Determination of pollution point source in parabolic system model

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Abstract: This paper considers an inverse problem for a partial differential equation to identify a pollution point source in a watershed. The mathematical model of the problem is a weakly coupled system of two linear parabolic equations for the concentrations $u(x, t)$ and $v(x, t)$ with an unknown point source $F(x, t) = \lambda(t)\delta(x - s)$ related to the concentration $u(x, t)$, where s is the point source location and $\lambda(t)$ is the amplitude of the pollution point source. Assuming that source F becomes inactive after time T^*, it is proved that it can be uniquely determined by the indirect measurements $\{v(0, t), v(a, t), v(b, t), v(l, t), 0 < t \leqslant T, T^* < T\}$, and, thus, the local Lipschitz stability for this inverse source problem is obtained. Based on the proof of its uniqueness, an inversion scheme is presented to determine the point source. Finally, two numerical examples are given to show the feasibility of the inversion scheme.

Key words: inverse source problem; parabolic system; uniqueness; local Lipschitz stability; pollution source

The environmental problem is one of the urgent issues for sustaining human life, where water pollution is one of the most important problems. In this paper, we consider the problem of identifying the pollution point source from the measurement data at some points in a watershed. The main application (but not only the one) of our study is the identification of the source pollution F of biological oxygen demand (BOD) in a river, from the concentration measures of dissolved oxygen (DO) at appropriate points. Here, the concentrations of BOD and DO are, respectively, denoted by u and v. In fact, the BOD measurements need 5 d in order to be available whereas the DO measurements are immediately available in general. Such a model is related to the determination of pollution sources causing water contamination in some finite region. The problem that we consider here is reduced to a linear parabolic system, and the concentrations $u(x, t)$, $v(x, t)$ satisfy the following initial-boundary problem with a zero-Neumann boundary condition,

$$\left.\begin{array}{ll} L[u](x,t) = F(x,t),\ L[v](x,t) = Ru(x,t) & x \in (0,l);0 < t < T \\ \dfrac{\partial u(0,t)}{\partial x} = \dfrac{\partial u(l,t)}{\partial x} = 0,\ \dfrac{\partial v(0,t)}{\partial x} = \dfrac{\partial v(l,t)}{\partial x} = 0 & t \in (0,T) \\ u(x,0) = \phi(x),\ v(x,0) = \psi(x) & x \in (0,l) \end{array}\right\} \tag{1}$$

where $L = \dfrac{\partial}{\partial t} - D\dfrac{\partial^2}{\partial x^2} + V\dfrac{\partial}{\partial x} + R$, $D > 0$ is a diffusion coefficient, $V > 0$ is the velocity of the watershed, $R > 0$ is the coefficient of the self-purifying function, $\phi(x)$ and $\psi(x)$ are the initial pollutions of the watershed, $F(x, t) = \lambda(t)\delta(x - s)$ is the pollution point source, s is the source location, and $\lambda(t) \in L^2[0, t]$ is the intensity of the pollution source.

Inverse source identification problems are important in many branches of the engineering sciences. Generally speaking, the reconstruction of unknown sources is ill-posed from practical noisy measurement data. Inverse source problems for parabolic equations were studied in Refs. [1 −8] and the references therein. When the initial conditions $\phi(x)$, $\psi(x)$ and the source $F(x, t)$ are known, the above-mentioned problem (1) is well-posed. This is the so-called direct problem. In this paper, the inverse problem we consider can be stated as follows. For given measurement data $\{v(0, t), v(a, t), v(b, t), v(l, t), 0 < t < T, a \neq b\}$, the point source $F(x, t) = \lambda(t)\delta(x - s)$ from problem (1) needs to be determined, that is, to find the source location s and the intensity $\lambda(t)$. Here, we assume that of the measurements of two points a and b, one chosen upstream from the source and the other downstream, are made based on the previous knowledge of the source location. Without loss of generality, we suppose that $0 < a < s < b < 0$.

The above inverse problem is different from that studied in our previous study[7], although we are all interested in the determination of pollution point source. In Ref. [6], only the first equation of (1) was considered to reconstruct the unknown source from the measurement data of the concentration u. However, in this paper we consider the weakly coupled equations of u and v where the measurement data are taken indirectly on the concentration v, which is naturally a different problem. The boundary conditions and the measurement data taken in this paper are also different from that in Ref. [8] for the same coupled equations.

1 Identifiability of the Point Source

Let

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$$u(x,t) = W_1(x,t)\exp\Big(\frac{V}{2D}x - (\frac{V^2}{4D}+R)t\Big) \tag{2}$$

$$v(x,t) = W_2(x,t)\exp\Big(\frac{V}{2D}x - (\frac{V^2}{4D}+R)t\Big) \tag{3}$$

Then problem (1) is equal to the following problem:

$$\left.\begin{array}{ll}\frac{\partial W_1}{\partial t} - D\frac{\partial^2 W_1}{\partial x^2} = F(x,t)\exp\Big(-\frac{V}{2D}x + (\frac{V^2}{4D}+R)t\Big),\ \frac{\partial W_2}{\partial t} - D\frac{\partial^2 W_2}{\partial x^2} = RW_1(x,t) & x\in(0,l);0<t<T \\ \frac{\partial W_1(x,t)}{\partial x} + \frac{V}{2D}W_1(x,t)\mid_{x=0,l} = 0,\ \frac{\partial W_2(x,t)}{\partial x} + \frac{V}{2D}W_2(x,t)\mid_{x=0,l} = 0 & 0<t<T \\ W_1(x,0) = \phi(x)\exp\Big(-\frac{V}{2D}x\Big),\ W_2(x,0) = \psi(x)\exp\Big(-\frac{V}{2D}x\Big) & x\in(0,l)\end{array}\right\} \tag{4}$$

Theorem 1 For given x_0, τ satisfying $0\leqslant x_0\leqslant l, 0\leqslant\tau\leqslant t_0<t_1<+\infty$, assume that $W_1(x,t)$ and $W_2(x,t)$ satisfy

$$\left.\begin{array}{ll}\frac{\partial W_1}{\partial t} - D\frac{\partial^2 W_1}{\partial x^2} = 0,\ \frac{\partial W_2}{\partial t} - D\frac{\partial^2 W_2}{\partial x^2} = RW_1(x,t) & x\in(0,l);\tau<t<t_1 \\ \frac{\partial W_1(x,t)}{\partial x} + \frac{V}{2D}W_1(x,t)\mid_{x=0,l} = 0,\ \frac{\partial W_2(x,t)}{\partial x} + \frac{V}{2D}W_2(x,t)\mid_{x=0,l} = 0 & \tau<t<t_1 \\ W_1(x,\tau),\ W_2(x,\tau)\in L^2(0,l) & \end{array}\right\} \tag{5}$$

If $W_2(x_0,t)\equiv 0$ for $t\in(t_0,t_1)$ and $\tan\frac{n\pi}{l}x_0 \neq \frac{2Dn\pi}{Vl}$ for $n=1,2,\ldots$, then $W_1(x,\tau)=0$ and $W_2(x,\tau)=0$ in the sense of $L^2(0,l)$.

Proof By the Fourier expansion, the solutions of (5) can be deduced such that

$$W_1(x,t) = C_0\exp\Big(\frac{V^2}{4D}(t-\tau)\Big)X_0(x) + \sum_{n=1}^{\infty} C_n\exp\Big(-D\Big(\frac{n\pi}{l}\Big)^2(t-\tau)\Big)X_n(x) \tag{6}$$

$$\begin{aligned} W_2(x,t) = {} & E_0\exp\Big(\frac{V^2}{4D}(t-\tau)\Big)X_0(x) + \sum_{n=1}^{\infty} E_n\exp\Big(-D\Big(\frac{n\pi}{l}\Big)^2(t-\tau)\Big)X_n(x) + \\ & RX_0(x)\int_\tau^t f_0(r)\exp\Big(\frac{V^2}{4D}(t-r)\Big)\mathrm{d}r + R\sum_{n=1}^{\infty} X_n(x)\int_\tau^t f_n(r)\exp\Big(-D\Big(\frac{n\pi}{l}\Big)^2(t-r)\Big)\mathrm{d}r \end{aligned} \tag{7}$$

where

$$C_n = \int_0^l W_1(x,\tau)X_n(x)\,\mathrm{d}x,\ E_n = \int_0^l W_2(x,\tau)X_n(x)\,\mathrm{d}x,\ f_n(t) = \int_0^l W_1(x,t)X_n(x)\,\mathrm{d}x$$

$$X_0(x) = d_0\exp\Big(-\frac{V}{2D}x\Big),\quad X_n(x) = d_n\Big(\cos\Big(\frac{n\pi}{l}x\Big) - \frac{V}{2D}\frac{l}{n\pi}\sin\Big(\frac{n\pi}{l}x\Big)\Big)$$

are normalized eigenfunctions; $d_n, n=0,1,\ldots$ are the normalized coefficients.

By substituting $W_1(x,t)$ given by Eq. (6) into the expression of $w_2(x,t)$, it follows that

$$W_2(x,t) = A_0(t)\exp\Big(\frac{V^2}{4D}(t-\tau)\Big)X_0(x) + \sum_{n=1}^{\infty} A_n(t)\exp\Big(-D\Big(\frac{n\pi}{l}\Big)^2(t-\tau)\Big)X_n(x) \tag{8}$$

where $A_n(t) = E_n + R(t-\tau)C_n$. Using the condition $W_2(x_0,t)=0, t\in(t_0,t_1)$, we have

$$A_0(t)\exp\Big(\frac{V^2}{4D}(t-\tau)\Big)X_0(x_0) + \sum_{n=1}^{\infty} A_n(t)\exp\Big(-D\Big(\frac{n\pi}{l}\Big)^2(t-\tau)\Big)X_n(x_0) \equiv 0 \qquad t\in(t_0,t_1)$$

Here, we consider the following function of a complex variable

$$\Phi(z) = A_0(z)X_0(x_0) + \sum_{n=1}^{\infty} A_n(z)\exp\Big(-\Big(D\Big(\frac{n\pi}{l}\Big)^2 + \frac{V^2}{4D}\Big)(z-\tau)\Big)X_n(x_0)$$

Then $\Phi(z)$ is analytic in Re $z\geqslant\tau+\alpha$ for any $\alpha>0$, where Re z indicates the real part of z. Since $\Phi(z)\equiv 0$ for z satisfying $0<t_0<\mathrm{Re}\,z<t_1$, and Im $z=0$, the unique extension theorem for analytic functions[9] yields that $\Phi(z)\equiv 0$ for all z such that Re $z\geqslant\tau+\alpha>0$. So, it follows that

$$A_0(t)X_0(x_0) + \sum_{n=1}^{\infty} A_n(t)\exp\Big(-\Big(D\Big(\frac{n\pi}{l}\Big)^2 + \frac{V^2}{4D}\Big)(t-\tau)\Big)X_n(x_0) = 0 \qquad t \in (\tau, +\infty)$$

Passing in this equality to the limit for $t \to +\infty$, we can obtain that

$$\lim_{t\to+\infty} A_0(t) = 0,\ \lim_{t\to+\infty} A_n(t)\Big[\cos\Big(\frac{n\pi}{l}x_0\Big) - \frac{V}{2D}\frac{l}{n\pi}\sin\Big(\frac{n\pi}{l}x_0\Big)\Big] = 0$$

Using the condition $\tan\frac{n\pi}{l}x_0 \neq \frac{2Dn\pi}{Vl}$ for $n = 1, 2, \ldots$, we have

$$\lim_{t\to+\infty} A_n(t) = \lim_{t\to+\infty}[E_n + R(t-\tau)C_n] = 0 \qquad n = 0, 1, 2, \ldots$$

Therefore, $C_n = 0$ and $E_n = 0$. Obviously, $W_1(x,\tau) = W_2(x,\tau) = 0$ in $L^2(0,l)$ because the system of the eigenfunctions $\{X_n(x)\}$ is complete in $L^2(0,1)$.

In view of (2) and (3), theorem 1 implies the following result.

Theorem 2 For $T^* < T$, assume that $u(x,t)$ satisfies

$$\left.\begin{array}{ll} L[u](x,t) = 0,\ L[v](x,t) = Ru(x,t) & x \in (0,l), t \in (T^*, T) \\ \dfrac{\partial u(0,t)}{\partial x} = \dfrac{\partial u(l,t)}{\partial x},\ \dfrac{\partial v(0,t)}{\partial x} = \dfrac{\partial v(l,t)}{\partial x} = 0 & t \in (T^*, T) \\ u(x,T^*), v(x,T^*) \in L^2(0,l) & x \in (0,l) \end{array}\right\} \tag{9}$$

Then the following results hold:

1) If $v(b,t) = 0$ for $t \in (T^*, T)$, where the observation point $b \in (0,l)$ satisfies $\tan\frac{n\pi}{l}b \neq \frac{2Dn\pi}{Vl}$ for all $n = 1, 2, \ldots$, then $u(x,T^*) = 0$ and $v(x,T^*) = 0$ in $L^2(0,l)$.

2) If $v(0,t) = 0$ or $v(l,t) = 0$ for $t \in (T^*, T)$, then $u(x,T^*) = 0$ and $v(x,T^*) = 0$ in $L^2(0,l)$.

In the following, we consider the uniqueness of the point source identification. Let $(u_i(x,t), v_i(x,t))$ be the solution of (1) with respect to $F_i(x,t) = \lambda_i(t)\delta(x - s_i)$, $i = 1, 2$.

Theorem 3 Let $F_i(x,t) = \lambda_i(t)\delta(x - s_i)$, where $s_i \in (a,b) \subset (0,l)$ and the nonnegative intensity $\lambda_i(t) \in L^2(0,T)$ satisfy $\lambda_i(t) = 0$ for $T^* < t < T$. Then

$$\{v_1(0,t) = v_2(0,t), v_1(a,t) = v_2(a,t), v_1(b,t) = v_2(b,t), v_1(l,t) = v_2(l,t), 0 < t < T\}$$

implies $s_1 = s_2$ and $\lambda_1(t) = \lambda_2(t)$ in $L^2(0,T)$.

Proof Let $U_1 = u_2 - u_1$, $U_2 = v_2 - v_1$. Then U_1 and U_2 satisfy

$$\left.\begin{array}{ll} L[U_1](x,t) = \lambda_2(t)\delta(x - s_2) - \lambda_1(t)\delta(x - s_1),\ L[U_2](x,t) = RU_1(x,t) & x \in (0,l); t \in (0,T) \\ \dfrac{\partial U_1(0,t)}{\partial x} = \dfrac{\partial U_1(l,t)}{\partial x} = 0,\ \dfrac{\partial U_2(0,t)}{\partial x} = \dfrac{\partial U_2(l,t)}{\partial x} = 0 & t \in (0,T) \\ U_1(x,0) = U_2(x,0) = 0 & x \in (0,l) \end{array}\right\} \tag{10}$$

and $\{U_2(0,t) = U_2(a,t) = U_2(b,t) = U_2(l,t) = 0, 0 < t < T\}$.

Step 1 Consider the solutions U_1, U_2 of (10) in $(0,l) \times (T^*, T)$. So, U_1 satisfy the homogeneous equation $L[U_1](x,t) = 0$, where the right-hand side of the first equation is zero because $\lambda(t) = 0$ for $t \in (T^*, T)$. Moreover, since $U_2(0,t) = U_2(l,t) = 0, t \in (T^*, T)$, from theorem 2 we immediately have $U_1(x,T^*) = 0$ and $U_2(x,T^*) = 0$ in $L^2(0,1)$.

Step 2 Consider the solutions U_1, U_2 of (10) in $(0,a) \times (0,T^*)$. Since $a < s_i < b$, it follows that

$$\left.\begin{array}{ll} L[U_1](x,t) = 0,\ L[U_2](x,t) = RU_1(x,t) & x \in (0,a); t \in (0,T^*) \\ \dfrac{\partial U_1(0,t)}{\partial x} = \dfrac{\partial U_2(0,t)}{\partial x} = 0,\ U_2(0,t) = U_2(a,t) = 0 & t \in (0,T^*) \\ U_1(x,0) = U_2(x,0) = 0 & x \in (0,a) \end{array}\right\} \tag{11}$$

Let $r_i(i = 1, 2)$ be the solutions of the characteristic equation $-Dr^2 - Vr + R = 0$, and then $r_1 = \frac{-V - \sqrt{V^2 + 4DR}}{2D}$, $r_2 = \frac{-V + \sqrt{V^2 + 4DR}}{2D}$. Let $\theta_i = e^{r_i x}$ and $h_i(x)$ be the solutions of the differential equation

$$Dh''_i(x) + Vh'_i(x) - Rh_i(x) = \theta_i(x) \qquad h_i(0) = h_i(a) = 0 \tag{12}$$

Multiplying the first equation of (11) by $h_1(x)$, the second equation by $\theta_1(x)$ and integrating with respect to x and t on $(0, a) \times (0, T^*)$, we obtain

$$\int_0^a h_1(x) U_1(x, T^*) \mathrm{d}x + \int_0^{T^*} \Big[-D \frac{\partial U_1}{\partial x} h_1 + D U_1 h_1' + V U_1 h_1 \Big]_0^a \mathrm{d}t = \int_0^{T^*} \int_0^a U_1(x, t) \theta_1(x) \mathrm{d}x \mathrm{d}t$$

$$\int_0^a \theta_1(x) U_2(x, T^*) \mathrm{d}x + \int_0^{T^*} \Big[-D \frac{\partial U_2}{\partial x} \theta_1 + D U_2 \theta_1' + V U_2 \theta_1 \Big]_0^a \mathrm{d}t = R \int_0^{T^*} \int_0^a U_1(x, t) \theta_1(x) \mathrm{d}x \mathrm{d}t$$

where $[f]_a^b := f(b) - f(a)$. Hence,

$$\begin{aligned} R D h_1'(a) \int_0^{T^*} U_1(a, t) \mathrm{d}t - R D h_1'(0) \int_0^{T^*} U_1(0, t) \mathrm{d}t &= -D \theta_1(a) \int_0^{T^*} \frac{\partial U_2(a, t)}{\partial x} \mathrm{d}t + \\ &\int_0^a \theta_1(x) U_2(x, T^*) \mathrm{d}x - R \int_0^a h_1(x) U_1(x, T^*) \mathrm{d}x \end{aligned} \tag{13}$$

On the other hand, multiplying the first equation of (11) by $h_2(x)$, the second equation by $\theta_2(x)$ and integrating with respect to x and t on $(0, a) \times (0, T^*)$, we have simultaneously

$$\begin{aligned} R D h_2'(a) \int_0^{T^*} U_1(a, t) \mathrm{d}t - R D h_2'(0) \int_0^{T^*} U_1(0, t) \mathrm{d}t &= -D \theta_2(a) \int_0^{T^*} \frac{\partial U_2(a, t)}{\partial x} \mathrm{d}t + \\ &\int_0^a \theta_2(x) U_2(x, T^*) \mathrm{d}x - R \int_0^a h_2(x) U_1(x, T^*) \mathrm{d}x \end{aligned} \tag{14}$$

Let $\mu_i(x) = \mathrm{e}^{-r_i x}$. Noting that $U_1(x, T^*) = U_2(x, T^*) = 0$ proven in step 1, we obtain from (13) and (14) that

$$[\mu_1(a) h_1'(a) - \mu_2(a) h_2'(a)] \int_0^{T^*} U_1(a, t) \mathrm{d}t = [\mu_1(a) h_1'(0) - \mu_2(a) h_2'(0)] \int_0^{T^*} U_1(0, t) \mathrm{d}t$$

Moreover, multiplying the differential equation (12) by $\mu_i(x)$, and integrating with x on $(0, a)$, we have $\mu_i(a) h_i'(a) = h_i'(0) + a/D$. Hence,

$$[h_1'(0) - h_2'(0)] \int_0^{T^*} U_1(a, t) \mathrm{d}t = [\mu_1(a) h_1'(0) - \mu_2(a) h_2'(0)] \int_0^{T^*} U_1(0, t) \mathrm{d}t \tag{15}$$

Let $\theta_a(x) = \theta_1(x) - \dfrac{\theta_1(a)}{\theta_2(a)} \theta_2(x)$, which satisfies $\theta_a(a) = 0$. Multiplying the first equation of (11) by $\theta_a(x)$, and integrating with respect to x and t over $(0, a) \times (0, T^*)$, we obtain

$$D \theta_a'(a) \int_0^{T^*} U_1(a, t) \mathrm{d}t = (D \theta_a'(0) + V \theta_a(0)) \int_0^{T^*} U_1(0, t) \mathrm{d}t \tag{16}$$

From (15) and (16), it follows that

$$\int_0^{T^*} U_1(0, t) \mathrm{d}t = \int_0^{T^*} U_1(a, t) \mathrm{d}t = 0$$

since $\Delta = (D \theta_a'(0) + V \theta_a(0)) [h_1'(0) - h_2'(0)] - D \theta_a'(a) [\mu_1(a) h_1'(0) - \mu_2(a) h_2'(0)] < 0$ which can be directly obtained from $\theta_i(x)$, $\theta_a(x)$, and $\mu_i(x)$.

Step 3 We proceed as in the process of step 2 and obtain $\int_0^{T^*} U_1(b, t) \mathrm{d}t = \int_0^{T^*} U_1(l, t) \mathrm{d}t = 0$.

Step 4 Multiplying the first equation of (10) by $\theta_i(x)$, $i = 1, 2$, and integrating with respect to x and t on $(0, l) \times (0, T^*)$, we have

$$\int_0^l \int_0^{T^*} L[U_1](x, t) \theta_i(x) \mathrm{d}t \mathrm{d}x = \theta_i(s_1) \int_0^{T^*} \lambda_1(t) \mathrm{d}t - \theta_i(s_2) \int_0^{T^*} \lambda_2(t) \mathrm{d}t \qquad i = 1, 2$$

By the properties of $U_1(x, t)$ and integrating by part, we find that

$$\bar{\lambda}_1 \mathrm{e}^{r_1 s_1} = \bar{\lambda}_2 \mathrm{e}^{r_1 s_2}, \ \bar{\lambda}_1 \mathrm{e}^{r_2 s_1} = \bar{\lambda}_2 \mathrm{e}^{r_2 s_2}$$

where $\bar{\lambda} = \int_0^{T^*} \lambda(t) \mathrm{d}t$. Since $r_1 \neq r_2$ and $\bar{\lambda}_i > 0$, there are $s_1 = s_2$ and $\bar{\lambda}_1 = \bar{\lambda}_2$.

Step 5 Let $s = s_1 = s_2$, $W_i(x, t) = U_i(x, t) \exp\Big(\dfrac{V}{2D} x - \Big(\dfrac{V^2}{4D} + R \Big) t \Big)$, $i = 1, 2$. By Eqs. (6) and (7) and the Duhamel principle, we obtain

$$W_1(x,t) = \exp\Big(-\frac{V}{2D}s\Big)X_0(s)X_0(x)\int_0^t \exp\Big(\Big(\frac{V^2}{4D}+R\Big)\tau\Big)(\lambda_2(\tau)-\lambda_1(\tau))\exp\Big(\frac{V^2}{4D}(t-\tau)\Big)\mathrm{d}\tau + \sum_{n=1}^{\infty}\exp\Big(-\frac{V}{2D}s\Big)X_n(s)X_n(x)\int_0^t \exp\Big(\Big(\frac{V^2}{4D}+R\Big)\tau\Big)(\lambda_2(\tau)-\lambda_1(\tau))\exp\Big(-D\Big(\frac{n\pi}{l}\Big)^2(t-\tau)\Big)\mathrm{d}\tau$$

$$W_2(x,t) = RX_0(x)\int_0^t f_0(r)\exp\Big(\frac{V^2}{4D}(t-r)\Big)\mathrm{d}r + R\sum_{n=1}^{\infty}X_n(x)\int_0^t f_n(r)\exp\Big(-D\Big(\frac{n\pi}{l}\Big)^2(t-r)\Big)\mathrm{d}r$$

where $f_n(t) = \int_0^l W_1(x,t)X_n(x)\,\mathrm{d}x$. Thus,

$$W_2(x,t) = R\exp\Big(-\frac{V}{2D}s\Big)X_0(s)X_0(x)\int_0^t\int_0^r \exp\Big(\Big(\frac{V^2}{4D}+R\Big)\tau\Big)(\lambda_2(\tau)-\lambda_1(\tau))\exp\Big(\frac{V^2}{4D}(t-\tau)\Big)\mathrm{d}\tau\mathrm{d}r + R\sum_{n=1}^{\infty}\exp\Big(-\frac{V}{2D}s\Big)X_n(s)X_n(x)\int_0^t\int_0^r \exp\Big(\Big(\frac{V^2}{4D}+R\Big)\tau\Big)(\lambda_2(\tau)-\lambda_1(\tau))\exp\Big(-D\Big(\frac{n\pi}{l}\Big)^2(t-\tau)\Big)\mathrm{d}\tau\mathrm{d}r \quad (17)$$

By exchanging the order of integration and simple computation, we rewrite (17) as

$$W_2(x,t) = \int_0^t \exp\Big(\Big(\frac{V^2}{4D}+R\Big)\tau\Big)(\lambda_2(\tau)-\lambda_1(\tau))K(x,t-\tau)\,\mathrm{d}\tau \quad (18)$$

where $K(x,t-\tau) = R(t-\tau)\exp\Big(-\frac{V}{2D}s\Big)\Big(X_0(s)X_0(x)\exp\Big(\frac{V^2}{4D}(t-\tau)\Big)+\sum_{n=1}^{\infty}X_n(s)X_n(x)\exp\Big(-D\Big(\frac{n\pi}{l}\Big)^2(t-\tau)\Big)\Big)$. The condition $U_2(l,t)=0$ implies

$$\int_0^t \exp\Big(\Big(\frac{V^2}{4D}+R\Big)\tau\Big)(\lambda_2(\tau)-\lambda_1(\tau))K(l,t-\tau)\,\mathrm{d}\tau = 0$$

According to Titchmarsh's convolution theorem[10] on the L^1 function, the functions $\lambda_2(t)-\lambda_1(t)$ and $K(l,t)$ must be zero identically at least in $(0,T')$ and $(0,T'')$, with T' and T'' such that $T'+T''\geqslant T$. If $K(l,t)=0$, $t\in(0,T'')$, $\forall T''>0$, by theorem 1 and the uniqueness of the analytic function we know that $K(l,t)=0$, $t\in(0,+\infty)$, that is, $X_n(s)X_n(l)=0$, $n=0,1,2,\ldots$. Obviously, it is impossible since $X_0(s)X_0(l)\neq 0$. Therefore, $\lambda_1(t)=\lambda_2(t)$, $t\in(0,T)$.

Remark 1 We have proven the uniqueness for our inverse problem. As for the local Lipschitz stability, it can be established by an analogous technique applied in Ref. [5].

2 Inversion Scheme for Determining the Point Source

Now we consider the inverse scheme of recovering $(s,\lambda(t))$. Our inversion scheme is based on the proof of theorem 3.

Step 1 Determining $u(x,T^*)$ and $v(x,T^*)$ by the measured data $v(b,t)$ or $v(l,t)$ from the following system

$$\left.\begin{array}{ll} L[u](x,t)=0,\ L[v](x,t)=Ru(x,t) & x\in(0,l);\ t\in(T^*,T) \\ \dfrac{\partial u(0,t)}{\partial x}=\dfrac{\partial u(l,t)}{\partial x}=0,\ \dfrac{\partial v(0,t)}{\partial x}=\dfrac{\partial v(l,t)}{\partial x}=0 & t\in(T^*,T)\end{array}\right\} \quad (19)$$

From Eqs. (2) to (8), the above backward problem is formulated to the first kind of Fredholm integral equation which can be solved by regularization methods, see Refs. [11 – 14].

Step 2 Computing $\int_0^{T^*} u(0,t)\,\mathrm{d}t$ and $\int_0^{T^*} u(l,t)\,\mathrm{d}t$. In view of (13) and (14), it follows that

$$\left.\begin{array}{l} [\mu_1(a)h_1'(a)-\mu_2(a)h_2'(a)]\displaystyle\int_0^{T^*}u(a,t)\,\mathrm{d}t-[\mu_1(a)h_1'(0)-\mu_2(a)h_2'(0)]\int_0^{T^*}u(0,t)\,\mathrm{d}t=\frac{\xi_1(a)}{RD} \\ D\theta_a'(a)\displaystyle\int_0^{T^*}u(a,t)\,\mathrm{d}t-(D\theta_a'(0)+V\theta_a(0))\int_0^{T^*}u(0,t)\,\mathrm{d}t=\xi_2(a)\end{array}\right\} \quad (20)$$

where

$$\xi_1(a) = \mu_1(a)\Big\{\int_0^a[\theta_1(x)v(x,T^*)-Rh_1(x)u(x,T^*)]\,\mathrm{d}x+\int_0^{T^*}[Dv\theta_1'+Vv\theta_1]_0^a\,\mathrm{d}t\Big\} - \mu_2(a)\Big\{\int_0^a[\theta_2(x)v(x,T^*)-Rh_2(x)u(x,T^*)]\,\mathrm{d}x+\int_0^{T^*}[Dv\theta_2'+Vv\theta_2]_0^a\,\mathrm{d}t\Big\}$$

$$\xi_2(a) = -\int_0^a \theta_a(x)u(x,T^*)\,\mathrm{d}x$$

Then, solve system (20) for the value $\int_0^{T^*} u(0, t)\,dt$. Compute $\int_0^{T^*} u(l, t)\,dt$ simultaneously from the following system,

$$\left.\begin{aligned} &[\mu_1(b)q_1'(l) - \mu_2(b)q_2'(l)]\int_0^{T^*} u(l,t)\,dt - [\mu_1(b)q_1'(b) - \mu_2(b)q_2'(b)]\int_0^{T^*} u(b,t)\,dt = \frac{\xi_3(b)}{RD} \\ &(D\theta_b'(l) + V\theta_b(l))\int_0^{T^*} u(l,t)\,dt - D\theta_b'(b)\int_0^{T^*} u(b,t)\,dt = \xi_4(b) \end{aligned}\right\} \tag{21}$$

where

$$\xi_3(b) = \mu_1(b)\left\{\int_b^l [\theta_1(x)v(x,T^*) - Rq_1(x)u(x,T^*)]\,dx + \int_0^{T^*}[Dv\theta_1' + Vv\theta_1]_b^l dt\right\} - \mu_2(b)\left\{\int_b^l [\theta_2(x)v(x,T^*) - Rq_2(x)u(x,T^*)]\,dx + \int_0^{T^*}[Dv\theta_2' + Vv\theta_2]_b^l dt\right\}$$

$$\xi_4(a) = -\int_b^l \theta_b(x)u(x,T^*)\,dx,\ \ \theta_b(x) = \theta_1(x) - \frac{\theta_1(b)}{\theta_2(b)}\theta_2(x)$$

and $q_i(x)$ is the solution of the differential equation

$$Dq_i''(x) + Vq_i'(x) - Rq_i(x) = \theta_i(x) \qquad q_i(b) = q_i(l) = 0 \tag{22}$$

Step 3 Identifying the location s. Multiplying the first equation of (1) by $e^{r_i x}$, $i = 1, 2$ and integrating with respect to x and t on $(0, l) \times (0, T^*)$, we obtain

$$\bar{\lambda} e^{r_i s} = \int_0^l [u(x,T^*) - \phi(x)]e^{r_i x}dx - D\int_0^{T^*}\left[e^{r_i x}\frac{\partial u(x,t)}{\partial x}\right]_0^l dt + D\int_0^{T^*}[r_i u(x,t)e^{r_i x}]_0^l dt + V\int_0^{T^*}[u(x,t)e^{r_i x}]_0^l dt \qquad i = 1, 2 \tag{23}$$

Then

$$s = \frac{1}{r_1 - r_2}\ln\left(\frac{M_1}{M_2}\right) \tag{24}$$

where M_i is the right hand of Eq. (23).

Step 4 Identifying the intensity $\lambda(t)$. In terms of (7) and (19), we have

$$v(x,t) = \exp\left(\frac{Vx}{2D} - \left(\frac{V^2}{4D} + R\right)t\right)\left(A_0\exp\left(\frac{V^2}{4D}t\right)X_0(x) + \sum_{n=1}^{\infty} A_n \exp\left(-D\left(\frac{n\pi}{l}\right)^2 t\right)X_n(x) + \int_0^t \exp\left(\left(\frac{V^2}{4D} + R\right)\tau\right)\lambda(\tau)K(x, t-\tau)\,d\tau\right) \tag{25}$$

where

$$C_n = \int_0^l \phi(x)X_n(x)\,dx,\ \ E_n = \int_0^l \psi(x)X_n(x)\,dx,\ \ A_n = E_n + RtC_n$$

$$K(x, t-\tau) = R(t-\tau)\exp\left(-\frac{V}{2D}s\right)\left(X_0(s)X_0(x)\exp\left(\frac{V^2}{4D}(t-\tau)\right) + \sum_{n=1}^{\infty} X_n(s)X_n(x)\exp\left(-D\left(\frac{n\pi}{l}\right)^2(t-\tau)\right)\right)$$

The problem we have to solve is to determine intensity $\lambda(t)$ satisfying Eq. (25) by one measurement data of $v(0, t)$, $v(a, t)$, $v(b, t)$ or $v(l, t)$. Taking into account the reconstruction accuracy of $\lambda(t)$, we prefer to select the measurement data $v(b, t)$ or $v(l, t)$ to determine $\lambda(t)$.

3 Numerical Examples

We test our inversion scheme from a classical model which is taken from Ref. [15]. More precisely, we consider the pollution process in a portion of a river of length $l = 1\,000$ m and during a period $T = 3.6$ h and $T^* = 3$ h, the diffusion coefficient $D = 29$ m^2/s, with the velocity of the watershed $V = 0.66$ m/s and the self-purifying coefficient $R = 1.01 \times 10^{-5}$ s. The initial pollution we take here is $\phi(x) = 0$. The source is located at $s = 400$ m and the measurement points chosen are $a = 200$ m and $b = 600$ m.

To test the stability of the inversion scheme, we use noisy data generated by

$$\tilde{u}(x, t_j) = u(x, t_j) + u(x, t_j)\epsilon(2\ \mathrm{rand}\ (1) - 1) \qquad x = 0, a, b, l$$

where $u(x, t_j)$ are exact data simulated by fully explicit finite difference approximation for the direct problem, rand (1) is a random number between (0, 1) and the magnitude ϵ indicates the relative error. The numerical results are shown in the original domain $(0, l) \times (0, T)$. Although there are many methods to solve the first kind of integral equation which is formulated from the backward problem (19), the reconstructions $u(x, T^*)$ and $v(x, T^*)$ obtained at the same time are not satisfied in step 4 of our numerical examples. There is also a gap between the theoretical background and the numerical implementation of the method. In other words, some new methods need to be developed to solve the backward problem (19). So, in the following computation we use the simulation data of $u(x, T^*)$ and $v(x, T^*)$ obtained by computing the direct problem to determine the location s. In our inverse scheme, we solve the discrete system by using a well known pseudo-inverse regularization method[16] for ill-conditional matrices with the tolerance max(size(A)) × norm(A) × eps, where "max", "size", "norm" and "eps" are Matlab functions.

Example 1 The intensity

$$\lambda(t) = \sum_{k=1}^{3} \alpha_k \exp(-\beta_k (t - \tau_k)^2)$$

where the configuration coefficients are $\alpha_1 = 120, \alpha_2 = 200, \alpha_3 = 160, \beta_1 = 10^{-6}, \beta_2 = 2 \times 10^{-6}, \beta_3 = 10^{-6}, \tau_1 = 3\,000$ s, $\tau_2 = 5\,000$ s, and $\tau_3 = 7\,000$ s.

We obtain $s = 434.3$ with $\epsilon = 0.005$, $s = 436.7$ with $\epsilon = 0.05$ by formula (24). Using these numerical results of s, we recover the intensity $\lambda(t)$. The inverse results of $\lambda(t)$ with noisy data are shown in Fig. 1.

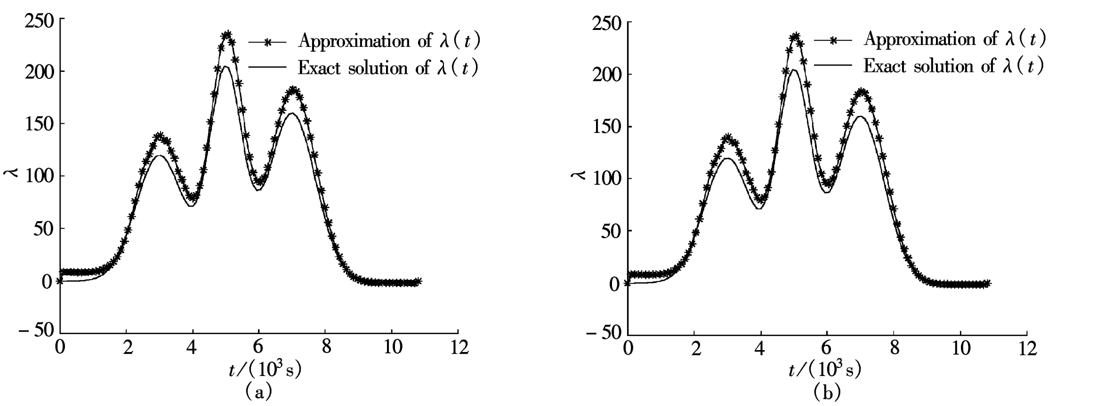

Fig. 1 $\lambda(t)$ recovered from data $u(b, t)$ at $b = 600$ with relative error for the case of example 1. (a) $\epsilon = 0.005$; (b) $\epsilon = 0.05$

Example 2 The intensity

$$\lambda(t) = \begin{cases} 0.001(t - 2\,160)(8\,640 - t) & 2\,160 < t < 8\,640 \\ 0 & \text{other} \end{cases}$$

In this example, we obtain $s = 433.9$ with $\epsilon = 0.005$, $s = 437.1$ with $\epsilon = 0.05$ by formula (24). Then using these numerical values of s, we recover the intensity $\lambda(t)$ which is shown in Fig. 2.

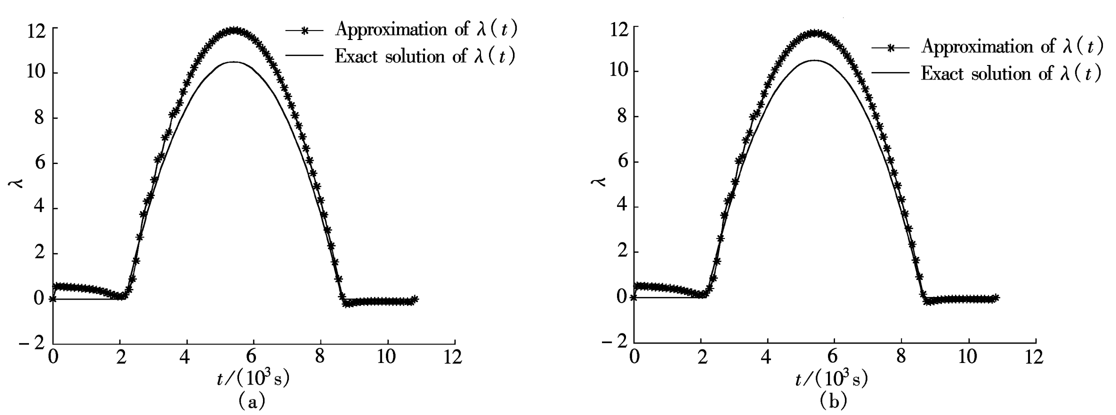

Fig. 2 $\lambda(t)$ recovered from data $u(b, t)$ at $b = 600$ with relative error for the case of example 2. (a) $\epsilon = 0.005$; $\epsilon = 0.05$

4 Conclusion and Discussion

In this paper, we consider an inverse problem modeling the point source detection in a watershed. The pollution diffusion process is governed by a one-dimensional linear parabolic system with the unknown source term $\lambda(t)\delta(x - s)$. The uniqueness and stability for determining the source (location and intensity) have been studied. Also we present an inversion scheme with two numerical examples. From the numerical implementations, we find that the intensity of the point source can be identified with a satisfactory accuracy, even if the input measurement data contain a large amount of noise. The source location can be successfully determined if the backward parabolic problem (19) is satisfactorily solved. Although we have proved that $u(x, T^*)$ and $v(x, T^*)$ can be uniquely determined in the backward problem (19), how to reconstruct them at the same time is an interesting problem and requires further studies.

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确定抛物型方程组模型中的污染点源

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摘要:考虑了在流域中确定单个污染点源的偏微分方程反问题. 该反问题的数学模型是关于浓度 $u(x,t)$ 和 $v(x,t)$ 的弱耦合线性抛物型方程组,其中关于浓度 $u(x,t)$ 的点源 $F(x,t)=\lambda(t)\delta(x-s)$ 是未知的,这里 s 表示点源位置,$\lambda(t)$ 表示污染点源的排放强度. 在已知污染源于时刻 T^* 停止排放的条件下,证明了 $F(x,t)=\lambda(t)\delta(x-s)$ 可由间接测量数据 $\{v(0,t), v(a,t), v(b,t), v(l,t), 0<t\leq T, T^*<T\}$ 惟一决定,且该反问题是局部 Lipschitz 稳定的. 基于惟一性的证明方法,提出了决定点源的反演算法. 最后,给出的 2 个数值例子表明了反演算法是可行的.

关键词:源项反演;抛物型方程组;惟一性;局部 Lipschitz 稳定;污染源
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