

Weak M -Armendariz rings

Zhang Cuiping^{1,2} Chen Jianlong¹

(¹Department of Mathematics, Southeast University, Nanjing 210096, China)

(²Department of Mathematics, Northwest Normal University, Lanzhou 730070, China)

Abstract: For a monoid M , this paper introduces the weak M -Armendariz rings which are a common generalization of the M -Armendariz rings and the weak Armendariz rings, and investigates their properties. Moreover, this paper proves that: a ring R is weak M -Armendariz if and only if for any n , the n -by- n upper triangular matrix ring $T_n(R)$ over R is weak M -Armendariz; if I is a semicommutative ideal of ring R such that R/I is weak M -Armendariz, then R is weak M -Armendariz, where M is a strictly totally ordered monoid; if a ring R is semicommutative and M -Armendariz, then R is weak $M \times N$ -Armendariz, where N is a strictly totally ordered monoid; if a finitely generated Abelian group G is torsion-free if and only if there exists a ring R such that R is weak G -Armendariz.

Key words: semicommutative rings; M -Armendariz rings; weak Armendariz rings; weak M -Armendariz rings

Throughout this paper R denotes an associative ring with identity, $\text{nil}(R)$ denotes the set of all nilpotent elements of R and M denotes a monoid with identity e . Rege and Chhawchharia^[1] introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \dots + a_mx^m$, $g(x) = b_0 + b_1x + \dots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i and j . The name “Armendariz ring” was chosen because Armendariz^[2] had noted that a reduced ring satisfies this condition. Some properties, examples and counterexamples of Armendariz rings were given in Refs. [1–5]. A monoid M is called a u. p. -monoid (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$, there exists an element $g \in M$ uniquely presented in the form ab where $a \in A$ and $b \in B$. Liu^[6] called a ring R M -Armendariz if whenever elements $\alpha = a_1g_1 + \dots + a_mg_m$, $\beta = b_1h_1 + \dots + b_nh_n \in R[M]$ satisfy $\alpha\beta = 0$, then $a_ib_j = 0$ for each i and j , which is a generalization of Armendariz rings. He showed that a finite generated Abelian group G is torsion-free if and only if there exists a ring R such that R is G -Armendariz. He also showed that if R is a reduced and M -Armendariz ring, then R is $M \times N$ -Armendariz, where N is a u. p. -monoid. Liu and Zhao^[7] called a ring R weak Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \dots + a_mx^m$, $g(x) = b_0 + b_1x + \dots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j \in \text{nil}(R)$ for each i and j . They showed that for a semicommutative ideal I such that R/I is weak Armendariz, then R is weak Armendariz, and R

is weak Armendariz if and only if for any n , the $n \times n$ upper triangular matrix ring over R is weak Armendariz.

In this paper, a ring R is said to be weak M -Armendariz if whenever elements $\alpha = a_1g_1 + \dots + a_mg_m$, $\beta = b_1h_1 + \dots + b_nh_n \in R[M]$ satisfy $\alpha\beta = 0$, then $a_ib_j \in \text{nil}(R)$ for each i and j . Clearly, M -Armendariz rings are weak M -Armendariz. Examples are given to show that the converse is not always true. If $M = \{N \cup \{0\}, +\}$, weak Armendariz rings are weak M -Armendariz. If $M = \{e\}$, then every ring is M -Armendariz, so it is weak M -Armendariz. Thus weak M -Armendariz rings need not be weak Armendariz. Hence weak M -Armendariz rings are a common generalization of M -Armendariz rings and weak Armendariz rings. If S is a semigroup with multiplication $st = 0$ for all $s, t \in S$ (for example, $S = \begin{bmatrix} 0 & \mathbf{Z} \\ 0 & 0 \end{bmatrix}$), and $M = S^1$, then any ring is not weak

M -Armendariz. We show that R is weak M -Armendariz if and only if for any n , the $n \times n$ upper triangular matrix ring over R is weak M -Armendariz. It is shown that a finitely generated Abelian group G is torsion-free if and only if there exists a ring R with $|R| \geq 2$ such that R is weak G -Armendariz. This result weakens the second condition of theorem 1.14 in Ref. [6]. An ordered monoid (M, \leq) is called a strictly ordered monoid if for any $g, g', h \in M$, $g < g'$ implies $gh < g'h$ and $hg < hg'$. For a strictly totally ordered monoid M , it is proved that if an ideal I is semicommutative such that R/I is weak M -Armendariz, then R is weak M -Armendariz. Moreover, for a monoid M and a strictly totally ordered monoid N , if R is a semicommutative and M -Armendariz ring, then R is weak $M \times N$ -Armendariz.

1 Weak M -Armendariz Rings

Let $T_n(R)$ be the $n \times n$ upper triangular matrix over a ring R . In Ref. [7], Liu and Zhao showed that a ring R is weak Armendariz if and only if $T_n(R)$ is weak Armendariz for any n . If $M = \{N \cup \{0\}, +\}$, then R is weak M -Armendariz if and only if R is weak Armendariz. Moreover, note that every M -Armendariz ring is weak M -Armendariz. In the following, we will give more examples of weak M -Armendariz rings which are not M -Armendariz.

Proposition 1 Let R be a ring and M a monoid. Then R is weak M -Armendariz if and only if, for any n , $T_n(R)$ is weak M -Armendariz.

Proof We note that any subring of weak M -Armendariz rings is weak M -Armendariz. Thus if $T_n(R)$ is a weak M -Armendariz ring, then R is a weak M -Armendariz ring.

Conversely, let $\alpha = A_1g_1 + A_2g_2 + \dots + A_pg_p$, and $\beta = B_1h_1 + B_2h_2 + \dots + B_qh_q$ be elements of $T_n(R)[M]$. Assume that $\alpha\beta = 0$. It is easy to see that there exists an isomorphism of rings $T_n(R)[M] \rightarrow T_n(R[M])$ defined by

Received 2008-08-27.

Biographies: Zhang Cuiping (1974—), female, graduate; Chen Jianlong (corresponding author), male, doctor, professor, jlchen@seu.edu.cn.

Foundation items: The National Natural Science Foundation of China (No. 10571026), the Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20060286006).

Citation: Zhang Cuiping, Chen Jianlong. Weak M -Armendariz rings [J]. Journal of Southeast University (English Edition), 2009, 25(1): 142–146.

$$\sum_{i=1}^p \begin{bmatrix} a_{11}^i & a_{12}^i & a_{13}^i & \dots & a_{1n}^i \\ 0 & a_{22}^i & a_{23}^i & \dots & a_{2n}^i \\ 0 & 0 & a_{33}^i & \dots & a_{3n}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn}^i \end{bmatrix} g_i \mapsto \begin{bmatrix} \sum_{i=1}^p a_{11}^i g_i & \sum_{i=1}^p a_{12}^i g_i & \sum_{i=1}^p a_{13}^i g_i & \dots & \sum_{i=1}^p a_{1n}^i g_i \\ 0 & \sum_{i=1}^p a_{22}^i g_i & \sum_{i=1}^p a_{23}^i g_i & \dots & \sum_{i=1}^p a_{2n}^i g_i \\ 0 & 0 & \sum_{i=1}^p a_{33}^i g_i & \dots & \sum_{i=1}^p a_{3n}^i g_i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sum_{i=1}^p a_{nn}^i g_i \end{bmatrix}$$

Assume that

$$A_i = \begin{bmatrix} a_{11}^i & a_{12}^i & a_{13}^i & \dots & a_{1n}^i \\ 0 & a_{22}^i & a_{23}^i & \dots & a_{2n}^i \\ 0 & 0 & a_{33}^i & \dots & a_{3n}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn}^i \end{bmatrix}$$

and

$$B_j = \begin{bmatrix} b_{11}^j & b_{12}^j & b_{13}^j & \dots & b_{1n}^j \\ 0 & b_{22}^j & b_{23}^j & \dots & b_{2n}^j \\ 0 & 0 & b_{33}^j & \dots & b_{3n}^j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{nn}^j \end{bmatrix}$$

Then we have

$$\begin{bmatrix} \sum_{i=1}^p a_{11}^i g_i & \sum_{i=1}^p a_{12}^i g_i & \sum_{i=1}^p a_{13}^i g_i & \dots & \sum_{i=1}^p a_{1n}^i g_i \\ 0 & \sum_{i=1}^p a_{22}^i g_i & \sum_{i=1}^p a_{23}^i g_i & \dots & \sum_{i=1}^p a_{2n}^i g_i \\ 0 & 0 & \sum_{i=1}^p a_{33}^i g_i & \dots & \sum_{i=1}^p a_{3n}^i g_i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sum_{i=1}^p a_{nn}^i g_i \end{bmatrix} \cdot \begin{bmatrix} \sum_{j=1}^p b_{11}^j h_j & \sum_{j=1}^p b_{12}^j h_j & \sum_{j=1}^p b_{13}^j h_j & \dots & \sum_{j=1}^p b_{1n}^j h_j \\ 0 & \sum_{j=1}^p b_{22}^j h_j & \sum_{j=1}^p b_{23}^j h_j & \dots & \sum_{j=1}^p b_{2n}^j h_j \\ 0 & 0 & \sum_{j=1}^p b_{33}^j h_j & \dots & \sum_{j=1}^p b_{3n}^j h_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sum_{j=1}^p b_{nn}^j h_j \end{bmatrix} = 0$$

It follows that

$$\left(\sum_{i=0}^p a_{ss}^i g_i \right) \left(\sum_{j=0}^p b_{ss}^j h_j \right) = 0 \quad s = 1, 2, \dots, n$$

Since R is weak M -Armendariz, there exists $m_{ijs} \in \mathbf{N}$ such that $(a_{ss}^i b_{ss}^j)^{m_{ijs}} = 0$ for any s, i and j . Let $m_{ij} = \max\{m_{ij1}, m_{ij2}, \dots, m_{ijn}\}$, then

$$(A_i B_j)^{m_{ij}} = \begin{bmatrix} a_{11}^i b_{11}^j & * & * & \dots & * \\ 0 & a_{22}^i b_{22}^j & * & \dots & * \\ 0 & 0 & a_{33}^i b_{33}^j & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn}^i b_{nn}^j \end{bmatrix}^{m_{ij}} = \begin{bmatrix} 0 & * & * & \dots & * \\ 0 & 0 & * & \dots & * \\ 0 & 0 & 0 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Thus $((A_i B_j)^{m_{ij}})^n = \mathbf{0}$. This shows that $T_n(R)$ is a weak M -Armendariz ring.

Corollary 1 Let M be a monoid. If a ring R is an M -Armendariz ring, then, for any n , $T_n(R)$ is a weak M -Armendariz ring.

Given a ring R and a bimodule ${}_R M_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication: $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$. This is isomorphic to the ring of all matrices $\begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Proposition 2 Let M be a monoid. Then R is a weak M -Armendariz ring if and only if the trivial extension $T(R, R)$ is a weak M -Armendariz ring.

Proof It follows from proposition 1.

In general, for any ring R , the $n \times n$ ($n \geq 2$) full matrix ring $M_n(R)$ over R need not be a weak M -Armendariz ring as shown by the following example.

Example 1 Let R be a ring and M a monoid with $|M| \geq 2$. Let $S = M_2(R)$. Take $e \neq g \in M$. Let $\alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} e + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} g$ and $\beta = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} e + \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} g$ in $S[M]$.

Then we have $\alpha\beta = 0$. But $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is not nilpotent. Thus S is not weak M -Armendariz.

Proposition 3 Let M be a cancellative monoid and N be an ideal of M . If a ring R is weak N -Armendariz, then R is weak M -Armendariz.

Proof Let $\alpha = a_1 g_1 + \dots + a_m g_m, \beta = b_1 h_1 + \dots + b_n h_n$ in $R[M]$ with $\alpha\beta = 0$. Set $g \in N$, then $gg_1, gg_2, \dots, gg_m, h_1 g, h_2 g, \dots, h_n g \in N$ and $gg_i \neq gg_j$ and $h_i g \neq h_j g$ when $i \neq j$. Now from $\left(\sum_{i=1}^m a_i g g_i \right) \left(\sum_{j=1}^n b_j h_j g \right) = 0$ and the hypothesis that R is weak N -Armendariz, it follows that $a_i b_j \in \text{nil}(R)$ for all i and j . Thus R is weak M -Armendariz.

Proposition 4 For a ring R and a monoid M , suppose that R/I is weak M -Armendariz for some ideal I of R . If $I \subseteq \text{nil}(R)$, then R is weak M -Armendariz.

Proof Let $\alpha = a_1 g_1 + a_2 g_2 + \dots + a_m g_m, \beta = b_1 h_1 + b_2 h_2 + \dots + b_n h_n \in R[M]$ such that $\alpha\beta = 0$. Then $\left(\sum_{i=1}^m \bar{a}_i g_i \right) \left(\sum_{j=1}^n \bar{b}_j h_j \right) = 0$. Thus, $(\bar{a}_i \bar{b}_j)^{n_{ij}} = 0$ for some

positive integer n_{ij} . Hence, $a_i b_j \in \text{nil}(R)$. This means that R is weak M -Armendariz.

Proposition 5 For a monoid M , if R is a finite subdirect sum of weak M -Armendariz rings, then R is weak M -Armendariz.

Proof Let $I_k (k = 1, 2, \dots, l)$ be ideals of R such that R/I_k is weak M -Armendariz and $\bigcap_{k=1}^l I_k = 0$. Suppose that $\alpha = \sum_{i=0}^m a_i g_i$ and $\beta = \sum_{j=0}^n b_j h_j \in R[M]$ are such that $\alpha\beta = 0$. Then there exists $n_{ijk} \in \mathbb{N}$ such that $(\bar{a}_i \bar{b}_j)^{n_{ijk}} = 0$ in R/I_k . Thus, $(a_i b_j)^{n_{ijk}} \in I_k$. Set $n_{ij} = \max\{n_{ij1}, n_{ij2}, \dots, n_{ijl}\}$, then, $(a_i b_j)^{n_{ij}} \in I_k$ for any k , which implies that $(a_i b_j)^{n_{ij}} = 0$. Thus, R is weak M -Armendariz.

Recall that R is semicommutative if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. An ideal I of R is semicommutative if it is semicommutative as a ring without identity. In Ref. [7], Liu and Zhao proved that if I is a semicommutative ideal of R such that R/I is weak Armendariz, then R is weak Armendariz. The following result is a generalization of this.

Theorem 1 For a ring R and a strictly totally ordered monoid M , suppose that R/I is weak M -Armendariz for some ideal I of R . If I is semicommutative, then R is weak M -Armendariz.

Proof Let $\alpha, \beta \in R[M]$ be such that $\alpha\beta = 0$. We write $\alpha = a_1 g_1 + a_2 g_2 + \dots + a_m g_m$, $\beta = b_1 h_1 + b_2 h_2 + \dots + b_n h_n$ with $g_1 < g_2 < \dots < g_m$ and $h_1 < h_2 < \dots < h_n$. We will use transfinite induction on the strictly totally ordered set (M, \leq) to show that $a_i b_j \in \text{nil}(R)$ for any i and j . Note that $(\bar{a}_1 g_1 + \bar{a}_2 g_2 + \dots + \bar{a}_m g_m)(\bar{b}_1 h_1 + \bar{b}_2 h_2 + \dots + \bar{b}_n h_n) = 0$ in $(R/I)[M]$. Since R/I is weak M -Armendariz, there exists a positive integer n_{ij} such that $(a_i b_j)^{n_{ij}} \in I$. Clearly, $g_1 h_1 < g_i h_j$ if $i \neq 1$ or $j \neq 1$. Hence $a_1 b_1 = 0 \in \text{nil}(R)$. Now suppose that $w \in M$ is such that for any g_i and h_j with $g_i h_j < w$, $a_i b_j \in \text{nil}(R)$. We will show that $a_i b_j \in \text{nil}(R)$ for any g_i and h_j with $g_i h_j = w$. Set $X = \{(g_i, h_j) \mid g_i h_j = w\}$. Then X is a finite set. We write X as $\{(g_i, h_j) \mid t = 1, 2, \dots, k\}$ such that $g_{i_1} < g_{i_2} < \dots < g_{i_k}$. Since M is cancellative, $g_{i_1} = g_{i_2}$ and $g_{i_1} h_{j_1} = g_{i_2} h_{j_2} = w$ imply $h_{j_1} = h_{j_2}$. Since \leq is a strict order, $g_{i_1} < g_{i_2}$ and $g_{i_1} h_{j_1} = g_{i_2} h_{j_2} = w$ imply $h_{j_2} < h_{j_1}$. Thus we have $h_{j_k} < \dots < h_{j_2} < h_{j_1}$. Now

$$\sum_{(g_i, h_j) \in X} a_i b_j = \sum_{t=1}^k a_{i_t} b_{j_t} = 0$$

For any $t \geq 2$, $g_{i_t} h_{j_t} < g_{i_1} h_{j_1} = w$, and thus, by induction hypothesis, we have $a_{i_t} b_{j_t} \in \text{nil}(R)$. Let $p = n_{i_1 j_1}$. Then $(a_{i_1} b_{j_1})^p \in I$. By hypothesis, $a_{i_1} b_{j_1} \in \text{nil}(R)$. Let $(a_{i_1} b_{j_1})^{p'} = 0$. Then $(b_{j_2} a_{i_1})^{p'+1} = 0$. Thus

$$(a_{i_2} b_{j_2})(a_{i_1} b_{j_1})^{p+1} a_{i_2} (b_{j_2} a_{i_1})^{p'+1} (b_{j_2} (a_{i_1} b_{j_1})^{p+1}) = 0$$

Since $(a_{i_2} b_{j_2})(a_{i_1} b_{j_1})^{p+1} a_{i_2} (b_{j_2} a_{i_1}) \in I$, $(b_{j_2} a_{i_1})^{p'} (b_{j_2} (a_{i_1} b_{j_1})^{p+1}) \in I$, $b_{j_2} (a_{i_1} b_{j_1})^p a_{i_2} \in I$, and I is semicommutative, it follows that

$$((a_{i_2} b_{j_2})(a_{i_1} b_{j_1})^{p+1} a_{i_2} (b_{j_2} a_{i_1})) (b_{j_2} (a_{i_1} b_{j_1})^p a_{i_2}) \cdot ((b_{j_2} a_{i_1})^{p'} (b_{j_2} (a_{i_1} b_{j_1})^{p+1})) = 0$$

That is,

$$((a_{i_2} b_{j_2})(a_{i_1} b_{j_1})^{p+1}) ((a_{i_2} b_{j_2})(a_{i_1} b_{j_1})^{p+1}) \cdot a_{i_2} (b_{j_2} a_{i_1})^{p'} (b_{j_2} (a_{i_1} b_{j_1})^{p+1}) = 0$$

$$((a_{i_2} b_{j_2})(a_{i_1} b_{j_1})^{p+1})^2 a_{i_2} (b_{j_2} a_{i_1})^{p'} (b_{j_2} (a_{i_1} b_{j_1})^{p+1}) = 0$$

Continuing this procedure, it yields that $((a_{i_2} b_{j_2})(a_{i_1} b_{j_1})^{p+1})^{p'+3} = 0$. Thus $(a_{i_2} b_{j_2})(a_{i_1} b_{j_1})^{p+1} \in \text{nil}(I)$. Similarly, we can show that $(a_i b_j)(a_i b_j)^{p+1} \in \text{nil}(I)$ for $t = 3, \dots, k$. By lemma 3.1 in Ref. [7], $\text{nil}(I)$ is an ideal of I since I is semicommutative. Thus, if we multiply the equation $\sum_{i=1}^k a_i b_{j_i} = 0$ on the right side by $(a_{i_1} b_{j_1})^{p+1}$, then

$$(a_{i_1} b_{j_1})^{p+2} = - \left(\sum_{i=2}^k a_i b_{j_i} (a_{i_1} b_{j_1})^{p+1} \right) \in \text{nil}(I)$$

Thus $a_{i_1} b_{j_1} \in \text{nil}(R)$. Let $q = n_{i_2 j_2}$, then $(a_{i_2} b_{j_2})^q \in I$. By analogy with the above proof, we have $\sum_{i=3}^k a_i b_{j_i} (a_{i_2} b_{j_2})^{q+1} \in \text{nil}(I)$. Suppose that $(a_{i_1} b_{j_1})^s = 0$. Then

$$(a_{i_2} b_{j_2})^{q+1} (a_{i_1} b_{j_1})^s (a_{i_2} b_{j_2})^{q+1} = 0$$

Since $(a_{i_2} b_{j_2})^{q+1} \in I$ and I is semicommutative, we have

$$((a_{i_1} b_{j_1})(a_{i_2} b_{j_2})^{q+1})^{s+1} = 0$$

Thus $(a_{i_1} b_{j_1})(a_{i_2} b_{j_2})^{q+1} \in \text{nil}(I)$. Hence, multiplying the equation $\sum_{i=1}^k a_i b_{j_i} = 0$ on the right side by $(a_{i_2} b_{j_2})^{q+1}$, we have

$$(a_{i_2} b_{j_2})^{q+2} = - \left(\sum_{i=3}^k a_i b_{j_i} (a_{i_2} b_{j_2})^{q+1} \right) - (a_{i_1} b_{j_1})(a_{i_2} b_{j_2})^{q+1} \in \text{nil}(I)$$

Hence, $a_{i_2} b_{j_2} \in \text{nil}(R)$. Similarly, we can show that $a_{i_t} b_{j_t} \in \text{nil}(R)$, \dots , $a_i b_{j_i} \in \text{nil}(R)$. Thus $a_i b_{j_i} \in \text{nil}(R)$ for any t with $g_i h_{j_i} = w$. Therefore, by transfinite induction, $a_i b_j \in \text{nil}(R)$ for any i and j . Thus, R is weak M -Armendariz.

Corollary 2 Let M be a strictly totally ordered monoid and R a semicommutative ring. Then R is weak M -Armendariz.

By corollary 3.4 in Ref. [7], semicommutative rings are weak Armendariz. In the following, we will see that semicommutative rings need not be weak M -Armendariz. Thus, the condition that M is a strictly totally ordered monoid in theorem 1 is not superfluous.

Proposition 6 If M is a finite monoid, then the complex field C is not weak M -Armendariz.

Proof Suppose that C is weak M -Armendariz. Let $M = \{e, g_1, \dots, g_n\}$. Let $\alpha = a_0 e + a_1 g_1 + \dots + a_n g_n$ and $\beta = b_0 e + b_1 g_1 + \dots + b_n g_n \in C[M]$ such that $\alpha\beta = 0$. Then $a_i b_j \in \text{nil}(C)$. Note that C is reduced, so $a_i b_j = 0$. Thus, C is M -Armendariz, which contradicts proposition 1.15 in Ref. [6].

Corollary 3 Let R be a semicommutative ring. Then R is weak \mathbf{Z} -Armendariz.

Recall that a monoid M is torsion-free if for any $g, h \in M$ and $k \geq 1$, $g^k = h^k$ implies $g = h$.

Corollary 4 Let M be a commutative, cancellable and torsion-free monoid. If one of the following conditions

holds, then R is a weak M -Armendariz ring.

1) R is semicommutative;

2) R/I is weak M -Armendariz for some ideal I of R and I is semicommutative.

Proof If M is commutative, cancellative and torsion-free, then there exists a compatible strict total order \leq on $M^{[8]}$. Now the results follow from theorem 1.

Lemma 1 Let M be a monoid and N be a submonoid of M . If R is a weak M -Armendariz ring, then R is weak N -Armendariz.

Lemma 2 Let M be a cyclic group of order $n \geq 2$ and R be a ring with $0 \neq 1$. Then R is not weak M -Armendariz.

Proof Suppose that $M = \{e, g, g^2, \dots, g^{n-1}\}$. Let $\alpha = 1e + 1g + 1g^2 + \dots + 1g^{n-1}$ and $\beta = 1e + (-1)g$, then $\alpha\beta = 0$, but $1 \cdot 1 = 1$ is not a nilpotent element. Thus, R is not weak M -Armendariz.

Let $T(G)$ be the set of elements of the finite order in an Abelian group G . Then $T(G)$ is a fully invariant subgroup of G . G is torsion-free if and only if $T(G) = \{e\}$. Liu^[6] showed that for a finitely generated Abelian group G , it is torsion-free if and only if there exists a ring R with $|R| \geq 2$ such that R is G -Armendariz. The following theorem will weaken the sufficient condition of this.

Theorem 2 Let G be a finitely generated Abelian group. Then the following conditions on G are equivalent:

1) G is torsion-free.

2) There exists a ring R with $|R| \geq 2$ such that R is weak G -Armendariz.

Proof 1) \Rightarrow 2) If G is a finitely generated Abelian group with $T(G) = \{e\}$, then $G \cong \mathbf{Z} \times \mathbf{Z} \times \dots \times \mathbf{Z}$, a finite direct product of group \mathbf{Z} . It is easy to see that $\mathbf{Z} \times \mathbf{Z} \times \dots \times \mathbf{Z}$ is a commutative, cancellative and torsion-free monoid, so is G . Let R be a semicommutative ring. Then, by corollary 4, R is weak G -Armendariz.

2) \Rightarrow 1) If $g \in T(G)$ and $g \neq e$, then $N = \langle g \rangle$ is a cyclic group of the finite order. If a ring $R \neq \{0\}$ is weak G -Armendariz, then, by lemma 1, R is weak N -Armendariz, contradicting lemma 2. Thus, every ring $R \neq \{0\}$ is not weak G -Armendariz.

Let M be a monoid and N be an ideal of M . Denote the Rees congruence induced by N by $\rho(N)$, which is defined by $g\rho(N)h \Leftrightarrow g, h \in N$ or $g = h$. If R is weak M -Armendariz, then, by lemma 1, R is weak N -Armendariz since $R[N]$ is a subring of $R[M]$. But R may not be weak $M/\rho(N)$ -Armendariz as shown by the following example.

Example 2^[6] Let $M = \{\mathbf{N} \cup \{0\}, +\}$ and $N = \{2, 3, \dots\}$. Then N is an ideal of M and $M/\rho(N)$ is a finite monoid. From proposition 6, it follows that \mathbf{C} is not weak $M/\rho(N)$ -Armendariz, but it is weak M -Armendariz because \mathbf{C} is M -Armendariz.

2 Monoid Rings

Lemma 3 Let R be a semicommutative ring and M a monoid. If $a_1, \dots, a_n \in \text{nil}(R)$, then $a_1g_1 + \dots + a_ng_n \in \text{nil}(R[M])$.

Proof The proof is similar to that of lemma 3. 7 in Ref. [7].

For a monoid M , we denote the largest subgroup of M by $G(M)$. In Ref. [7], Liu and Zhao proved that if R is semicommutative, then $R[x]$ is weak Armendariz. In Ref. [6], Liu proved that if M is a commutative and cancellative

monoid with $G(M) = \{e\}$, and R is Armendariz and M -Armendariz, then $R[M]$ is Armendariz. For the above monoid M , we do not know whether $R[M]$ is weak Armendariz if R is weak Armendariz and weak M -Armendariz. However, if R is semicommutative, we have the following proposition.

Proposition 7 Let M be a commutative and cancellative monoid with $G(M) = \{e\}$. If R is a semicommutative and weak M -Armendariz ring, then $R[M]$ is weak Armendariz.

Proof Suppose that $\left(\sum_{i=0}^m \alpha_i x^i\right) \left(\sum_{j=0}^n \beta_j x^j\right) = 0$, where $\alpha_i = \sum a_{ip} g_{ip}, \beta_j = \sum b_{jq} h_{jq} \in R[M]$. Set $g = \left(\prod_i \prod_p g_{ip}\right) \left(\prod_j \prod_q h_{jq}\right)$. Clearly for any $r \in R$ and $h \in M$, $(rh)(1g^2) = (1g^2)(rh)$. Thus from $\left(\sum_{i=0}^m \alpha_i x^i\right) \left(\sum_{j=0}^n \beta_j x^j\right) = 0$, it follows that $\left(\sum_{i=0}^m \alpha_i (1g^2)^i\right) \left(\sum_{j=0}^n \beta_j (1g^2)^j\right)$. Thus we have

$$\left(\sum_i \sum_p b_{ip} h_{ip} g^{2i}\right) \left(\sum_j \sum_q b_{jq} h_{jq} g^{2j}\right) = 0$$

Suppose that $g_{i'p'} g^{2i'} = g_{r'p'} g^{2r'}$ for some i' and r' . If $i' = r'$, then $g_{i'p'} = g_{r'p'}$ since M is cancellative, and so $p' = p''$. Thus without loss of generality, we can assume that $i' > r'$. Then $g_{i'p'} g^{2(i'-r')} = g_{r'p'}$ since M is cancellative. Thus it is easy to see that g_{ip} and h_{jq} are in $G(M)$ for all i, j, p, q . Hence $g_{ip} = h_{jq} = e$ by hypothesis, and then we may assume that $\alpha_i = a_i e$ and $\beta_j = b_j e$ for all i, j . So we have

$$\left(\sum_i (a_i e) x^i\right) \left(\sum_j (b_j e) x^j\right) = 0$$

from which it follows that $\left(\sum_i a_i x^i\right) \left(\sum_j b_j x^j\right) = 0$. Thus, $a_i b_j \in \text{nil}(R)$ for all i and j since R is weak Armendariz. Assume that $(a_i b_j)^{n_{ij}} = 0$ for some $n_{ij} \in \mathbf{N}$. Then $(\alpha_i \beta_j)^{n_{ij}} = (a_i b_j)^{n_{ij}} e = 0$. So $\alpha_i \beta_j \in \text{nil}(R[M])$. If $h_{j'q'} g^{2j'} = h_{r'q'} g^{2r'}$ for some j' and r' , then by analogy with the above proof, it follows that $\alpha_i \beta_j = (a_i e)(b_j e) \in \text{nil}(R[M])$ for all i, j . Now suppose that each pair of $g_{ip} g^{2i}$ is distinct and each pair of $h_{jq} g^{2j}$ is distinct. Then $a_{ip} b_{jq} \in \text{nil}(R)$ for all i, j, p, q since R is weak M -Armendariz. Thus, $\alpha_i \beta_j = \sum_p \sum_q (a_{ip} b_{jq})(g_{ip} h_{jq}) \in \text{nil}(R[M])$ by lemma 3.

We can easily obtain the following fact.

Lemma 4 Let M be a monoid. If R is a semicommutative and M -Armendariz ring, then $R[M]$ is semicommutative.

Let $M = \{e\}$ and $N = \{\mathbf{N} \cup \{0\}, +\}$. Let R be a semicommutative ring. Then R is M -Armendariz. But $R[M]$ need not be N -Armendariz by example 3. 5 in Ref. [1]. However, we have the following result.

Proposition 8 Let M be a monoid and N a strictly totally ordered monoid. If R is a semicommutative and M -Armendariz ring, then $R[M]$ is a weak N -Armendariz ring.

Proof Since R is semicommutative and M -Armendariz, by lemma 4, $R[M]$ is semicommutative. The assertion holds according to corollary 2.

Corollary 5 Let N be a strictly totally ordered monoid.

If R is a semicommutative and Armendariz ring, then $R[x]$ is a weak N -Armendariz ring.

Proposition 9 Let M be a monoid and N a strictly totally ordered monoid. If R is a semicommutative and M -Armendariz ring, then $R[N]$ is a weak M -Armendariz ring.

Proof It is easy to see that there exists an isomorphism of rings $R[N][M] \rightarrow R[M][N]$ defined by

$$\sum_p \left(\sum_i a_{ip} n_i \right) m_p \mapsto \sum_i \left(\sum_p a_{ip} n_p \right) n_i.$$

Now suppose that $\alpha_i, \beta_j \in R[N]$ are such that $\left(\sum_i \alpha_i m_i \right) \left(\sum_j \beta_j m'_j \right) = 0$. We will show that $\alpha_i \beta_j \in \text{nil}(R[N])$ for all i, j . Assume that $\alpha_i = \sum_p a_{ip} n_p$ and $\beta_j =$

$\sum_q b_{jq} n'_q$ where $n_p, n'_q \in N$ for all p and q . Then

$$\left(\sum_i \left(\sum_p a_{ip} n_p \right) m_i \right) \left(\sum_j \sum_q b_{jq} n'_q \right) m'_j = 0. \text{ Thus, in}$$

$R[M][N]$ we have $\left(\sum_p \left(\sum_i a_{ip} m_i \right) n_p \right) \left(\sum_q \sum_j b_{jq} m'_j \right) n'_q = 0$. By proposition 8, $R[M]$ is weak N -Armendariz,

$\left(\sum_i a_{ip} m_i \right) \left(\sum_j b_{jq} m'_j \right) \in \text{nil}(R[M])$ for all p, q . Since R

is M -Armendariz, $a_{ip} b_{jq} \in \text{nil}(R)$ for all i, j, p, q according to proposition 1.6 in Ref. [6]. Hence $\alpha_i \beta_j \in \text{nil}(R[N])$ by lemma 3. This means that $R[N]$ is weak M -Armendariz.

Corollary 6 Let M be a monoid and R be a semicommutative ring. If R is M -Armendariz, then $R[x]$ and $R[x, x^{-1}]$ are weak M -Armendariz.

Proof Note that $R[x] \cong R[N \cup \{0\}]$ and $R[x, x^{-1}] \cong R[\mathbb{Z}]$.

In Ref. [6], Liu showed that if R is reduced and M -Armendariz, then R is $M \times N$ -Armendariz, where N is a u. p. -monoid. For weak M -Armendariz rings, we have the following result.

Theorem 3 Let M be a monoid and N be a strictly totally ordered monoid. If R is a semicommutative and M -Armendariz ring, then R is weak $M \times N$ -Armendariz.

Proof Suppose that $\sum_{i=1}^s a_i(m_i, n_i)$ is in $R[M \times N]$. For

any $1 \leq p \leq s$, denote $A_p = \{i \mid 1 \leq i \leq s, n_i = n_p\}$. Then

$$\sum_{p=1}^t \left(\sum_{i \in A_p} a_i m_i \right) n_p \in R[M][N]. \text{ Note that } m_i \neq m_{i'} \text{ for any } i, i' \in A_p \text{ with } i \neq i'. \text{ Now it is easy to see that there exists an isomorphism of rings } R[M \times N] \rightarrow R[M][N] \text{ defined by}$$

$$\sum_{i=1}^s a_i(m_i, n_i) \mapsto \sum_{p=1}^t \left(\sum_{i \in A_p} a_i m_i \right) n_p.$$

Suppose that $\left(\sum_{i=1}^s a_i(m_i, n_i) \right) \left(\sum_{j=1}^{s'} b_j(m'_j, n'_j) \right) = 0$ in $R[M \times N]$. Then from the above isomorphism, it follows that

$$\left(\sum_{p=1}^t \left(\sum_{i \in A_p} a_i m_i \right) n_p \right) \left(\sum_{q=1}^{t'} \left(\sum_{j \in B_q} b_j m'_j \right) n'_q \right) = 0$$

By proposition 8, $R[M]$ is weak N -Armendariz, thus we have $\left(\sum_{i \in A_p} a_i m_i \right) \left(\sum_{j \in B_q} b_j m'_j \right) \in \text{nil}(R[M])$ for all p, q .

Since R is M -Armendariz, $a_i b_j \in \text{nil}(R)$ for any $i \in A_p$ and $j \in B_q$ by proposition 1.6 in Ref. [6]. Hence, $a_i b_j \in \text{nil}(R)$ for all $1 \leq i \leq s$ and $1 \leq j \leq s'$. The proof is completed.

References

- [1] Rege M B, Chhawchharia S. Armendariz rings [J]. *Proc Japan Acad Ser A Math Sci*, 1997, **73**(1): 14–17.
- [2] Armendariz E P. A note on extension of Bear and pp-rings [J]. *J Austral Math Soc*, 1974, **18**(2): 470–473.
- [3] Anderson D D, Camillo V. Armendariz rings and Gaussian rings [J]. *Comm Algebra*, 1998, **26**(7): 2265–2272.
- [4] Huh C, Lee Y, Smoktunowicz A. Armendariz rings and semicommutative rings [J]. *Comm Algebra*, 2002, **30**(2): 751–761.
- [5] Kim N K, Lee Y. Armendariz rings and reduced rings [J]. *J Algebra*, 2000, **223**(2): 477–488.
- [6] Liu Z K. Armendariz rings relative to a monoid [J]. *Comm Algebra*, 2005, **33**(3): 649–661.
- [7] Liu Z K, Zhao R Y. On weak Armendariz rings [J]. *Comm Algebra*, 2006, **34**(7): 2607–2616.
- [8] Ribenboim P. Noetherian rings of generalized power series [J]. *J Pure Appl Algebra*, 1992, **79**(3): 293–312.

弱 M -Armendariz 环

张翠萍^{1,2} 陈建龙¹

(¹ 东南大学数学系, 南京 210096)

(² 西北师范大学数学系, 兰州 730070)

摘要: 对于么半群 M , 引入了弱 M -Armendariz 环的概念, 此概念是 M -Armendariz 环和弱 Armendariz 环的共同推广. 研究了这类环的性质, 并且证明了: R 是弱 M -Armendariz 环当且仅当对任意的 n , R 的 n 阶上三角矩阵环 $T_n(R)$ 是弱 M -Armendariz 环; 如果 I 是环 R 的半交换理想, 使得 R/I 是弱 M -Armendariz 环, 则 R 是弱 M -Armendariz 环, 其中 M 是严格全序么半群; 如果 R 是半交换的 M -Armendariz 环, 则 R 是弱 $M \times N$ -Armendariz 环, 其中 N 是严格全序么半群; 有限生成 Abelian 群 G 是 torsion-free 的当且仅当存在一个环 R , 使得 R 是弱 G -Armendariz 环.

关键词: 半交换环; M -Armendariz 环; 弱 Armendariz 环; 弱 M -Armendariz 环

中图分类号: O153.3