Journal of Southeast University (English Edition) Vol. 25, No. 1, pp. 138 – 141 Mar. 2009 ISSN 1003—7985

Application of martingale theory in enterprise investment decision making

Huang Chao[1,2] Da Qingli[2]

([1] School of Economics and Management, Nanjing University of Information Science and Technology, Nanjing 210044, China)
([2] School of Economics and Management, Southeast University, Nanjing 210096, China)

Abstract: From the point of view of the basic option model, enterprise investment decision making under uncertainty is studied based on the martingale method. The study shows that investment options and yields are increasing functions of time, and when the option equals the yield, the investment opportunity cost is the least, which is the appropriate time for the enterprise investment. Under the condition that the investment yield is an increasing function of time, the investment opportunity cost is also an increasing function of time after the time when the investment option equals the investment yield. So the investors should invest as soon as possible, otherwise they should stop investment forever in this project. It is impossible to acquire more investment yields by indefinitely delaying the investment. Meanwhile, the study also shows that the martingale method, used widely in financial investment theory, is a powerful tool for enterprise investment decision making.

Key words: enterprise investment theory; investment option; investment yield; martingale

During the early development of modern investment theory, research was mainly centered on the field of macro-economics. From the point of view of gross analysis, the multiplier and acceleration theory, the economic growth theory, and the international investment theory gradually come into being after Keynes published *The General Theory of Employment, Interest and Money*. All of them made great contributions in many respects, such as the theory model, the influence factor, the investment analysis method, etc. But, as the micro foundations of the macro investment theory, the enterprise investment theory developed very slowly, and did not make great breakthroughs until the 1960s[1]. In 1963, Jorgenson[2] analyzed the relationships among investment elements by introducing an optimization method into an enterprise investment theory. He studied the investment behavior of firms as an investment body from the point of view of microeconomics. Jorgenson's most significant contribution to the enterprise investment theory was the establishment of the optimal capital function, which indicated that modern investment theory was coming into being.

Many researchers further developed and perfected the enterprise investment theory after Jorgenson. Arrow put forward the irreversibility theory of investment, and Tobin presented the q theory, and so on. The q theory among these theories has gained much attention. Tobin regards investment decision making as a function of q, which is the ratio of market value of capital to its product cost or replacement cost. If q is greater than 1, firms should invest, otherwise not[3]. Tobin divided q into two types, (one was an average q; the other was a marginal q), and deemed that the marginal q really influenced investment decision making, but it could not be measured. Tobin's q theory made great progress in the development of the enterprise investment theory, since Tobin introduced an adjustment function about investment cost into the investment theory and made up for the shortcomings of Jorgenson's theory.

Both Jorgenson's investment theory and Tobin's q theory assumed that the motion routes of exogenous variables in the model are known; i. e. they studied the investment behavior of firms under certainty. Dixit and Pindyck even pointed out that these two theories were both based on the NPV investment principle[4]. Apparently, the assumption was inconsistent with the actual conditions in a market environment, so the investment decision making of firms under uncertainty became the main content of modern investment theory. Lucas et al. [5] introduced uncertainty into investment theory very early, while the effect of uncertainty on the investment behavior of firms had been thoroughly studied after real options came into being. When Myers[6] studied real project investment by the NPV method, he found that the NPV method lacked manageable flexibility during investment decision making. Then, he developed the real option concept, which was similar to the concept of the financial option. The real option concept emphasizes the effects of time value and management flexibility on the enterprise investment, and there also exist investment options in real projects under uncertainty.

With the development of the research methods, such as the optimization and the asset pricing, the investment model is becoming very delicate and precise. Meanwhile, the enterprise investment theory mainly focuses on the investment return, but it ignores investment opportunities. The real option theory considers the effect of timing on investment, but there is no study on the investment opportunities. As a mathematical model of fair gambling, martingale has been widely used in financial investment theory for twenty to thirty years, especially in finance asset pricing, but there is not a relative application in the enterprise investment theory. This paper will introduce the martingale method into the enterprise investment theory and analyze investment behaviors of firms including the problem of investment opportunities.

1 Investment Decision Making Model of Firm

In the market environment, the effect factor on enterprise investment decision making is uncertainty. It is difficult for a firm to decide whether to invest or not, while it also makes it

Received 2008-07-22.
Biography: Huang Chao(1968—), male, doctor, lecturer, cchhuang@sina.com.
Citation: Huang Chao, Da Qingli. Application of martingale theory in enterprise investment decision making[J]. Journal of Southeast University (English Edition), 2009, 25(1): 138 – 141.

flexible to select an investment opportunity. The investor can wait in order to obtain more information about an investment and dynamically adjust investment strategies in order to avoid deciding whether to invest immediately or not. The traditional investment theories, such as the Marshall theory, the Jorgenson theory, Tobin's q theory, do not consider the effects of uncertainty on investment, so they cannot be applied to analyze the problem of investment decision making of firms under uncertainty.

Option theory effectively solves the uncertainty in investment decision making. Because of the uncertainty of the market concerning investment yields, the investment yield is uncertain and fluctuates stochastically, so investors always keep waiting before exercising investment. They only invest when the investment return equals the expected value. It indicates that the investor has a right to invest in the future, which is essentially an option.

Considering an investment project: I is the investment cost, V is the investment yield, V is a statistic, and it follows the following stochastic lognormal function

$$\mathrm{d}V=\alpha V\mathrm{d}t+\sigma V\mathrm{d}w \tag{1}$$

where α and σ are the draft and volatility parameters, respectively; w is the standard Brownian motion in a complete probability space (Ω, F, P); $\{F(t)\}$ is the σ-algebra of the natural filtration of $\{w(t)\}$.

Assuming that F is the investment option of the project, V^* is the investment return threshold and T^* is the investment time when V first reaches V^*. Because of the stochastic nature of V, T^* is also stochastic. According to the option pricing theory, F is given by

$$F(t)=\max_{t\leqslant T^*}\{V(t)-I, E[\exp(-r(T^*-t))F(T^*)\mid F(0)]\} \tag{2}$$

Before time T^*, there is no investment, and now $F(T^*)=V^*-I$. Therefore, Eq. (2) can be transformed into

$$F(t)=\begin{cases}(V^*-I)E[\exp(-r(T^*-t))\mid F(0)] & t<T^*\\ V^*-I & t\geqslant T^*\end{cases} \tag{3}$$

Eq. (3) is a basic investment option model. In the following, V^* and F can be solved by the martingale method.

Because of Brownian motion's spatial homogeneity[7−9], assume that V_0 is the initial value of the investment return. Let $\mu=(\alpha-\sigma^2/2)$ and then we have

$$\sigma w(t)=\ln V(t)-\mu t$$

$$w(0)=\frac{\ln V_0}{\sigma}$$

Define the following stochastic process

$$Z(t)=\exp\left\{\xi\ln V(t)-\left(\mu\xi+\frac{1}{2}\sigma^2\xi^2\right)t\right\} \qquad \xi\in\mathbf{R}, t\geqslant 0 \tag{4}$$

Hence,

$$Z(t)=\exp\left(\xi\sigma w(t)-\frac{1}{2}\sigma^2\xi^2 t\right) \tag{5}$$

Eq. (5) shows that the stochastic process $Z(t)$ is a Wald martingale. Letting $q(\xi)=\mu\xi+\frac{1}{2}\sigma^2\xi^2$, then substituting it into Eq. (4), we obtain

$$Z(t)=\exp(\xi\ln V(t)-q(\xi)t)$$

so

$$E[Z(T^*)\mid F(0)]=\exp(\xi\ln V_0)$$

Let $q(\xi)=r$, then we can obtain

$$E[\exp(\xi\ln V^*-rT^*)\mid F(0)]=\exp(\xi\ln V_0)$$

$$E[\exp(-rT^*)\mid F(0)]\exp(\xi\ln V^*)=\exp(\xi\ln V_0)$$

$$E[\exp(-rT^*)\mid F(0)]=\frac{\exp(\xi\ln V_0)}{\exp(\xi\ln V^*)}$$

Hence,

$$E[\exp(-rT^*)\mid F(0)]=\left(\frac{V_0}{V^*}\right)^{\xi} \tag{6}$$

where ξ can be obtained by solving the following equation:

$$\frac{1}{2}\sigma^2\xi^2+\left(\alpha-\frac{\sigma^2}{2}\right)\xi-r=0$$

If $E[\exp(-rT^*)\mid F(0)]$ is reasonable, ξ should equal the position foot of the quadratic equation, i. e. $\xi=\frac{1}{2}-\frac{\alpha}{\sigma^2}+\sqrt{\left(\frac{\alpha}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}$.

V^*-I is the investment return at T^* when the investor invests, and let Y equal the expected present value of V^*-I. Hence, we have

$$Y=(V^*-I)E[\exp(-rT^*)\mid F(0)]=(V^*-I)\left(\frac{V_0}{V^*}\right)^{\xi}$$

In order to obtain the investment return threshold V^*, let $\frac{\mathrm{d}Y}{\mathrm{d}V^*}=0$, and we can obtain

$$\left(\frac{V_0}{V^*}\right)^{\xi}-(V^*-I)\xi\left(\frac{V_0}{V^*}\right)^{\xi-1}\frac{V_0}{(V^*)^2}=0$$

$$V^*=\frac{\xi}{\xi-1}I \tag{7}$$

Reforming the option expression of Eq. (3) when $t<T^*$, the following form is obtained:

$$F(t)=(V^*-I)\exp(rt)E[\exp(-rT^*)\mid F(0)]$$

Substituting Eqs. (6) and (7) into the expression above, we find that

$$F(t)=\left(\frac{\xi}{\xi-1}I-I\right)\exp(rt)\left(\frac{V_0}{V^*}\right)^{\xi}=$$
$$\frac{I}{\xi-1}[\exp(rt)(V_0)^{\xi}]\left(\frac{\xi-1}{\xi I}\right)^{\xi}=(V(t))^{\xi}\xi^{-\xi}(\xi-1)^{\xi-1}I^{1-\xi}$$

Then, constituting Eq. (7) into the option expression of Eq. (3) when $t \geqslant T^*$, we can obtain

$$F(t) = \begin{cases} (V(t))^{\xi} \xi^{-\xi} (\xi - 1)^{\xi - 1} I^{1-\xi} & t < T^* \\ \dfrac{I}{\xi - 1} & t \geqslant T^* \end{cases} \tag{8}$$

Eq. (8) is completely consistent with the option expression in traditional real option theory.

We know from Eq. (7) that an investor should obtain a maximal investment return if he/she invests at V^* where $V^* = \xi I/(\xi - 1)$, but the investor does not know when V will reach V^*, which is often ignored because real option theory usually pays more attention to investment return thresholds. In fact, investment return and investment time are two aspects of the same problem, and, furthermore, it is the exact problem that the investor should face directly. Just as mentioned above, T^* indicates the investment time, which is a stochastic variable, so selecting the investment time is actually to obtain the expectation of T^*. Because V follows geometric Brownian motion, we can obtain the following equation according to the Ito lemma:

$$E[\ln V^* - \ln V_0] = E[\mu T^* + \sigma w(T^*) - \sigma w(0)]$$

$$E[T^*] = \frac{1}{\mu} \ln \frac{V^*}{V_0}$$

Substituting Eq. (7) into the above equation, we can obtain

$$E[T^*] = \frac{1}{\mu} \ln \frac{\xi I}{(\xi - 1) V_0} \tag{9}$$

where μ and ξ are both functions of the draft α and the volatility σ. α is the reason of investment return V showing certainty in its motion process, and σ the reason of V showing uncertainty. So Eq. (9) indicates that investment time has a stochastic nature. Thus, for an investor, the selection of investment timing is actually to find the expected value of T^*.

2 Numerical Example

We know that the stochastic property of investment return V has an important effect on investment decision making, and investors should pay more attention to the uncertainty part in V's motion. The following example studies the effects of the uncertainty on the investment option and investment time.

We can observe in Fig. 1 the value of the investment option or return at different Ts, where σ may also equal different values, and other parameters are fixed ($V_0 = 20, I = 40, r = 0.09, \alpha = 0.12$). Fig. 1 shows that the investment option F and return $V - I$ are both increasing functions of time T, and there is a tangency point between curves F and $V - I$; the time at this point is just the expected value of investment exercise time T^*. When σ equals 0.1, 0.2 or 0.35, the expected value of T^* is roughly 25, 32 or 89. It is found that the investor trends to delay investing and wishes to obtain a great return when uncertainty is high. This is explained by the fact that the higher the threshold value of the investment return is, the greater σ is.

While the traditional real option theory is applied to ana-

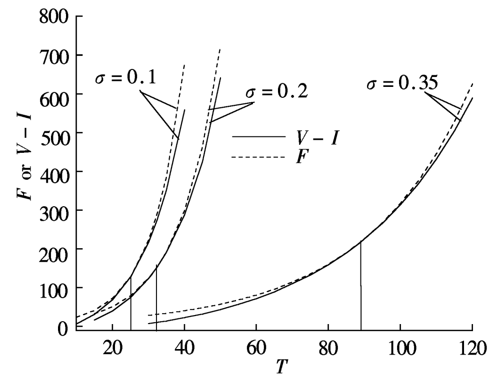

Fig. 1 Investment option or return as a function of time

lyze the relationship between investment option and return by numerical examples in much of the literature, it is found that the investment option curve and the return curve will incorporate a curve after time T^* due to the fact that the investment option equals the investment return. From Fig. 1, we can see that the higher the investment return is, the longer the time when an investor delays investing is. Because the investment return is an increasing function of time, we may get the wrong conclusion that an investor always indefinitely delays investing. In order to explain why an investor invests at tangency points but not indefinitely delay investment, we extend the investment option curve according to the relationship between investment option and time when $T < T^*$. Fig. 1 shows that the extended part of the investment option curve does not overlap the investment return curve after $T > T^*$, and the option curve is always above the return curve except when they are tangent at T^*; the difference between the two curves is just the opportunity cost of the investment. As increasing functions of time, the curve slopes of the investment option and return are greater than zero, and the curve slope of the investment option is smaller than the one of the investment return before time T^*, but greater after T^*. It indicates that waiting for investment is the optimal strategy for an investor before time T^*, because if an investor exercises investment too early, there is a bigger investment opportunity cost, which is a decreasing function of T when $T < T^*$. But the investment opportunity cost would become an increasing function of T if $T \geqslant T^*$, and it equals zero when $T = T^*$; therefore, T^* is the exact time for an investor to invest. If $T \geqslant T^*$ and the investor does not invest at T, he/she should invest as soon as possible, otherwise he/she should not.

3 Conclusion

In a complicated and rapidly changing market environment, the enterprise investment theory should consider more factors and much more information in order to be freed from a lot of assumptions and restrictions, and the theory should become very delicate and precise through deep development by using powerful mathematical methods. In modern investment theory, uncertainty is a basic factor with important effects on investments. As a mathematical method to describe uncertainty, the stochastic method is widely applied in investment theory, but as an important content of the stochastic process, martingale is still not used in the real option theory despite its wide application in financial investment theory. The martingale method is introduced into the enterprise investment theory to analyze a firm's investment behavior in this paper. The study shows that the martingale method has the same function in analyzing investments compared with other mathematical methods, but it is more delicate and simple. The problem of investment timing is also studied in this paper, and the result indicates that the investment opportunity cost is the smallest at the time when investment option equals investment return, which is only the optimal investment time.

References

[1] Wang Duan. The investment theory in modern macroeconomics and its newest development[J]. *Economic Research Journal*, 2000, **52**(12): 54 − 77. (in Chinese)

[2] Jorgenson D W. Capital theory and investment behavior[J]. *American Economic Review*, 1963, **53**(2): 247 − 259.

[3] Tobin J. A general equilibrium approach to monetary theory [J]. *Journal of Money*, *Credit and Banking*, 1969, **1**(1): 15 − 29.

[4] Dixit A K, Pindyck R S. *Investment under uncertainty* [M]. Princeton: Princeton University Press, 1994.

[5] Lucas Robert E Jr, Prescott Edward C. Investment under uncertainty[J]. *Econometrics*, 1971, **39**(5): 659 − 681.

[6] Myers S C. Determinants of corporate borrowing[J]. *Journal of Financial Economics*, 1977, **5**(2): 147 − 175.

[7] Lin Yuanlie. *The application of stochastic process*[M]. Beijing: Tsinghua University Press, 2002. (in Chinese)

[8] Zhang Po, Zhang Jingxiao. *The application of stochastic process* [M]. Beijing: Tsinghua University Press, 2004. (in Chinese)

[9] Jin Zhiming. *The fundamentals and application of stochastic analysis*[M]. Beijing: National Defence Industry Press, 2004. (in Chinese)

鞅理论在企业投资决策中的应用

黄　超[1,2]　　达庆利[2]

([1] 南京信息工程大学经济管理学院,南京 210044)

([2] 东南大学经济管理学院,南京 210096)

摘要:从基本的期权模型出发,利用鞅方法分析研究了不确定情况下的企业投资决策问题.研究表明投资期权和投资收益均是时间的增函数,且投资期权曲线相切于投资收益曲线,在切点处,企业的投资机会成本最小,故该时刻是企业的最佳投资时刻.在投资收益是时间的增函数情况下,进一步分析指出若当前时刻位于投资期权等于投资收益的时刻之后,投资机会成本也是时间的增函数,因此,企业要么立即投资,要么永不投资,而不会无限期地延迟投资以获取较大投资收益.研究同时表明广泛应用于金融投资理论中的鞅方法在企业投资决策研究中也具有强大的分析功能.

关键词:企业投资理论;投资期权;投资收益;鞅

中图分类号:F830. 59