 - \\ \|u^k - \alpha(u^k - \tilde{v}^k)\|^2 &= \|u^k\|^2 - \|u^k - \alpha(u^k - (u^k - \beta_k M v^k))\|^2 = \\ \|u^k\|^2 - \|u^k - \alpha \beta_k M v^k\|^2 &= (2\alpha - \alpha^2) \beta_k^2 \|v^k\|^2 \end{aligned}$$

and

$$\begin{aligned} \Phi_1(\alpha, v^k) &= \Phi(\alpha, v^k) = 2\alpha \|u^k - \tilde{v}^k\|^2 - \alpha^2 \|u^k - \tilde{v}^k\|^2 = \\ (2\alpha - \alpha^2) \beta_k^2 \|v^k\|^2 \end{aligned}$$

so

$$\Theta_1(\alpha, v^k) = \Phi_1(\alpha, v^k)$$

For algorithm 2, in this special example, we have

$$\begin{aligned} \Theta_2(\alpha, v^k) &= \|u^k\|^2 - \|u_2^{k+1}(\alpha, v^k)\|^2 = \\ \|u^k\|^2 - \|u^k - \alpha \beta_k M v^k\|^2 &= (2\alpha - \alpha^2) \beta_k^2 \|v^k\|^2 \\ \Phi_2(\alpha, v^k) &= \Phi(\alpha, v^k) = (2\alpha - \alpha^2) \beta_k^2 \|v^k\|^2 \end{aligned}$$

We also have  $\Theta_2(\alpha, v^k) = \Phi_2(\alpha, v^k)$ .

Nevertheless, the following theorem indicates that in each iterative step, we may expect algorithm 2 to get more progress than algorithm 1.

**Theorem 4** Let  $\Phi_1(\alpha, v^k)$  and  $\Phi_2(\alpha, v^k)$  be defined as in Eqs. (24) and (28). We have

$$\Phi_2(\alpha, v^k) - \Phi_1(\alpha, v^k) \geq \|u_2^{k+1}(\alpha, v^k) - u_1^{k+1}(\alpha, v^k)\|^2 \quad (41)$$

**Proof** It follows from Eqs. (24) and (28) that

$$\begin{aligned} \Phi_2(\alpha, v^k) - \Phi_1(\alpha, v^k) &\geq \|u^k - \alpha(u^k - \tilde{v}^k) - u_2^{k+1}(\alpha, v^k)\|^2 - \\ \|u^k - \alpha(u^k - \tilde{v}^k) - u_1^{k+1}(\alpha, v^k)\|^2 \end{aligned} \quad (42)$$

Note that  $u_2^{k+1}(\alpha, v^k) \in \Omega$ , setting  $v = u^k - \alpha(u^k - \tilde{v}^k)$  and  $u = u_2^{k+1}(\alpha, v^k)$  in inequality (20) we obtain

$$\begin{aligned} \|u_1^{k+1}(\alpha, v^k) - u_2^{k+1}(\alpha, v^k)\|^2 &\leq \|u^k - \alpha(u^k - \tilde{v}^k) - u_2^{k+1}(\alpha, v^k)\|^2 - \\ \|u^k - \alpha(u^k - \tilde{v}^k) - u_1^{k+1}(\alpha, v^k)\|^2 \end{aligned} \quad (43)$$

The assertion of this theorem follows directly from inequalities (42) and (43).

## 4 Numerical Experiments and Conclusion

The detailed algorithms 1 and 2 are as follows:

**Step 1** Let  $\beta_k \equiv 0.1$ ,  $u^0 \in \Omega$ ,  $0 < \nu < 1$ ,  $\gamma = 1.8$ ,  $\varepsilon = 10^{-6}$ .

**Step 2** Find an approximate solution of inequality (4), i. e., find  $v^k$  in the sense that

$$v^k \approx P_\Omega[v^k - \beta_k F(v^k)] \quad (44)$$

under inexactness restriction (14).

**Step 3** Compute the new iterate

$$u_1^{k+1}(\alpha, v^k) = P_\Omega[u^k - \gamma_k \alpha_k^* (u^k - \tilde{v}^k)]$$

or

$$\boldsymbol{u}_2^{k+1}(\boldsymbol{\alpha}, \boldsymbol{v}^k) = P_{\Omega}[\boldsymbol{u}^k - \gamma_k \boldsymbol{\alpha}_k^* F(\boldsymbol{v}^k)]$$

To test the proposed algorithms, we consider the nonlinear complementarity problems (NCP): Find  $\boldsymbol{u} \in \mathbf{R}^n$  such that

$$\boldsymbol{u} \geq \mathbf{0}, \quad F(\boldsymbol{u}) \geq \mathbf{0}, \quad \boldsymbol{u}^T F(\boldsymbol{u}) = \mathbf{0} \tag{45}$$

In our test problems we take

$$F(\boldsymbol{u}) = D(\boldsymbol{u}) + \boldsymbol{M}\boldsymbol{u} + \boldsymbol{q} \tag{46}$$

where  $D(\boldsymbol{u})$  and  $\boldsymbol{M}\boldsymbol{u} + \boldsymbol{q}$  are the nonlinear part and the linear part of  $F(\boldsymbol{u})$ , respectively. The matrix  $\boldsymbol{M} = \boldsymbol{A}^T \boldsymbol{A} + \boldsymbol{B}$ , where  $\boldsymbol{A}$  is an  $n \times n$  matrix whose entries are randomly generated in the interval  $(-5, 5)$  and a skew-symmetric matrix  $\boldsymbol{B}$  is generated in the same way. The vector  $\boldsymbol{q}$  is generated from a uniform distribution in the interval  $(-500, 500)$ . In  $D(\boldsymbol{u})$ , the nonlinear part of  $F(\boldsymbol{u})$ , the components are  $D_j(\boldsymbol{u}) = d_j \arctan(u_j)$ , where  $d_j$  is a random variable in  $(0, 1)$ .

All iterations start with  $\boldsymbol{u}^0$ , whose entries are randomly generated in the interval  $(0, 10)$ . We take  $\beta_k \equiv 0.1$  and  $\gamma_k \equiv 1.8$ . We use the improved extra-gradient method<sup>[8]</sup> to solve the sub-problem (4) approximately. For comparison, we use the same inexactness restriction (14) with  $\nu = 0.3$ . The relaxation factor  $\gamma_k$  in Eq. (32) should lie in the interval  $[1, 2]$ . All codes are written in Matlab and run on a TCL T51 notebook computer. The computation stops as soon as  $\|e(\boldsymbol{u}^k, 1)\|_{\infty} \leq 10^{-6}$  (see Eq. (3)). We report the iteration numbers and CPU time of algorithms 1 and 2 in Tabs. 1 and 2, respectively. Since the sub-problems are solved approximately by the iterative method in Ref. [8], the number of total inner iterations is also reported.

Tab.1 Numerical results of algorithm 1

$n$	Number of outer iterations	Number of total inner iterations	CPU time/s
100	37	551	0.169 901
200	39	1 035	0.629 120
300	40	1 436	1.695 101
400	42	1 763	3.754 972
500	43	2 020	5.639 794

Tab.2 Numerical results of algorithm 2

$n$	Number of outer iterations	Number of total inner iterations	CPU time/s
100	37	482	0.155 687
200	38	958	0.594 977
300	39	1 406	1.673 128
400	41	1 649	3.581 007
500	42	1 914	5.437 483

The numerical results show that algorithm 2 outperforms algorithm 1.

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求解单调变分不等式的两类近似邻近点算法比较

陶 敏

(南京邮电大学数理学院, 南京 210046)

摘要: 将 Solodov 和 Svaiter 于 2000 年发表的 Error bounds for proximal point subproblems and associated inexact proximal point algorithms 一文中提出的方法进行推广, 得到 2 类近似邻近点算法. 这 2 类算法都是预测-校正方法, 预测点满足相同的非精确准则, 不同之处在于校正步的下降方向. 为了使每次迭代产生的迭代点更加靠近解点, 在校正步均采用了最优步长的技巧. 在一定条件下, 可以证明这 2 种邻近点算法是全局收敛的. 并且, 从理论上证明了采用算法 2 每一步所产生的下降量的下界大于算法 1 的, 所以算法 2 比算法 1 能更快地收敛到解点. 数值试验也表明了这一点.

关键词: 单调变分不等式; 近似邻近点算法; 非精确准则

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