

# On existence and uniqueness of the solution of elastoplastic contact problems

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**Abstract:** Contact problems and elastoplastic problems are unified and described by the variational inequality formulation, in which the constraints of the constitutional relations for elastoplastic materials and the contact conditions are relaxed totally. First, the coerciveness of the functional is proved. Then the uniqueness of the solution of variational inequality for the elastoplastic contact problems is demonstrated. The existence of the solution is also demonstrated according to the sufficient conditions for the solution of the elliptic variational inequality. A mathematical foundation is developed for the variational extremum principle of elastoplastic contact problems. The developed variational extremum forms can give an effective and strict mathematical modeling to solve contact problems with mathematical programming.

**Key words:** elastoplastic; contact problem; existence; uniqueness; coerciveness; variational extremum form

The contact problem belongs to the variational problems with indefinite boundaries, and widely exists in practical engineering. Worldwide, many works focus on the problems using theoretical and numerical analysis. Strong non-linearity of the contact state equations decides the difficulties involved in solving the contact problems, and also brings forth difficulties in engineering applications and some mathematical problems such as non-lubricity, existence, non-uniqueness of the solutions, etc. Take other non-linearity (such as elastoplastic contact problems) into consideration, it is more difficult to solve them.

Mathematical programming is an efficient way to solve nonlinear problems. Minimizing the variational formulation of contact problems with serial linear programming or quadratic programming techniques can be considered as another way to solve the problem except for the typical analytic methods and iterative methods. The establishment of the variational extremum principle is the theoretical foundation used in solving the nonlinear problems based on the application of mathematical programming. Aiming at the variational inequality models for the elastoplastic contact problems, the author of this paper proves, under certain conditions, the existence and uniqueness of the solution to the problems. Correspondingly, the mathematical foundation for the variational extremum principle of the problems is established.

## 1 Variational Inequality Formulation of Elastoplastic Contact Problems<sup>[1]</sup>

As for the elastoplastic problems, the physical variables, such as displacements  $u_i (i = 1, 2, 3)$ , strains  $\varepsilon_{ij}$ , stresses  $\sigma_{ij}$ , contact displacements  $\bar{\varepsilon}_i$  and contact force  $P_{ci}$ , are the state variables of the system. Some of them can be derived from the basic variables, so that the displacements are chosen as the state variables and the flow parameters  $\lambda$  and  $\bar{\lambda}$  are selected as the control variables.

Define the following spaces<sup>[2]</sup>:

$$H_1^1(\Omega) = \{u \mid u \in H_1(\Omega), u|_{\Gamma_c} = u^0\}, \quad H_1^0(\Omega) = \{u \mid u \in H_1(\Omega), u|_{\Gamma_c} = 0\}$$

$$\bar{H}_1^1(\Omega) = (H_1^1(\Omega))^3, \quad \bar{H}_1^0(\Omega) = (H_1^0(\Omega))^3, \quad \bar{L}_2(\Omega) = (L_2(\Omega))^m, \quad \bar{L}_2(\Omega) = (L_2(\Omega))^2$$

where  $H_1(\Omega)$  is the Sobolev space, and  $L_2(\Omega)$  is the Hilbert space.

Redefine the solution space  $\tilde{K}$ :

$$\tilde{K} = \{\{u, \lambda, \bar{\lambda}\} \mid \{u, \lambda, \bar{\lambda}\} \in \bar{H}_1^1(\Omega) \bar{L}_2(\Omega) \bar{L}_2(\Omega); \lambda_\alpha \geq 0; \bar{\lambda}_\beta \geq 0; \alpha = 1, 2, \dots, m; \beta = 1, 2\}$$

According to Ref. [1], we can obtain the equivalent variational inequality formulation, from which the functional with the relaxed constitutive relation and the contact restriction is obtained.

Find  $\{u, \lambda, \bar{\lambda}\} \in \tilde{K}$ , such that

$$a(u, v - u) - b(v - u, \lambda) + \tilde{a}(u, v - u) - \tilde{b}(v - u, \bar{\lambda}) + c(r - \lambda, \lambda) - d(u, r - \lambda) + j(r - \lambda) + \tilde{c}(\bar{r} - \bar{\lambda}, \bar{\lambda}) - \tilde{d}(u, \bar{r} - \bar{\lambda}) + \tilde{j}(\bar{r} - \bar{\lambda}) \geq L(v - u) \quad \forall \{v, r, \bar{r}\} \in \tilde{K} \quad (1)$$

where

$$a(u, v) = \int_{\Omega} \varepsilon_{ij}(u) D_{ijkl} \varepsilon_{kl}(v) d\Omega, \quad b(v, r) = \int_{\Omega} r_\alpha \left( \frac{\partial g_\alpha}{\partial \sigma_{ij}} \right) D_{ijkl} \varepsilon_{kl}(v) d\Omega$$

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$$\begin{aligned}
c(\mathbf{\lambda}, \mathbf{r}) &= \int_{\Omega} \lambda_{\alpha} \left[ \left( \frac{\partial f_{\alpha}}{\partial \sigma_{ij}} \right) D_{ijkl} \left( \frac{\partial g_{\beta}}{\partial \sigma_{kl}} \right) + t_{\alpha\beta} \right] r_{\beta} d\Omega, \quad d(\mathbf{v}, \mathbf{r}) = \int_{\Omega} r_{\alpha} \left( \frac{\partial f_{\alpha}}{\partial \sigma_{ij}} \right) D_{ijkl} \varepsilon_{kl}(\mathbf{v}) d\Omega, \quad j(\mathbf{r}) = - \int_{\Omega} r_k f_k^0 d\Omega \\
\tilde{a}(\mathbf{u}, \mathbf{v}) &= \int_{\Gamma_c} \tilde{\mathbf{e}}(\mathbf{v}) \tilde{\mathbf{D}} \tilde{\mathbf{e}}(\mathbf{u}) d\Gamma, \quad \tilde{b}(\mathbf{v}, \tilde{\mathbf{\lambda}}) = \int_{\Gamma_c} \sum_k \left( \frac{\partial \tilde{g}_k}{\partial \mathbf{P}_c} \right)^T \tilde{\mathbf{D}} \tilde{\lambda}_k \tilde{\mathbf{e}}(\mathbf{v}) d\Gamma \\
\tilde{c}(\tilde{\mathbf{\lambda}}, \tilde{\mathbf{r}}) &= \int_{\Gamma_c} \sum_{k,j} \tilde{\lambda}_j \left( \frac{\partial \tilde{f}_k}{\partial \mathbf{P}_c} \right)^T \tilde{\mathbf{D}} \left( \frac{\partial \tilde{g}_j}{\partial \mathbf{P}_c} \right) \tilde{r}_k d\Gamma, \quad \tilde{d}(\mathbf{v}, \tilde{\mathbf{\lambda}}) = \int_{\Gamma_c} \sum_k \tilde{\lambda}_k \left( \frac{\partial \tilde{f}_k}{\partial \mathbf{P}_c} \right)^T \tilde{\mathbf{D}} \tilde{\mathbf{e}}(\mathbf{v}) d\Gamma \\
\tilde{j}(\tilde{\mathbf{r}}) &= - \int_{\Gamma_c} \sum_k \tilde{f}_k^0 \tilde{r}_k d\Gamma, \quad L(\mathbf{v}) = \int_{\Omega} \mathbf{v}^T \mathbf{f} d\Omega + \int_{\Gamma_r} \mathbf{v}^T \mathbf{P} d\Gamma
\end{aligned}$$

where  $f_k$  is the loading function,  $g_k$  is the plastic potential function,  $\tilde{f}_k$  is the sliding function, and  $\tilde{g}_k$  is the sliding potential function.

From the defined solution space  $\tilde{K}$ , the nature of the formulation lies in: With all the possibilities to satisfy the continuity equations and the displacement boundary conditions, and the real solution  $\{\mathbf{u}, \mathbf{\lambda}, \tilde{\mathbf{\lambda}}\}$  satisfies Eq. (1), including the equilibrium equation, the compatibility equation and the boundary conditions effectively relax the constraint conditions of the constitutional relations for elastoplastic material and the contact conditions.

## 2 Discussion on the Existence and Uniqueness for the Solution

The structure of Eq. (1) releases the elastoplastic constitutive equations and the contact state equations. The variational inequality, which conceals two constraint relations, effectively solves the solutions to the variational problems of the elastoplastic contact problems; and thus it enables us to discuss the characteristics of the solutions to this problem.

The establishment of the variational extremum principle is the theoretical foundation involved in solving the nonlinear problems based on the application of mathematical programming. However, it is very difficult to establish the variational extremum principle to the elastoplastic contact problems because of some mathematical problems.

Though the variational extremum principle for the elastoplastic contact problems with friction has been successfully set up, it is difficult to prove the characteristics of the solutions in mathematics. The following cases are discussed.

Assume that ① Materials are subjected to the Drucker postulate; ② There is no friction on contact surfaces. Let

$$a^{\#}((\mathbf{u}, \mathbf{\lambda}, \tilde{\mathbf{\lambda}}), (\mathbf{v}, \mathbf{r}, \tilde{\mathbf{r}})) = a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, \mathbf{\lambda}) - d(\mathbf{u}, \mathbf{r}) + c(\mathbf{r}, \mathbf{\lambda}) + \tilde{a}(\mathbf{u}, \mathbf{v}) - \tilde{b}(\mathbf{v}, \tilde{\mathbf{\lambda}}) - \tilde{d}(\mathbf{u}, \tilde{\mathbf{r}}) + \tilde{c}(\tilde{\mathbf{r}}, \tilde{\mathbf{\lambda}}) \quad (2)$$

When the materials satisfy the Drucker postulate, we can prove that the loading surfaces are outer-convex and orthogonal for the plastic displacement increments, which is valid for both stabilizing and non-stabilizing materials<sup>[3]</sup>. Under this condition, the plastic potential surfaces and the loading surfaces are superposed, i. e.  $f = g$ , thus  $b(\mathbf{u}, \mathbf{\lambda}) = d(\mathbf{u}, \mathbf{\lambda})$ .

When assumption ② is satisfied, from the definition of  $\tilde{f}$ ,  $\tilde{g}$ , we can obtain  $\frac{\partial \tilde{f}_k}{\partial \mathbf{P}_c} = \frac{\partial \tilde{g}_k}{\partial \mathbf{P}_c}$ , i. e.  $\tilde{b}(\mathbf{u}, \tilde{\mathbf{\lambda}}) = \tilde{d}(\mathbf{u}, \tilde{\mathbf{\lambda}})$ .

The following proof is done with the above conditions mentioned, where  $a^{\#}$  is coercive. It means that  $\exists \alpha_0 > 0$ , which leads to

$$a^{\#}((\mathbf{u}, \mathbf{\lambda}, \tilde{\mathbf{\lambda}}), (\mathbf{u}, \mathbf{\lambda}, \tilde{\mathbf{\lambda}})) \geq \alpha_0 (\|\mathbf{u}\|^2 + \|\mathbf{\lambda}\|^2 + \|\tilde{\mathbf{\lambda}}\|^2) \quad (3)$$

Expanding  $a^{\#}$ , we obtain

$$\begin{aligned}
a^{\#}((\mathbf{u}, \mathbf{\lambda}, \tilde{\mathbf{\lambda}}), (\mathbf{u}, \mathbf{\lambda}, \tilde{\mathbf{\lambda}})) &= \int_{\Omega} \mathbf{\varepsilon}^T(\mathbf{u}) \mathbf{D} \mathbf{\varepsilon}(\mathbf{u}) d\Omega - 2 \int_{\Omega} \sum_k \lambda_k \left( \frac{\partial f_k}{\partial \sigma} \right)^T \mathbf{D} \mathbf{\varepsilon}(\mathbf{u}) d\Omega + \int_{\Omega} \sum_{k,j} \lambda_j \left[ \left( \frac{\partial f_k}{\partial \sigma} \right)^T D \left( \frac{\partial f_j}{\partial \sigma} \right) + t_{kj} \right] \lambda_k d\Omega + \\
&\int_{\Gamma_c} \tilde{\mathbf{e}}^T(\mathbf{u}) \tilde{\mathbf{D}} \tilde{\mathbf{e}}(\mathbf{u}) d\Gamma - 2 \int_{\Gamma_c} \sum_k \left( \frac{\partial \tilde{f}_k}{\partial \mathbf{P}_c} \right)^T \tilde{\mathbf{D}} \tilde{\lambda}_k \tilde{\mathbf{e}}(\mathbf{u}) d\Gamma + \int_{\Gamma_c} \sum_{k,j} \tilde{\lambda}_j \left( \frac{\partial \tilde{f}_k}{\partial \mathbf{P}_c} \right)^T \tilde{\mathbf{D}} \left( \frac{\partial \tilde{f}_j}{\partial \mathbf{P}_c} \right) \tilde{\lambda}_k d\Gamma
\end{aligned} \quad (4)$$

With the matrices  $[t_{kj}]$  and  $\mathbf{D}$ ,  $\tilde{\mathbf{D}}$  is positive definite<sup>[1]</sup>, and Eq. (4) can be written as

$$\begin{aligned}
a^{\#} \geq \int_{\Omega} \left\{ d_0 \left[ \mathbf{\varepsilon}(\mathbf{u}) - \sum_k \lambda_k \left( \frac{\partial f_k}{\partial \sigma} \right) \right]^T \left[ \mathbf{\varepsilon}(\mathbf{u}) - \sum_j \lambda_j \left( \frac{\partial f_j}{\partial \sigma} \right) \right] + t_0 \mathbf{\lambda}^T \mathbf{\lambda} \right\} d\Omega + \int_{\Gamma_c} \left\{ \tilde{d}_0 \left[ \tilde{\mathbf{e}}(\mathbf{u}) - \sum_k \tilde{\lambda}_k \left( \frac{\partial \tilde{f}_k}{\partial \mathbf{P}_c} \right) \right]^T \cdot \right. \\
\left. \left[ \tilde{\mathbf{e}}(\mathbf{u}) - \sum_j \tilde{\lambda}_j \left( \frac{\partial \tilde{f}_j}{\partial \mathbf{P}_c} \right) \right] \right\} d\Gamma
\end{aligned} \quad (5)$$

where  $d_0 > 0$ ,  $\tilde{d}_0 > 0$ ,  $t_0 > 0$ .

$$\begin{aligned}
a^{\#} \geq \int_{\Omega} d_0 \left\{ \theta_1 \mathbf{\varepsilon}^T(\mathbf{u}) \mathbf{\varepsilon}(\mathbf{u}) + (1 - \theta_1) \mathbf{\varepsilon}^T(\mathbf{u}) \mathbf{\varepsilon}(\mathbf{u}) - 2 \mathbf{\varepsilon}^T(\mathbf{u}) \left( \frac{\partial f_k}{\partial \sigma} \right) \lambda_k + \frac{1}{1 - \theta_1} \left[ \sum \lambda_k \left( \frac{\partial f_k}{\partial \sigma} \right) \right]^T \left[ \sum \lambda_j \left( \frac{\partial f_j}{\partial \sigma} \right) \right] + \right. \\
\left. \frac{-\theta_1}{1 - \theta_1} \left[ \sum \lambda_k \left( \frac{\partial f_k}{\partial \sigma} \right) \right]^T \left[ \sum \lambda_j \left( \frac{\partial f_j}{\partial \sigma} \right) \right] \right\} d\Omega + t_0 \int_{\Omega} \mathbf{\lambda}^T \mathbf{\lambda} d\Omega + \int_{\Gamma_c} \tilde{d}_0 \left\{ \theta_2 \left[ \sum \tilde{\lambda}_k \left( \frac{\partial \tilde{f}_k}{\partial \mathbf{P}_c} \right) \right]^T \left[ \sum \tilde{\lambda}_j \left( \frac{\partial \tilde{f}_j}{\partial \mathbf{P}_c} \right) \right] + \right.
\end{aligned}$$

$$(1 - \theta_2) \left[ \sum \bar{\lambda}_k \left( \frac{\partial \tilde{f}_k}{\partial \mathbf{P}_c} \right) \right]^T \left[ \sum \bar{\lambda}_j \left( \frac{\partial \tilde{f}_j}{\partial \mathbf{P}_c} \right) \right] - 2 \tilde{\mathbf{e}}^T(\mathbf{u}) \sum_k \left( \frac{\partial \tilde{f}_k}{\partial \mathbf{P}_c} \right) \lambda_k + \frac{1}{1 - \theta_2} \tilde{\mathbf{e}}^T(\mathbf{u}) \tilde{\mathbf{e}}(\mathbf{u}) + \frac{-\theta_2}{1 - \theta_2} \tilde{\mathbf{e}}^T(\mathbf{u}) \tilde{\mathbf{e}}(\mathbf{u}) \} d\Gamma \quad (6)$$

where  $0 < \theta_1 < 1, 0 < \theta_2 < 1$ .

Additionally, we can obtain

$$\begin{aligned} a^\# \geq & \int_\Omega \left\{ d_0 \theta_1 \mathbf{e}^T(\mathbf{u}) \mathbf{e}(\mathbf{u}) + t_0 \boldsymbol{\lambda}^T \boldsymbol{\lambda} - \frac{\theta_1 d_0}{1 - \theta_1} \left[ \sum \lambda_k \left( \frac{\partial f_k}{\partial \boldsymbol{\sigma}} \right) \right]^T \left[ \sum \lambda_j \left( \frac{\partial f_j}{\partial \boldsymbol{\sigma}} \right) \right] \right\} d\Omega + \\ & \int_{\Gamma_c} \left\{ \tilde{d}_0 \theta_2 \left[ \sum \bar{\lambda}_k \left( \frac{\partial \tilde{f}_k}{\partial \mathbf{P}_c} \right) \right]^T \left[ \sum \bar{\lambda}_j \left( \frac{\partial \tilde{f}_j}{\partial \mathbf{P}_c} \right) \right] - \frac{\tilde{d}_0 \theta_2}{1 - \theta_2} \tilde{\mathbf{e}}^T(\mathbf{u}) \tilde{\mathbf{e}}(\mathbf{u}) \right\} d\Gamma \geq \int_\Omega d_0 \theta_1 \mathbf{e}^T(\mathbf{u}) \mathbf{e}(\mathbf{u}) d\Omega + \\ & \int_\Omega \left( t_0 - \frac{M_1 \theta_1 d_0}{1 - \theta_1} \right) \boldsymbol{\lambda}^T \boldsymbol{\lambda} d\Omega + \int_{\Gamma_c} \tilde{d}_0 \theta_2 M_2 \bar{\boldsymbol{\lambda}}^T \bar{\boldsymbol{\lambda}} d\Gamma - \int_{\Gamma_c} \frac{\tilde{d}_0 \theta_2}{1 - \theta_2} \tilde{\mathbf{e}}^T(\mathbf{u}) \tilde{\mathbf{e}}(\mathbf{u}) d\Gamma \end{aligned} \quad (7)$$

From the Poincare inequality and the Korn inequality<sup>[4]</sup>, Eq. (7) is written as

$$\begin{aligned} a^\# \geq & d_0 \theta_1 C_1 \|\mathbf{u}\|^2 + \left( t_0 - \frac{M_1 \theta_1 d_0}{1 - \theta_1} \right) C_2 \|\boldsymbol{\lambda}\|^2 + \tilde{d}_0 \theta_2 M_2 C_3 \|\bar{\boldsymbol{\lambda}}\|^2 - \frac{\tilde{d}_0 \theta_2 M_2}{1 - \theta_2} C_4 \|\mathbf{u}\|^2 = \\ & \left( t_0 - \frac{M_1 \theta_1 d_0}{1 - \theta_1} \right) C_2 \|\boldsymbol{\lambda}\|^2 + \tilde{d}_0 \theta_2 M_2 C_3 \|\bar{\boldsymbol{\lambda}}\|^2 + \left( d_0 \theta_1 C_1 - \frac{\tilde{d}_0 \theta_2 M_2}{1 - \theta_2} C_4 \right) \|\mathbf{u}\|^2 \end{aligned} \quad (8)$$

where  $M_1 > 0, M_2 > 0, C_i > 0$  ( $i = 1, 2, 3, 4$ ).

Considering  $t_0 - \frac{M_1 \theta_1 d_0}{1 - \theta_1} > 0$ , we obtain  $\theta_1 < \frac{t_0}{t_0 + M_1 d_0}$ . From  $d_0 \theta_1 C_1 - \frac{\tilde{d}_0 \theta_2 M_2 C_4}{1 - \theta_2} > 0$ , we have  $\theta_2 < \frac{d_0 \theta_1 C_1}{d_0 \theta_1 C_1 + M_2 C_4 \tilde{d}_0}$ .

Certainly,  $\theta_1, \theta_2$  can totally satisfy  $0 < \theta_i < 1$ . We can select  $\theta_1 = \frac{t_0}{2(t_0 + M_1 d_0)}, \theta_2 = \frac{d_0 \theta_1 C_1}{2(d_0 \theta_1 C_1 + M_2 C_4 \tilde{d}_0)}$  to make the coefficients of module for  $\mathbf{u}, \boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}$  be positive.

After that, we take the smallest among the three coefficients and obtain  $a^\#((\mathbf{u}, \boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}), (\mathbf{u}, \boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}})) \geq \alpha_0 (\|\mathbf{u}\|^2 + \|\boldsymbol{\lambda}\|^2 + \|\bar{\boldsymbol{\lambda}}\|^2)$ .

Accordingly we can make a conclusion that the solution of variational inequality(1) is unique under the conditions that the materials are subjected to the associated flow law and there is no friction on contact surfaces.

**Proof** Assume that there exist two different sets of the solution for inequality (1) as  $\{\mathbf{u}_1, \boldsymbol{\lambda}_1, \bar{\boldsymbol{\lambda}}_1\}$  and  $\{\mathbf{u}_2, \boldsymbol{\lambda}_2, \bar{\boldsymbol{\lambda}}_2\}$ . Substituting them into inequality (1), for  $\forall \{\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}}\} \in \tilde{K}$ , we have

$$\begin{aligned} a(\mathbf{u}_1, \mathbf{v} - \mathbf{u}_1) - b(\mathbf{v} - \mathbf{u}_1, \boldsymbol{\lambda}_1) + \tilde{a}(\mathbf{u}_1, \mathbf{v} - \mathbf{u}_1) - \tilde{b}(\mathbf{v} - \mathbf{u}_1, \bar{\boldsymbol{\lambda}}_1) + c(\mathbf{r} - \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1) - d(\mathbf{u}_1, \mathbf{r} - \boldsymbol{\lambda}_1) + \\ j(\mathbf{r} - \boldsymbol{\lambda}_1) + \tilde{c}(\bar{\mathbf{r}} - \bar{\boldsymbol{\lambda}}_1, \bar{\boldsymbol{\lambda}}_1) - \tilde{d}(\mathbf{u}_1, \bar{\mathbf{r}} - \bar{\boldsymbol{\lambda}}_1) + \tilde{j}(\bar{\mathbf{r}} - \bar{\boldsymbol{\lambda}}_1) \geq L(\mathbf{v} - \mathbf{u}_1) \end{aligned} \quad (9)$$

$$\begin{aligned} a(\mathbf{u}_2, \mathbf{v} - \mathbf{u}_2) - b(\mathbf{v} - \mathbf{u}_2, \boldsymbol{\lambda}_2) + \tilde{a}(\mathbf{u}_2, \mathbf{v} - \mathbf{u}_2) - \tilde{b}(\mathbf{v} - \mathbf{u}_2, \bar{\boldsymbol{\lambda}}_2) + c(\mathbf{r} - \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_2) - d(\mathbf{u}_2, \mathbf{r} - \boldsymbol{\lambda}_2) + \\ j(\mathbf{r} - \boldsymbol{\lambda}_2) + \tilde{c}(\bar{\mathbf{r}} - \bar{\boldsymbol{\lambda}}_2, \bar{\boldsymbol{\lambda}}_2) - \tilde{d}(\mathbf{u}_2, \bar{\mathbf{r}} - \bar{\boldsymbol{\lambda}}_2) + \tilde{j}(\bar{\mathbf{r}} - \bar{\boldsymbol{\lambda}}_2) \geq L(\mathbf{v} - \mathbf{u}_2) \end{aligned} \quad (10)$$

Setting  $\{\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}}\} = \{\mathbf{u}_2, \boldsymbol{\lambda}_2, \bar{\boldsymbol{\lambda}}_2\}$  in Eq. (9),  $\{\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}}\} = \{\mathbf{u}_1, \boldsymbol{\lambda}_1, \bar{\boldsymbol{\lambda}}_1\}$  in Eq. (10) and adding the two inequalities, we obtain

$$\begin{aligned} a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_2 - \mathbf{u}_1) - b(\mathbf{u}_2 - \mathbf{u}_1, \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) + \tilde{a}(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_2 - \mathbf{u}_1) - \tilde{b}(\mathbf{u}_2 - \mathbf{u}_1, \bar{\boldsymbol{\lambda}}_1 - \bar{\boldsymbol{\lambda}}_2) + c(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1) - \\ d(\mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1) + \tilde{c}(\bar{\boldsymbol{\lambda}}_1 - \bar{\boldsymbol{\lambda}}_2, \bar{\boldsymbol{\lambda}}_2 - \bar{\boldsymbol{\lambda}}_1) - \tilde{d}(\mathbf{u}_1 - \mathbf{u}_2, \bar{\boldsymbol{\lambda}}_2 - \bar{\boldsymbol{\lambda}}_1) \geq 0 \end{aligned} \quad (11)$$

Here the bilinearity of  $a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$  is considered. From (11), it follows that

$$a^\#((\mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2, \bar{\boldsymbol{\lambda}}_1 - \bar{\boldsymbol{\lambda}}_2), (\mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2, \bar{\boldsymbol{\lambda}}_1 - \bar{\boldsymbol{\lambda}}_2)) \leq 0 \quad (12)$$

From Eq. (3), we can obtain  $\exists \alpha_0 > 0$ , and it leads to

$$\alpha_0 (\|\mathbf{u}_1 - \mathbf{u}_2\|^2 + \|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|^2 + \|\bar{\boldsymbol{\lambda}}_1 - \bar{\boldsymbol{\lambda}}_2\|^2) \leq a^\#((\mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2, \bar{\boldsymbol{\lambda}}_1 - \bar{\boldsymbol{\lambda}}_2), (\mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2, \bar{\boldsymbol{\lambda}}_1 - \bar{\boldsymbol{\lambda}}_2)) \quad (13)$$

Consequently, from (12) and (13) we have  $\|\mathbf{u}_1 - \mathbf{u}_2\|^2 + \|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|^2 + \|\bar{\boldsymbol{\lambda}}_1 - \bar{\boldsymbol{\lambda}}_2\|^2 \leq 0$  or  $\{\mathbf{u}_1, \boldsymbol{\lambda}_1, \bar{\boldsymbol{\lambda}}_1\} = \{\mathbf{u}_2, \boldsymbol{\lambda}_2, \bar{\boldsymbol{\lambda}}_2\}$ .

The uniqueness of the solution for inequality (1) is proved.

For the existence of the solutions to the variational inequality (1), Lions<sup>[5]</sup> stated clearly the sufficient conditions for the solution to the elliptic variational inequality. Thus we can obtain the existing conditions for inequality (1):

- 1)  $a^\#((\mathbf{u}, \boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}), (\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}}))$  is continuous;
- 2)  $a^\#((\mathbf{u}, \boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}), (\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}}))$  is coercive;
- 3)  $j$  and  $\tilde{j}$  are lower convex and strongly lower semi-continuous.

According to the definition of  $a, b, c, d, j, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{j}$  in inequality (1) and the proof of the coerciveness of  $a^\#((\mathbf{u}, \boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}), (\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}}))$ , we can prove that  $a^\#((\mathbf{u}, \boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}), (\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}}))$  is continuous, and  $j$  and  $\tilde{j}$  are lower convex and strongly lower semi-continuous, respectively. Hence, the sufficient conditions are satisfied. Thus, the solution of inequality (1) exists. Of course,

the condition mentioned in this paper is a sufficient condition, rather than a necessary condition.

### 3 Variational Extremum Form

Furthermore, it can also be proved that inequality (1) is equivalent to the following minimization problem under the coerciveness of  $a^\#((\mathbf{u}, \boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}), (\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}}))$ . Find  $\{\mathbf{u}, \boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}\} \in \tilde{K}$  and let

$$J(\mathbf{u}, \boldsymbol{\lambda}, \bar{\boldsymbol{\lambda}}) = \min[J(\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}})] \quad (14)$$

where

$$J(\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) + \frac{1}{2}\bar{a}(\mathbf{v}, \mathbf{v}) + \frac{1}{2}c(\mathbf{r}, \mathbf{r}) + \frac{1}{2}\bar{c}(\bar{\mathbf{r}}, \bar{\mathbf{r}}) - b(\mathbf{v}, \mathbf{r}) - \bar{b}(\mathbf{v}, \bar{\mathbf{r}}) + j(\mathbf{r}) + \bar{j}(\bar{\mathbf{r}}) - L(\mathbf{v})$$

Because the maximal degree of  $\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}}$  about  $J(\mathbf{v}, \mathbf{r}, \bar{\mathbf{r}})$  is 2, the problems are linear. The definition of  $\tilde{K}$  is a close convex and the objective function of this problem is a convex function. Thus the original problem is a convex programming problem and has convex characteristics<sup>[6]</sup>. The convex characteristics have two advantages:

1) Possibilities of linearization;

2) The smallness in specific means the smallness in general. So it is only necessary to concentrate study on the functional. The restriction in specific means less data, which can minimize a computer's storage and create benefits for computational speed.

There is no better way to prove the existence and the uniqueness of the solution for the elastoplastic contact problems with friction. Cocu<sup>[7]</sup>, in 1984, did some research on the uniqueness of the solution of the unilateral boundary value problems and analyzed the frictional contact problem. He obtained a uniqueness and existence result for the Signorini problem with the Coulomb friction. In 1988, Curnier and Alart<sup>[8]</sup> gave necessary and sufficient conditions for bifurcation of finite-dimensional contact incremental problems. The examples of non-uniqueness of the solution for quasistatic contact problems with friction were presented by Klarbring<sup>[9]</sup> in 1990. Additionally, Khludnev<sup>[10]</sup> studied the problems of the convergence for the solutions of the contact between two plates. In 2000, Fu and Wu<sup>[11]</sup> elucidated the existence and uniqueness of solutions of generalized variational inequalities arising from elasticity with friction, which is equivalent to the corresponding elemental problems.

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## 一类弹塑性接触问题解的存在唯一性

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**摘要:** 针对弹塑性接触问题所构造的等价变分不等式, 解除了弹塑性本构状态约束方程和接触状态约束方程的约束. 首先证明了所构造泛函的强制性, 从而证明了所构造的等价变分不等式解的唯一性, 并根据椭圆型变分不等式解存在的充分条件论证了弹塑性接触问题解的存在性, 为该问题的变分极值原理的建立奠定了数学理论基础. 所构造的变分极值形式为运用数学规划法求解弹塑性接触问题提供了理论保证.

**关键词:** 弹塑性; 接触问题; 存在性; 唯一性; 强制性; 变分极值形式

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