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Rings satisfying UR-stable range

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Abstract: A ring R is said to be satisfying P-stable range provided that whenever $aR+bR=R$, there exists $y\in P(R)$ such that $a+by$ is a unit of R, where $P(R)$ is the subset of R which satisfies the property that up, $pu\in P(R)$ for every unit u of R and $p\in P(R)$. By studying this ring, some known results of rings satisfying unit-1 stable range, $(S,2)$-stable range, weakly unit 1-stable range and stable range one are unified. An element of a ring is said to be UR if it is the sum of a unit and a regular element and a ring is said to be satisfying UR-stable range if R has P-stable range and $P(R)$ is the set of all UR-elements of R. Some properties of this ring are studied and it is proven that if R satisfies UR-stable range then so does any $n\times n$ matrix ring over R.

Key words: stable range condition; UR-ring; matrix extension

Throughout, all rings are associative with identity. For a ring R, let $P(R)$ be the subset of R which satisfies the property that up, $pu\in P(R)$ for every $p\in P(R)$ and $u\in U(R)$, where $U(R)$ stands for the group of units. We call a ring R satisfying P-stable range provided that whenever $aR+bR=R$, there exists $y\in P(R)$ such that $a+by\in U(R)$. In particular, a ring R is said to have stable range one[1-2], unit 1-stable range[3-6], $(S,2)$-stable range[7] or weakly unit 1-stable range[8] if R satisfies P-stable range and $P(R)=R$, $P(R)=U(R)$, $P(R)=\{u_1+u_2: u_1,u_2\in U(R)\}$ or $P(R)=\{y\in R$: there exists a positive integer n such that $y=e+u_1+\dots+u_2$, where $e^2=e$ is a central idempotent of R and $u_1,\dots,u_n\in U(R)\}$, respectively. Many researchers studied these rings with units since these conditions are useful in the study of algebraic K-theory and the cancellation theory in categories of modules over rings. We unify some known results of rings satisfying unit-1 stable range, $(S,2)$-stable range, weakly unit 1-stable range and stable range one.

Recall that an element a of a ring R is said to be (von Neumann) regular if there exists $b\in R$ such that $a=aba$, and a ring R is regular if each element is regular. Following Ref. [9], an element is called a UR-element if it is the sum of a unit and a regular element, and a ring is called a UR-ring if every element is UR. UR-rings are shown to be a unifying generalization of three important classes of rings: regular rings, $(S,2)$-rings (i. e., every element is the sum of two units) and clean rings (i. e., every element is the sum of an idempotent and a unit). Clearly, if $a\in R$ is a UR-element, then au and ua are both UR-elements for any $u\in U(R)$. So we can call a ring R satisfying UR-stable range if R has P-stable range and $P(R)$ is the set of all UR-elements of R. Some properties of rings satisfying UR-stable range are studied in this paper.

For a ring R, let $\boldsymbol{M}_n(R)$ and $\boldsymbol{T}_n(R)$ be the rings of all $n\times n$ matrices and all $n\times n$ upper triangular matrices over R, respectively. The symbol $J(R)$ stands for the Jacobson radical. For convenience, we write the elementary $\boldsymbol{B}_{ij}(x)=\mathrm{diag}(1,1)+xe_{ij}(1\leqslant i,j\leqslant 2$ with $i\neq j)$, where e_{ij} are matrix units.

1 Rings Satisfying *P*-Stable Range

In this section, we give some equivalent characterizations of a ring satisfying P-stable range.

Proposition 1 For a ring R, the following are equivalent:

1) R satisfies P-stable range.

2) Whenever $ax+b=1$, there exists $y\in P(R)$ such that $a+by\in U(R)$.

3) Whenever $ax+b=1$, there exists $y\in R$ such that $a+by\in U(R)$ and $1-xy\in P(R)$.

Proof 1)⇒2) is trivial.

2)⇒3). Suppose that $ax+b=1$ with $a,x,b\in R$. From $xa+(1-xa)=1$, we have $z\in P(R)$ such that $x+(1-xa)z=u\in U(R)$. Then

$$\boldsymbol{B}_{21}(-z)\begin{bmatrix} a & -b\\ 1 & x\end{bmatrix}=\begin{bmatrix} a & -b\\ 1-za & x+zb\end{bmatrix}=\begin{bmatrix} u^{-1} & -b\\ 0 & x+zb\end{bmatrix}\boldsymbol{B}_{21}(u^{-1}-azu^{-1})$$

So we have

$$\begin{bmatrix} a & -b\\ 1 & x\end{bmatrix}\boldsymbol{B}_{21}(azu^{-1}-u^{-1})=\boldsymbol{B}_{21}(z)\begin{bmatrix} u^{-1} & -b\\ 0 & x+zb\end{bmatrix}=\begin{bmatrix} u^{-1} & -b\\ zu^{-1} & x\end{bmatrix}$$

This shows that $a+by=u^{-1}\in U(R)$ and $1-xy=zu^{-1}\in P(R)$ by the definition of $P(R)$, where $y=u^{-1}-azu^{-1}$.

3)⇒2). Given $ax+b=1$ in R. Since $xa+(1-xa)=1$, we have some $y\in R$ such that $x+(1-xa)y=u\in U(R)$ and $1-ay\in P(R)$. Hence,

$$\boldsymbol{B}_{21}(y)\begin{bmatrix} a & b\\ -1 & x\end{bmatrix}=\begin{bmatrix} a & b\\ ya-1 & yb+x\end{bmatrix}=\begin{bmatrix} u^{-1} & b\\ 0 & yb+x\end{bmatrix}\boldsymbol{B}_{21}((ay-1)u^{-1})$$

This implies that

$$\begin{bmatrix} a & b\\ -1 & x\end{bmatrix}\boldsymbol{B}_{21}((1-ay)u^{-1})=\boldsymbol{B}_{21}(-y)\begin{bmatrix} u^{-1} & b\\ 0 & yb+x\end{bmatrix}=\begin{bmatrix} u^{-1} & b\\ -yu^{-1} & x\end{bmatrix}$$

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It shows that $a+b(1-ay)u^{-1}=u^{-1}\in U(R)$, where $(1-ay)u^{-1}\in P(R)$.

2)$\Rightarrow$1). Given $aR+bR=R$, then $ax+by=1$ for some $x, y\in R$. By hypothesis, there exists $y_1\in P(R)$ such that $b+axy_1=u\in U(R)$. Hence, $axy_1u^{-1}+bu^{-1}=1$. By hypothesis again, there exists $y_2\in P(R)$ such that $a+bu^{-1}y_2\in U(R)$, where $u^{-1}y_2\in P(R)$.

Theorem 1 For a ring R, the following are equivalent:

1) Whenever $ax+b=1$, there exists $y\in P(R)$ such that $a+by\in U(R)$.

2) Whenever $ax+b=1$, there exists $y\in P(R)$ such that $x+yb\in U(R)$.

Proof 1)$\Rightarrow$2). Let $ax+b=1$. By proposition 1, there exists $w\in R$ such that $a+bw=u\in U(R)$ and $1-xw\in P(R)$. Let $y=(1-xw)u^{-1}$ and $v=x+yb$. Then $y\in P(R)$ and it can be verified that $v\in U(R)$ with the inverse $a+w(1-xa)$.

2)$\Rightarrow$1). Given $ax+b=1$. As the proof in proposition 1, there exists $w\in R$ such that $x+wb\in U(R)$ and $1-wa\in P(R)$. By symmetry, we can complete the proof.

From theorem 1, we obtain that a ring R satisfies P-stable range if and only if the opposite ring R^o of R satisfies P-stable range. This actually shows that the notion of a ring satisfying P-stable range is left-right symmetric. In other words, a ring R satisfies P-stable range if and only if whenever $Ra+Rb=R$, there exists $y\in P(R)$ such that $a+yb\in U(R)$.

Proposition 2 For a ring R, the following are equivalent:

1) R satisfies P-stable range.

2) Whenever $aR+bR=dR$ with $a, b, d\in R$, there exist $u\in U(R)$ and $v\in P(R)$ such that $au+bv=d$.

3) Whenever $a_1R+\ldots+a_nR=dR$ with $n\geqslant 2$, $a_1,\ldots,a_n, d\in R$, there exist $u_1\in U(R)$ and $u_2,\ldots,u_n\in P(R)$ such that $a_1u_1+\ldots+a_nu_n=d$.

Proof Both 3)$\Rightarrow$2) and 2)$\Rightarrow$1) are trivial.

1)$\Rightarrow$2). Since R satisfies P-stable range, R satisfies stable range one. Whenever $aR+bR=dR$ with $a, b, d\in R$, the sets (a, b) and $(d, 0)$ generate the same R-submodule of R^2. Therefore, there exists $U=(u_{ij})\in \mathrm{GL}_2(R)$ such that $(a, b)=(d, 0)U$ from lemma 2.1 in Ref. [4]. Clearly, $u_{11}R+u_{12}R=R$. Since R satisfies P-stable range, there exists some $y\in P(R)$ such that $u_{11}+u_{12}y=v\in U(R)$. This implies $a+by=dv$, so $av^{-1}+byv^{-1}=d$, where $v^{-1}\in U(R)$ and $yv^{-1}\in P(R)$.

2)$\Rightarrow$3). Given $a_1R+\ldots+a_nR=dR$ with $n\geqslant 2$, $a_1,\ldots,a_n$, $d\in R$. If $n=2$, then the result holds from 2). Assume that the result holds for all $k\leqslant n$ $(k\geqslant 2)$. Let $n=k+1$. Take $x_1,\ldots,x_{k+1}\in R$ with $\sum_{i=1}^{k+1}a_ix_i=d$. Then $a_1R+\ldots+a_{k-1}R+(a_kx_k+a_{k+1}x_{k+1})R=dR$. Hence there exist $u_1\in U(R)$ and $u_2,\ldots,u_k\in P(R)$ such that $a_1u_1+\ldots+a_{k-1}u_{k-1}+(a_kx_k+a_{k+1}x_{k+1})u_k=d$. This gives $(a_1u_1+a_{k+1}x_{k+1}u_k)R+a_2R+\ldots+a_{k-1}R+a_kR=dR$. And so there exist $v_1\in U(R)$ and $v_2,\ldots,v_k\in P(R)$ such that $(a_1u_1+a_{k+1}x_{k+1}u_k)v_1+a_2v_2+\ldots+a_kv_k=d$. This implies that $(a_1u_1v_1+a_2v_2)R+\ldots+a_kR+a_{k+1}R=dR$. Then there exist $w_1\in U(R)$ and $w_2,\ldots,w_k\in P(R)$ such that $a_1u_1v_1w_1+a_2v_2w_1+\ldots+a_kw_{k-1}+a_{k+1}w_k=d$. Clearly, $u_1v_1w_1\in U(R)$ and $v_2w_1, w_2,\ldots,w_k\in P(R)$. This shows that 3) holds.

We should point out that there are some special cases of the above results. For example, proposition 1 generalizes lemma 2.1 in Ref. [5] and lemma 3.1 in Ref. [7]; theorem 1 generalizes theorem 4.5 in Ref. [8]; and proposition 2 generalizes theorem 2.2 in Ref. [4] and theorem 5.3 in Ref. [8].

2 Rings Satisfying UR-Stable Range

As we discussed in the introduction, the results above on rings satisfying P-stable range are also true for rings satisfying UR-stable range. Clearly, we have the inclusion {Rings satisfying unit 1-stable range} $\subseteq$ {$(S, 2)$-rings} $\cap$ {Rings having stable range one} $\subseteq$ {Rings satisfying $(S, 2)$-stable range} $\subseteq$ {Rings satisfying UR-stable range}. Let $R=\mathbf{Z}_2$ be the ring of integers modulo 2. Then R satisfies UR-stable range, but not $(S, 2)$-stable range.

It was proven in Ref. [2] that a ring R satisfies stable range one if and only if so does $R/J(R)$. Now we consider a similar case for rings satisfying UR-stable range.

Proposition 3 If R satisfies UR-stable range, then so does $R/J(R)$, and the converse holds if idempotents can be lifted modulo $J(R)$.

Proof Given $\bar{a}\bar{x}+\bar{b}=\bar{1}$ in $\bar{R}=R/J(R)$, then $ax+b=1+j$ for some $j\in J(R)$, which implies $ax(1+j)^{-1}+b(1+j)^{-1}=1$. Thus, there exists a UR-element $y\in R$ such that $a+by\in U(R)$. This shows that $\bar{a}+\bar{b}\bar{y}\in U(\bar{R})$ and $\bar{y}$ is a UR-element in $\bar{R}$.

Next, suppose that $\bar{R}$ satisfies UR-stable range and idempotents can be lifted modulo $J(R)$. Given $ax+b=1$ in R, then $\bar{a}\bar{x}+\bar{b}=\bar{1}$ in $\bar{R}$. So there exists a UR-element $\bar{y}$ in $\bar{R}$ such that $\bar{a}+\bar{b}\bar{y}=\bar{u}\in U(\bar{R})$. Thus, there exists $\bar{v}\in U(\bar{R})$ such that $\bar{a}\bar{v}+\bar{b}\bar{y}\bar{v}=\bar{1}$. Since units can always be lifted modulo $J(R)$ and UR-elements can also be lifted modulo $J(R)$ by lemma 2.4 in Ref. [9], we may assume that $v\in U(R)$ and y is a UR-element in R. Thus $av+byv=1+j$ for some $j\in J(R)$, which implies that $a+by=(1+j)v^{-1}\in U(R)$. The proof is completed.

In the following, we investigate the extension property of rings satisfying UR-stable range.

Lemma 1 Let $e_1, e_2,\ldots,e_n$ be idempotents of a ring R. If all e_iRe_i satisfy UR-stable range, then so does the ring.

$$\boldsymbol{S}_n=\begin{bmatrix} e_1Re_1 & \ldots & e_1Re_n \\ \vdots & & \vdots \\ e_nRe_1 & \ldots & e_nRe_n \end{bmatrix}$$

Proof We can complete the proof by making some necessary modifications in the proof of theorem 3.2 in Ref. [7].

Theorem 2 The following are equivalent for a ring R:

1) R satisfies UR-stable range.

2) There exists a complete set $\{e_1, e_2,\ldots,e_n\}$ of orthogonal idempotents such that all e_iRe_i satisfy UR-stable range.

Proof 1)$\Rightarrow$2). It is obvious.

2)$\Rightarrow$1). Let S_n be the same as that in lemma 1. Construct a map $\varphi: R\to \boldsymbol{S}_n$ given by $\boldsymbol{\varphi}(r)=\begin{bmatrix} e_1re_1 & \ldots & e_1re_n \\ \vdots & & \vdots \\ e_nre_1 & \ldots & e_nre_n \end{bmatrix}$. Since $\{e_1, e_2,\ldots,e_n\}$ is a complete set of orthogonal idempotents, it is easy to prove that φ is a ring isomorphism. By virtue of lemma 1, R satisfies UR-stable range.

The following results are immediately clear by theorem 2.

Corollary 1 If a ring R satisfies UR-stable range, then so does $\boldsymbol{M}_n(R)$ for any positive integer n.

Corollary 2 Let $M, M_1, \ldots, M_n$ be modules. If $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ and each $\mathrm{End}(M_i)$ satisfies UR-stable range, then $\mathrm{End}(M)$ also satisfies UR-stable range.

Corollary 3 A ring R satisfies UR-stable range iff $\boldsymbol{T}_n(R)$ satisfies UR-stable range for any positive integer n.

Proof The "only if" part is by induction and theorem 2, and the "if" part follows by the fact that $(r_{ij}) \in \boldsymbol{T}_n(R)$ is UR iff r_{ii} is UR in R for every $i = 1, 2, \ldots, n$.

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具有 UR-稳定度的环

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摘要：环 R 具有 P 稳定度是指若有 $aR + bR = R$，则存在 $y \in P(R)$ 使得 $a + by$ 是 R 中的可逆元. 其中 $P(R)$ 是环 R 的子集并满足如下性质：对于任意的可逆元 u 和 $p \in P(R)$ 都有 $up, pu \in P(R)$. 通过对环 R 的研究，统一了关于具有可逆-1 稳定度、$(S,2)$-稳定度、弱可逆-1 稳定度和稳定度为 1 的环的一些已知结果. 当环的一个元素是一个可逆元和一个正则元之和，则称这个元素为 UR. 如果环 R 具有 P 稳定度且 $P(R)$ 是环中所有 UR 元素组成的集合，则称环 R 具有 UR-稳定度. 研究了该环的性质，并证明了如果 R 具有 UR-稳定度，则 R 上的任意 n 阶矩阵环也具有 UR-稳定度.

关键词：稳定度条件；UR-环；矩阵扩张

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