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# Nonlocal controllability for semilinear problems in Banach spaces

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**Abstract:** If $A$: $D(A) \subset X \to X$ is a densely defined and closed linear operator, which generates a linear semigroup $S(t)$ in Banach space $X$. The nonlocal controllability for the following nonlocal semilinear problems: $u'(t) = Au(t) + Bx(t) + f(t, u(t))$, $0 \leqslant t \leqslant T$ with nonlocal initial condition $u(0) = u_0 + g(u)$ is discussed in Banach space $X$. The results show that if semigroup $S(t)$ is strongly continuous, the functions $f$ and $g$ are compact and the control $B$ is bounded, then it is nonlocally controllable. The nonlocal controllability for the above nonlocal problem is also studied when $B$ and $W$ are unbounded and the semigroup $S(t)$ is compact or strongly continuous. For illustration, a partial differential equation is worked out.

**Key words:** nonlocal problem; nonlocal controllability; mild solution; completely continuous

Let $X$ be a real Banach space and $A$: $D(A) \subset X \to X$ be the infinitesimal generator of a strongly continuous semigroup $S(t)$ of bounded linear operators. We discuss the controllability for the following nonlocal Cauchy problem:

$$\left.\begin{aligned} &u'(t) = Au(t) + Bx(t) + f(t, u(t)) \qquad 0 \leqslant t \leqslant T \\ &u(0) = u_0 + g(u) \end{aligned}\right\} \quad (1)$$

where $f$: $[0, T] \times X \to X$, $g$: $C([0, T]; X) \to X$ are given $X$-valued functions; the operator $B$: $L^2(0, T; U) \to L(0, T; X)$ is linear, $U$ is a real Banach space, and $x$ is a control function.

Recently, as it can be applied in physics with better effect than the classical initial conditions, the nonlocal differential equations and their controllability in Banach spaces have been discussed by many researchers (see Refs. [1 – 10] and their references). Some of the controllability results are given under the hypotheses: the semigroup $S(t)$ generated by $A$ is compact and the linear operator $W$ (defined in hypothesis $H_3$ below) has a bounded inverse. But, in fact, if the above two hypotheses are both true, then the space $X$ must be finite dimensional (see remark 1).

In this paper, we will discuss the nonlocal controllability for the Cauchy problem (1) when the semigroup $S(t)$ is not compact.

Let $X$, $U$ be real Banach spaces with norms $\|\cdot\|$. Denote $E$ by the Banach space $C([0, T]; X)$ of $X$-valued continuous functions defined on $[0, T]$ with the norm $|u| = \sup\limits_{0 \leqslant t \leqslant T}\{\|u(t)\|\}$; and $L(0, T; X)$ by the Banach space of $X$-valued Bochner integrable functions defined on $[0, T]$ with the norm $\|u\|_1 = \int_0^T \|u(t)\| \mathrm{d}t$. Denote by the Banach space $L^2(0, T; U) = \left\{x: [0, T] \to U; x(\cdot) \text{ measurable and } \int_0^T \|x(t)\|^2 \mathrm{d}t < +\infty\right\}$, with the norm $\|x\|_2 = \left(\int_0^T \|x(t)\|^2 \mathrm{d}t\right)^{1/2}$.

In this paper, we suppose that the semigroup $S(t)$ generated by $A$ is strongly continuous in Banach space $X$. (The definition and properties of semigroups can be seen in Ref. [11]).

**Definition 1** By the mild solution $u \in E$ of problem (1), we mean that the function $u$ satisfies:

$$u(t) = S(t)(u_0 + g(u)) + \int_0^t S(t-s)(Bx(s) + f(s, u(s)))\mathrm{d}s \qquad 0 \leqslant t \leqslant T \quad (2)$$

**Definition 2** The nonlocal problem (1) is said to be nonlocally controllable if for every $u_0$ and $u_1 \in X$, there exists a control function $x \in L^2(0, T; U)$, such that there is a mild solution $u$ satisfying $u(T) = u_1 + g(u)$.

## 1 Main Results

In this section, we will study the controllability of the nonlocal Cauchy problem (1). First we give some hypotheses:

$\mathbf{H}_1$ 1) The function $f(t, \cdot)$ is continuous for $0 \leqslant t \leqslant T$ a. e., and $f(\cdot, x)$ is measurable for any $x \in X$;

2) There exists a function $a \in L(0, T; \mathbf{R}^+)$ and a nondecreasing function $\Omega$: $\mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\|f(t, x)\| \leqslant a(t)\Omega(\|x\|)$$

for any $x \in X$ and $0 \leqslant t \leqslant T$ a. e..

3) The function $f$: $[0, T] \times X \to X$ is compact.

$\mathbf{H}_2$ 1) The function $g$: $E \to X$ is completely continuous (i. e., compact and continuous);

2) There exists a nondecreasing function $\Lambda$: $\mathbf{R}^+ \to \mathbf{R}^+$ such that $\|g(u)\| \leqslant \Lambda(|u|)$ for any $u \in E$.

$\mathbf{H}_3$ The linear operator $B$: $L^2(0, T; U) \to L(0, T; X)$ is bounded; and the operator $W$: $L^2(0, T; U) \to X$, defined by $Wx = \int_0^T S(T-s)Bx(s)\mathrm{d}s$ for $x \in L^2(0, T; U)$, has a bounded inverse $W^{-1}$: $X \to L^2(0, T; U)/\ker W$, where the kernel space of $W$ is defined by $\ker W = \{x \in L^2(0, T; U); Wx = 0\}$.

$$\mathbf{H}_4 \qquad \|a\|_1 < \limsup_{r \to +\infty} \frac{r - c_1 - c_2\Lambda(r)}{c_3\Omega(r)} \quad (3)$$

where

$$c_1 = M(\|u_0\| + \sqrt{T}\|B\| \|W^{-1}\| + \sqrt{T}M\|B\| \|W^{-1}\| \|u_0\|)$$
$$c_2 = M(1 + \sqrt{T}\|B\| \|W^{-1}\| + \sqrt{T}M\|B\| \|W^{-1}\|)$$
$$c_3 = M(1 + \sqrt{T}M\|B\| \|W^{-1}\|)$$

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$$M=\sup\{\|S(t)\|, 0\leqslant t\leqslant T\}$$

**Remark 1** If the semigroup $S(t)$ is compact and the linear operator $B$ is bounded, and the operator $W$ must be compact. So the Banach space $X$ must be finite dimensional if $W$ has a bounded inverse.

The operator $W$ is bounded if the operator $B$ is bounded, and due to the famous inverse mapping theorem, it has a bounded inverse if it is surjective. Ref. [7] gave an example for the suspension bridge model such that hypothesis $H_3$ is satisfied.

**Remark 2** If $\Omega(r)=o(r)$, $\Lambda(r)=o(r)$ for a large $r$, then hypothesis $H_4$ is true.

If we suppose that hypothesis $H_3$ is satisfied, then for $u\in E$, we define the map $F: E\to E$ as

$$Fu(t)=S(t)(u_0+g(u))+\int_0^t S(t-s)(Bx(s)+f(s,u(s)))\,ds \qquad 0\leqslant t\leqslant T$$

where the control $x$ is defined as

$$x=W^{-1}\left[u_1+g(u)-S(T)(u_0+g(u))-\int_0^T S(T-s)f(s,u(s))\,ds\right] \tag{4}$$

Then problem (1) is nonlocally controllable if and only if the fixed points of map $F$ exist.

Now we give the main results.

**Theorem 1** If hypotheses $H_1$ to $H_4$ are satisfied, then problem (1) is nonlocally controllable.

**Proof** We prove this theorem by the following steps:

**Step 1** There exists $r>0$ such that $F(B_r)\subset B_r$, where $B_r=\{u\in E, |u|\leqslant r\}$.

Clearly, $Fu\in E$ for any $u\in E$. By hypothesis $H_4$, there exists $r>0$ such that

$$\|a\|_1\leqslant\frac{r-c_1-c_2\Lambda(r)}{c_3\Omega(r)}$$

Then for any $u\in B_r$ the control $x$, defined in Eq. (4), satisfies that

$$\|x\|_2\leqslant\|W^{-1}\|\left[\|u_1\|+\|g(u)\|+M\left(\|u_0\|+\|g(u)\|+\int_0^T\|f(s,u(s))\|\,ds\right)\right]\leqslant$$
$$\|W^{-1}\|[\|u_1\|+\Lambda(r)+M(\|u_0\|+\Lambda(r)+\|a\|_1\Omega(r))]$$

So we obtain that

$$\|Fu(t)\|\leqslant M\left[\|u_0\|+\Lambda(r)+\sqrt{t}\|B\|\,\|x\|_2+\Omega(r)\int_0^T a(s)\,ds\right]\leqslant c_1+c_2\Lambda(r)+c_3\Omega(r)\|a\|_1\leqslant r$$

for any $t\in[0,T]$, i. e., $F(B_r)\subset B_r$.

**Step 2** The map $F: B_r\to B_r$ is continuous.

We suppose that $u_n\to u$ in $E$, then we obtain that

$$\|x_n-x\|_2\leqslant\|W^{-1}\|\left[(1+M)\sqrt{T}\|g(u_n)-g(u)\|+M\left(\int_0^T\|f(s,u_n(s))-f(s,u(s))\|^2 ds\right)^{1/2}\right]$$

for any $t\in[0,T]$, where

$$x_n=W^{-1}\left[u_1+g(u_n)-S(T)(u_0+g(u_n))-\int_0^T S(T-s)f(s,u_n(s))\,ds\right]$$

By hypotheses $H_1$ to $H_3$ and the Lebesgue convergence theorem, we know that $x_n\to x$ in $L^2(0,T;U)$, and

$$\|Fu_n(t)-Fu(t)\|\leqslant M\|g(u_n)-g(u)\|+\sqrt{T}M\|B\|\,\|W^{-1}\|\,\|x_n-x\|_2+M\int_0^T\|f(s,u_n(s))-f(s,u(s))\|\,ds \qquad t\in[0,T]$$

So $F: B_r\to B_r$ is continuous due to hypotheses $H_1$, $H_2$, and the Lebesgue convergence theorem.

**Step 3** $F(B_r)(t)\subset X$ is precompact for any $t\in[0,T]$.

By hypothesis 3) of $H_1$ and the strong continuity of the semigroup $S(t)$, we know that $S(\cdot)f(\cdot,\cdot):[0,T]\times[0,T]\times X\to X$ is compact, and then the set $S=\{S(t)f(s,u(s)), u\in B_r, 0\leqslant t, s\leqslant T\}\subset X$ is precompact. So, for any $t\in[0,T]$,

$$S'_t=\left\{\int_0^t S(t-s)f(s,u(s))\,ds, u\in B_r\right\}\subset t\,\overline{\mathrm{conv}\,S}$$

is precompact in $X$, where $\overline{\mathrm{conv}S}$ means the closure of the convex hull of $S$ in $X$. By hypotheses 1) of $H_2$ and $H_3$, we obtain that

$$S''=\left\{x=W^{-1}\left[u_1+g(u)-S(T)(u_0+g(u))-\int_0^T S(T-s)f(s,u(s))\,ds\right], u\in B_r\right\}$$

is precompact in $L^2(0,T;U)$. As $B: L^2(0,T;U)\to L(0,T;X)$ is bounded, it implies that $BS''\subset L(0,T;X)$ is precompact. So we know that

$$\left\{\int_0^t S(t-s)Bx(s)\,ds; x\in S''\right\}\subset X$$

is precompact as the map $y\mapsto\int_0^t S(t-s)y(s)\,ds: L(0,T;X)\to X$ is continuous. As $S(t)$ is bounded, and for any $t\in[0,T]$,

$$F(B_r)(t)\subset S(t)(u_0+g(B_r))+\left\{\int_0^t S(t-s)Bx(s)\,ds; x\in S''\right\}+S'_t$$

we have that $F(B_r)(t)$ is precompact in $X$ for every $t\in[0,T]$.

**Step 4** $F: B_r\to B_r$ is compact.

By step 3 we should only prove that $F(B_r)\subset E$ is equicontinuous.

For any fixed $t\in[0,T]$, any $h>0$, and $u\in B_r$,

$$\|Fu(t+h)-Fu(t)\|\leqslant\left\|\int_t^{t+h}S(t-s)(Bx(s)+f(s,u(s)))\,ds\right\|+\left\|(S(h)-I)\left[S(t)(u_0+g(u))+\int_0^t S(t-s)(Bx(s)+f(s,u(s)))\right]ds\right\|$$

$\leqslant MC_r h + \| (S(h) - I)Fu(t) \|$

where $C_r = \sup\{ \| Bx(s) + f(s, u(s)) \| ; u \in B_r, s \in [0, T] \}$. As $F(B_r)(t)$ is precompact and the semigroup $S(t)$ is strongly continuous, we have that $\| Fu(t+h) - Fu(t) \|$ is uniformly convergent to 0 at any $t \in [0, T]$ when $h \to 0^+$ in $B_r$. It implies that $F(B_r) \subset E$ is equicontinuous.

From the above discussions we know that the map $F: B_r \to B_r$ is completely continuous, and then there exists at least one fixed point of $F$ due to the famous Schauder's fixed point theorem; i. e., the nonlocal Cauchy problem (1) is nonlocally controllable. The proof is complete.

Next, we give some hypotheses which are weaker than hypotheses $H_3$ and $H_4$, such that problem (1) is also nonlocally controllable.

$\mathbf{H}'_3$ The operator $B$: $L^2(0, T; U) \to L(0, T; X)$ is linear, the linear operator $W$: $L^2(0, T; U)/\ker W \to X$ is invertible, and the linear operator $BW^{-1}: X \to L^2(0, T; X)$ is bounded.

$\mathbf{H}'_4$
$$\| a \|_1 < \limsup_{r \to +\infty} \frac{r - d_1 - d_2 \Lambda(r)}{d_3 \Omega(r)}$$

where

$$d_1 = M( \| u_0 \| + \sqrt{T} \| BW^{-1} \| + \sqrt{T} M \| BW^{-1} \| \| u_0 \| )$$
$$d_2 = M(1 + \sqrt{T} \| BW^{-1} \| + \sqrt{T} M \| BW^{-1} \| )$$
$$d_3 = M(1 + \sqrt{T} M \| BW^{-1} \| )$$

**Remark 3** Here, as the operator $B$ is not assumed to be bounded, then the operator $W$ is not compact even when the $C_0$-semigroup $S(t)$ is compact. So we can discuss the controllability of problem (1) in infinite dimensional Banach spaces under hypothesis $H'_3$ when the semigroup $S(t)$ is strongly continuous or compact.

We can prove:

**Theorem 2** Let hypotheses $H_1, H_2, H'_3$, and $H'_4$ be satisfied, then the nonlocal problem (1) is nonlocally controllable.

**Theorem 3** If $A$ generates a compact $C_0$-semigroup $S(t)$, and hypotheses 1) and 2) of $H_1, H_2, H'_3$, and $H'_4$ are true, then the nonlocal problem (1) is nonlocally controllable.

## 2 An Example

In this section, an example is presented for the controllability of the following partial differential equation.

Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ with a smooth boundary $\partial\Omega$. Now we discuss the following semilinear elliptic equation:

$$\left.\begin{aligned} &\frac{\partial u(\xi, t)}{\partial t} + \sum_{|\alpha| \leqslant 2m} a_\alpha(\xi) D^\alpha u(\xi, t) = \\ &\qquad \int_\Omega h(t, \xi, \eta, u(\eta, t))\,\mathrm{d}\eta + b(\xi) x(t) \\ &\qquad (\xi, t) \in \Omega \times [0, 1] \\ &u(\xi, t) = 0 \qquad (\xi, t) \in \partial\Omega \times [0, 1] \\ &u(\xi, 0) = u_0(\xi) + \int_\Omega \int_0^1 k(t, \xi, \eta, u(\eta, t))\,\mathrm{d}t\,\mathrm{d}\eta \\ &\qquad \xi \in \Omega \end{aligned}\right\} \quad (5)$$

where $a_\alpha(\xi)$ is a smooth real function on $\overline{\Omega}$, $h, k: [0, 1] \times \Omega \times \Omega \times \mathbf{R} \to \mathbf{R}$, $b: \Omega \to \mathbf{R}$ are given functions, $x \in L^2(0, 1; L^2(\partial\Omega))$ and $u_0 \in L^2(\Omega)$.

We suppose that:

1) The differential operator $\sum_{|\alpha| \leqslant 2m} a_\alpha(\xi) D^\alpha$ is strongly elliptic[11].

2) The function $h$: $[0, 1] \times \Omega \times \Omega \times \mathbf{R} \to \mathbf{R}$ is continuous and satisfies that

a) For any $N > 0$, there exists $w_N \in L^2((0, 1) \times \Omega \times \Omega \times \mathbf{R}; \mathbf{R}^+)$, such that

$$| h(t, \xi, \eta, r) - h(t, \xi', \eta, r) | \leqslant w_N(t, \xi, \xi', \eta)$$

for $t \in (0, 1)$ a. e., $\xi, \xi', \eta \in \Omega$, $|r| \leqslant N$, and for any $t \in [0, 1]$,

$$\lim_{\Delta\xi \to 0} \int_\Omega (w_N(t, \xi, \xi + \Delta\xi, \eta))^2 \mathrm{d}\eta = 0$$

uniformly for $\xi \in \Omega$;

b) There exist $0 < \alpha < 1$, $a(\cdot) \in L(0, 1)$, and $d(\cdot) \in L(0, 1; L^2(\Omega \times \Omega))$, such that

$$| h(t, \xi, \eta, r) | \leqslant a(t) |r|^\alpha + d(t, \xi, \eta)$$

for $t \in (0, 1)$ a. e., $\xi, \eta \in \Omega$ a. e., and any $r \in \mathbf{R}$.

3) The function $k$: $[0, 1] \times \Omega \times \Omega \times \mathbf{R} \to \mathbf{R}$ is continuous and satisfies that

a) For any $N > 0$, there is $\beta_N \in L^2((0, 1) \times \Omega \times \Omega \times \mathbf{R}; \mathbf{R}^+)$ such that

$$| k(t, \xi, \eta, r) - k(t, \xi', \eta, r) | \leqslant \beta_N(t, \xi, \xi', \eta)$$

for $t \in (0, 1)$ a. e., $\xi, \xi', \eta \in \Omega$, $|r| \leqslant N$, and

$$\lim_{\Delta\xi \to 0} \int_0^1 \int_\Omega (w_N(t, \xi, \xi + \Delta\xi, \eta))^2 \mathrm{d}\eta \mathrm{d}t = 0$$

uniformly for $\xi \in \Omega$;

b) There are $\overline{a}(\cdot) \in L(0, 1)$ and $\overline{d}(\cdot) \in L(0, 1; L^2(\Omega \times \Omega))$, such that

$$| k(t, \xi, \eta, r) | \leqslant \overline{a}(t) |r|^\alpha + \overline{d}(t, \xi, \eta)$$

for $t \in (0, 1)$ a. e., $\xi, \eta \in \Omega$ a. e., and $r \in \mathbf{R}$.

Let $D(A) = H^{2m}(\Omega) \cap H_0^m(\Omega)$ and $Au(\xi) = -\sum_{|\alpha| \leqslant 2m} a_\alpha(\xi) D^\alpha u(\xi, \cdot)$; then $A$ generates an analytic semigroup on $X = L^2(\Omega)$[11]. We suppose that

$$f(t, u)(\xi) = \int_\Omega h(t, \xi, \eta, u(\eta, t))\,\mathrm{d}\eta \qquad \xi \in \Omega$$

$$g(u)(\xi) = \int_\Omega \int_0^1 k(t, \xi, \eta, u(\eta, t))\,\mathrm{d}t\,\mathrm{d}\eta \qquad \xi \in \Omega$$

Then we can prove that hypotheses $H_1$ and $H_2$ are satisfied for the above functions $f$ and $g$[12]; let

$$Bx(t)(\xi) = b(\xi) x(t) \qquad t \in [0, 1]; \ \xi \in \Omega$$

then $B: L^2(0, 1; L^2(\partial\Omega)) \to L(0, 1; L^2(\Omega))$ is bounded. If we suppose that the operator $W$ is surjective, then hypothesis $H_3$ is true. As $\alpha < 1$, by remark 2, we know that hypothesis $H_4$ is satisfied. So, by using theorem 1, problem (5) is nonlocally controllable.

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# Banach 空间中半线性问题的非局部可控性

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**摘要：**设 $A:D(A)\subset X\rightarrow X$ 是 Banach 空间 $X$ 上的线性稠定的闭算子，它是 $X$ 上的强连续有界线性算子半群 $S(t)$ 的无穷小生成元. 对于 Banach 空间 $X$ 中的含非局部初值条件 $u(0)=u_0+g(u)$ 的半线性 Cauchy 问题：$u'(t)=Au(t)+Bx(t)+f(t,u(t))$，在 $A$ 生成的线性算子半群 $S(t)$ 是非紧，映射 $f$ 和 $g$ 满足一定的紧性条件，控制算子 $B$ 是有界线性算子时，证明了该问题是非局部可控的. 并分别在半群是紧或强连续的条件下，证明了在控制算子 $B$ 和 $W$ 不是有界情形时上面的非局部 Cauchy 问题是非局部可控的. 同时给出了在偏微分方程中的可控性问题的一个应用.

**关键词：**非局部问题；非局部可控；适度解；全连续
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