Journal of Southeast University (English Edition) Vol. 25, No. 3, pp. 419 – 422 Sept. 2009 ISSN 1003—7985

Periodic solutions of non-autonomous differential delay equations with superlinear properties

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Abstract: The existence of periodic solutions of a class of non-autonomous differential delay equations with the form $x'(t) = -\sum_{k=1}^{n-1} f(t, x(t-kr))$ is considered, where $r>0$ is a given constant and $f \in \mathbf{C}(\mathbf{R}\times\mathbf{R}, \mathbf{R})$ is odd in x, r-periodic in t and satisfies some superlinear conditions at origin and at infinity. First, the delay system is changed to an equivalent Hamiltonian system. Then the existence of periodic solutions of the Hamiltonian system is studied. Periodic solutions of the Hamiltonian system can be obtained by critical points of a functional defined on a Hilbert space, i. e. , points satisfying $\varphi'(z) = 0$. By using a linking theorem in critical point theory, the existence of critical points of the functional is obtained. Therefore, the existence of periodic solutions for the Hamiltonian system and its equivalent differential delay equation is established.

Key words: periodic solution; delay equation; Hamiltonian system; linking theorem

In the fields of applications, a variety of practical problems, such as communication systems[1], population growth models[2], the operation of a control system working with potentially explosive chemical reactions[3], and even in economic studies of business cycles[4], can be described by the following differential delay equation,

$$x'(t) = -\alpha f(x(t-1)) \tag{1}$$

where α is a real parameter and f is odd.

Eq. (1) was first studied by Jones[3] on the existence of periodic solutions in the 1970s. After that, according to dozens of applications, various questions on periodic solutions of Eq. (1) were considered by many researchers[1,5-6]. In recent years, the reduction method introduced by Kaplan and Yorke[5] was applied to study the existence of multiple periodic solutions of a more general form of Eq. (1) as follows[7-10]:

$$x'(t) = -\sum_{k=1}^{n-1} f(x(t-kr)) \tag{2}$$

Under the asymptotically linear conditions at origin and at infinity,

$$\lim_{x\to 0}\frac{f(x)}{x} = \alpha_0, \quad \lim_{x\to\infty}\frac{f(x)}{x} = \alpha_\infty \tag{3}$$

where $\alpha_0, \alpha_\infty \in \mathbf{R}$ and $|\alpha_0|, |\alpha_\infty| < +\infty$.

More specifically, they reduced the study of multiple periodic solutions of Eq. (2) to the problem of seeking multiple periodic solutions for a related system of the Hamiltonian system, which is called the coupled system of Eq. (2). The condition (3) plays a crucial role in the study of periodic solutions of Eq. (2) in Refs. [7 – 10], since the principle tools employed in their papers are various index theories. Therefore, $|\alpha_0|$, $|\alpha_\infty| < +\infty$ is necessary, since $|\alpha_0|$ or $|\alpha_\infty| = +\infty$ will bring difficulty to the discussion of the problem.

Now there are many results based on the autonomous differential equation (1). However, Belair and Mackey[4] studied a class of non-autonomous equation (1) where the delay depends on t instead of the constants. In this paper, we consider the following non-autonomous differential delay equation,

$$x'(t) = -\sum_{k=1}^{n-1} f(t, x(t-kr)) \tag{4}$$

where $r = 2\pi/n$ is a given constant and $f(t, x)$ satisfies the following assumptions:

f1) $f(t, x) \in \mathbf{C}(\mathbf{R}\times\mathbf{R}, \mathbf{R})$ is odd with respect to x, r-periodic with respect to t and $\int_0^x f(t,\xi)\,\mathrm{d}\xi > 0$ for $x \in \mathbf{R}$;

f2) There exist constants $\mu > 2$, $1 < \lambda < 2$, $c_1 > 0$ and $R > 0$ such that $f(t,x) < c_1|x|^\lambda$, $xf(t,x) \geqslant \mu\int_0^x f(t,\xi)\,\mathrm{d}\xi$, $\forall\ |x| > R$, $\forall\ t \in [0, r]$;

f3) The limits $\lim_{x\to 0}\frac{f(t,x)}{x} = 0$, $\lim_{x\to\infty}\frac{f(t,x)}{x} = \infty$ exist uniformly in t.

Then our main result reads as follows.

Theorem 1 Suppose that $f(t, x)$ satisfies f1) to f3). Then Eq. (4) possesses a nontrivial 2π-periodic solution.

We use the reduction method[5] and a linking theorem in critical point theory [11] to prove our result. The ideas come from Refs. [11 – 13]. Theorem 1 will be proved in section 2.

1 Some Preliminaries

Let E be a Hilbert space with $E = E_1 \oplus E_2$. Let P_1 and P_2 be the projections of E onto E_1 and E_2, respectively. Write

$$\Lambda = \{\psi \in C([0,r]\times E) \mid \psi(0,u) = u, P_2(t,u) = P_2u - K(t,u)\}$$

where K: $[0, r]\times E \to E$ is compact.

Definition 1 Let S, $Q \subset E$ and Q have boundaries. We call S and ∂Q link whenever $\psi \in \Lambda$ and $\psi(t, \partial Q)\cap S = \varnothing$ for all t, then $\psi(t, Q)\cap S \neq \varnothing$.

Definition 2 A functional $\varphi \in C^1(E, R)$ satisfies (PS)-condition, if every sequence such that $\{z_m\}\subset E$, $\varphi'(z_m)\to 0$ and $\varphi(z_m)$ being bounded possesses a convergent subsequence.

Received 2009-01-20.

Biography: Cheng Rong (1977—), male, doctor, mathchr@163.com.

Citation: Cheng Rong. Periodic solutions of non-autonomous differential delay equations with superlinear properties[J]. Journal of Southeast University (English Edition), 2009, 25(3): 419 – 422.

The following theorem is Theorem 5.29 of Ref. [11] and is used in our discussion.

Theorem 2 Let E be a real Hilbert space with $E=E_1\oplus E_2$, $E_2=E_1^{\perp}$ and inner product $\langle\cdot\rangle$. Suppose that $\varphi\in C^1(E,R)$ satisfies (PS)-condition, and

1) $\varphi(z)=\frac{1}{2}\langle Az,z\rangle+J(z)$, where $A(z)=A_1P_1z+A_2P_2z$ and $A_i: E_i\to E_i$ is bounded and self-adjoint, $i=1,2$;

2) J' is compact;

3) There exist a subspace $\tilde{E}\subset E$ and sets $S\subset E$, $Q\subset\tilde{E}$ and constants $\alpha>\bar{\omega}$ such that ① $S\subset E_1$ and $\varphi\mid_S\geqslant\alpha$; ② Q is bounded and $\varphi\mid_{\partial Q}\leqslant\bar{\omega}$; ③ S and ∂Q link. Then φ possesses a critical value $c\geqslant\alpha$.

2 Proof of the Main Result

Now we reduce Eq. (4) to an equivalent Hamiltonian system. Precisely, for $n=2N\in\mathbf{Z}^+$, if a 2π-periodic solution $\boldsymbol{X}(t)=\{x_1(t),x_2(t),\ldots,x_n(t)\}$ of the following system,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(t)=\boldsymbol{A}_n\boldsymbol{F}(t,\boldsymbol{X}(t))$$

where

$$\boldsymbol{A}_n=\begin{bmatrix}0 & -1 & \ldots & -1 & -1\\ 1 & 0 & \ldots & -1 & -1\\ 1 & 1 & & \vdots & \vdots\\ \vdots & \vdots & \ddots & 0 & -1\\ 1 & 1 & \ldots & 1 & 0\end{bmatrix}\tag{5}$$

satisfies the following symmetric structure

$$x_1(t)=-x_n(t-r),x_2(t)=x_1(t-r),\ldots,x_n(t)=x_{n-1}(t-r)\tag{6}$$

then $x(t)=x_1(t)$ is a 2π-periodic solution of Eq. (4) and satisfies $x(t-nr)=-x(t)$. Here $\boldsymbol{F}(t,\boldsymbol{X}(t))=(f(t,x_1),f(t,x_2),\ldots,f(t,x_n))$. Since $\boldsymbol{A}_{2N}$ is a nonsingular skew symmetric Hamiltonian matrix, system (5) can be written as the following Hamiltonian system,

$$z'(t)=\boldsymbol{A}_{2N}H_z(t,z)\tag{7}$$

where $H(t,z)=\int_0^{z_1}f(t,x)\,\mathrm{d}x+\ldots+\int_0^{z_{2N}}f(t,x)\,\mathrm{d}x$, $\forall z=(z_1,z_2,\ldots,z_{2N})\in\mathbf{R}^{2N}$, and $H_z(t,z)$ denotes the gradient of $H(t,z)$ with respect to z. In this paper, we always assume that $n=2N$ is even.

Lemma 1 Under the conditions f1) to f3), the Hamiltonian function $H(t,z)$ satisfies

H1) $H(t,z)\in C^1([0,r]\times\mathbf{R}^{2N},\mathbf{R}^{2N})$ is even, r-periodic in t, and $H(t,z)\geqslant0$ for all $(t,z)\in[0,r]\times\mathbf{R}^{2N}$;

H2) The limits $\lim\limits_{|z|\to0}\frac{H(t,z)}{|z|^2}=0$, $\lim\limits_{|z|\to\infty}\frac{H(t,z)}{|z|^2}=\infty$ exist uniformly in t;

H3) There exist constants $c_2>0$, $L>0$ such that for all with $z_i>R$, $i=1,2,\ldots,2N$, and $t\in[0,r]$, $0<\mu H(t,z)<zH_z(t,z)$, $|H_z(t,z)|\leqslant c_2|z|^{\lambda}$.

Proof From f1) and f3) and the definition of H, we can check easily that H1) and H2) hold. By $xf(t,x)\geqslant\mu\int_0^x f(t,\xi)\,\mathrm{d}\xi$, $\forall\,|x|>R$, we have a constant $L=\sqrt{2N}R$ such that $0<\mu H(t,z)<zH_z(t,z)$ for $|z|>L$ with $z_i>R$. Now we prove $|H_z(t,z)|\leqslant c_2|z|^{\lambda}$ for $z_i>R$; i. e., $f^2(t,z_1)+f^2(t,z_2)+\ldots+f^2(t,z_{2N})\leqslant c_2^2(z_1^2+z_2^2+\ldots+z_{2N}^2)^{\lambda}$.

First, it follows from $0\leqslant f(t,z_1)<c_2|z_1|^{\lambda}$ that $f^2(t,z_1)<c_2^2|z_1|^{2\lambda}$. Now we show $f^2(t,z_1)+f^2(t,z_2)<c_2^2(|z_1|^{2\lambda}+|z_2|^{2\lambda})$. Let $|z_1|^{\lambda}=\tau\cos\theta$, $|z_2|^{\lambda}=\tau\sin\theta$. Then $(z_1^2+z_2^2)^{\lambda}=\tau^2(\cos^{2/\lambda}\theta+\sin^{2/\lambda}\theta)\geqslant\tau^2=|z_1|^{2\lambda}+|z_2|^{2\lambda}$, since for $1<\lambda<2$, $1-\sin^2\theta\geqslant(1-\sin^{2/\lambda}\theta)^{\lambda}$; i. e., $(\cos^{2/\lambda}\theta+\sin^{2/\lambda}\theta)\geqslant1$. By the reducing method, we have $z_1^{2\lambda}+z_2^{2\lambda}+\ldots+z_{2N}^{2\lambda}\leqslant(z_1^2+z_2^2+\ldots+z_{2N}^2)^{\lambda}=|z|^{2\lambda}$. Thus, the inequality $|H_z(t,z)|\leqslant c_2|z|^{\lambda}$ since $z_i>R$ holds.

In the sequel, we work in the Hilbert space $E=W^{\frac{1}{2},2}(S^1,\mathbf{R}^{2N})$ with the inner product $\langle\cdot\rangle$ and the norm $\|\cdot\|$. The simplest way to introduce this space is shown as follows. Every function $z\in L^2(S^1,\mathbf{R}^{2N})$ has a Fourier expansion $z(t)=a_0+\sum_{m=1}^{\infty}(a_m\cos mt+b_m\sin mt)$, where a_m, b_m are $2N$-vectors. E is the set of such functions that $\|z\|^2=|a_0|^2+\sum_{m=1}^{\infty}m(|a_m|^2+|b_m|^2)<+\infty$.

With the norm $\|\cdot\|$, E is a Hilbert space. For $\forall z,y\in E$, we define an operator A on E by

$$\langle Az,y\rangle=\int_0^{2\pi}A_{2N}^{-1}z'(t)y(t)\,\mathrm{d}t\tag{8}$$

It is not difficult to verify that A is a bounded self-adjoint linear operator on E and $\ker A=\mathbf{R}^{2N}$. For any $\forall z\in E$, we define

$$J(z)=\int_0^{2\pi}H(t,z(t))\,\mathrm{d}t\tag{9}$$

Then critical points of the following functional on E defined by $\varphi(z)=\frac{1}{2}\langle Az,z\rangle-J(z)$ are periodic solutions of the Hamiltonian system (7). In order to obtain periodic solutions of Eq. (7) with the symmetric structure (6), we define the following action matrix,

$$\boldsymbol{T}_{2N}=\begin{bmatrix}0 & 0 & \ldots & 0 & -1\\ 1 & 0 & \ldots & 0 & 0\\ 0 & 1 & & \vdots & \vdots\\ \vdots & \vdots & \ddots & 0 & 0\\ 0 & 0 & \ldots & 1 & 0\end{bmatrix}$$

Then by $\boldsymbol{T}_{2N}$, for any $\forall z(t)\in E$, define an action on z by

$$\delta z(t)=\boldsymbol{T}_{2N}z(t-r)\tag{10}$$

Then by direct computation we obtain that $\delta^{2N}z(t)=-z(t-2Nr)$, $\delta^{4N}z(t)=z(t)$ and $G=\{\delta,\delta^2,\ldots,\delta^{4N}\}$ is a compact group action over E. If $\delta x(t)=x(t)$ holds, then through a direct check, we obtain that $z(t)$ has the symmetric structure (6).

Write $E_0=\{x\in E\mid\delta x(t)=x(t)\}$. Then from $z(t)=-z(t-2Nr)$ and direct computation, we obtain that $E_0=\left\{z(t)\mid z(t)=\sum_{m=1}^{\infty}a_m\cos(2m-1)t+b_m\sin(2m-1)t\right\}$.

Lemma 2 Critical points of $\varphi\mid_{E_0}$ over E_0 are critical points of φ on E, where $\varphi\mid_{E_0}$ is the restriction of φ over E_0.

Proof The definition of $H(t, z)$, together with a direct computation implies that

$$H(t, T_{2N}z) = H(t, z), H_z(t, T_{2N}z) = T_{2N}H_z(t, z)$$

Combining these with (10) and the fact that any $z(t) \in E$ is 2π-periodic, we can verify that $\varphi(\delta z) = \varphi(z)$, $\varphi'(\delta z) = \delta\varphi'(z) = \varphi'(z)$.

That means φ is G-invariant and φ' is G-equivariant. Moreover, $\varphi'(z) \in E_0$. Therefore, if $\varphi'(z) = 0$ on E_0, then $\varphi'(z) = 0$ on E. Hence, the conclusion of lemma 2 holds.

Note that $\ker A = \mathbf{R}^{2N}$. Let E_0^+ and E_0^- denote the positive definite and negative definite subspace of A in E_0, respectively. Then $E_0 = E_0^+ \oplus E_0^-$. Letting $E_1 = E_0^+$, $E_2 = E_0^-$, we see that φ satisfies 1) of theorem 2. By Lemma 4. 1 of Ref. [14], J' is compact. Hence, 2) holds. Now we establish 3) of theorem 2 by the following three lemmas.

Lemma 3 If H satisfies H1) to H3), then ① of 3) holds for $\varphi|_{E_0}$.

Proof From H1) to H3), one has $H(t, z) \leqslant c_3 + c_4 |z|^{\lambda+1}$, $\forall (t, z) \in [0, r] \times \mathbf{R}^{2N}$. By H2), for any $\varepsilon > 0$, there is a $\eta > 0$ such that $H(t, z) \leqslant \varepsilon |z|^2$, $\forall (t, z) \in [0, r]$, $|z| < \eta$.

Therefore, there is an $M = M(\varepsilon) > 0$ such that

$$H(t, z) \leqslant \varepsilon |z|^2 + M|z|^{\lambda+1} \quad \forall (t, z) \in [0, r] \times \mathbf{R}^{2N} \tag{11}$$

Since E_0 is compactly embedded in $L^s(S^1, \mathbf{R}^{2N})$ for all $s \geqslant 1$ and by (11), we have

$$\int_0^{2\pi} H(t, z)\,dt \leqslant \varepsilon\|z\|_{L^2}^2 + M\|z\|_{L^{\lambda+1}}^{\lambda+1} \leqslant (\varepsilon c_5 + Mc_6\|z\|^{\lambda-1})\|z\|^2$$

Consequently, for $z \in E_1$, $\varphi(z) \geqslant \|z\|^2 - (\varepsilon c_5 + Mc_6\|z\|^{\lambda-1}) \cdot \|z\|^2$. Choose $\varepsilon = (3c_5)^{-1}$ and ρ such that $3Mc_6\rho^{\lambda-1} = 1$. Then for any $z \in \partial B_\rho \cap E_1$, $\varphi_{E_0} \geqslant \frac{1}{3}\rho^2 = \alpha$. Thus, φ_{E_0} satisfies ① of 3) with $S = \partial B_\rho \cap E_1$.

Lemma 4 If H satisfies H1) and H3), then φ_{E_0} satisfies ② of 3).

Proof Set $e \in S = \partial B_\rho \cap E_1$ and let $Q = \{se: 0 \leqslant s \leqslant 2s_1\} \oplus B_{2s_1} \cap E_2$, where s_1 is free for the moment. Let $\tilde{E}_0 = E_0^- \oplus \operatorname{span}\{e\}$. Denote

$$\Gamma = \{z \in \tilde{E}_0: \|z\| = 1\}, \quad \lambda^- = \inf_{z \in E_0^-, \|z\|=1} |\langle Az^-, z^-\rangle|$$

$$\lambda^+ = \inf_{z \in E_0^+, \|z\|=1} |\langle Az^+, z^+\rangle|$$

Case 1 If $\|z^-\| > \gamma\|z^+\|$ with $\gamma = \sqrt{\lambda^+/\lambda^-}$, one has

$$\varphi_{E_0}(sz) = \frac{1}{2}\langle Asz^+, sz^+\rangle + \frac{1}{2}\langle Asz^-, sz^-\rangle - \int_0^{2\pi} H(t, sz)\,dt \leqslant -\frac{1}{2}\lambda^- s^2\|z^-\|^2 + \frac{1}{2}\lambda^+ s^2\|z^+\|^2 \leqslant 0$$

Case 2 If $\|z^-\| \leqslant \gamma\|z^+\|$, we have $1 = \|z\|^2 = \|z^+\|^2 + \|z^-\|^2 \leqslant (1+\gamma^2)\|z^+\|^2$, that is, $\|z^+\|^2 \geqslant \frac{1}{1+\gamma^2} > 0$. Denote $\tilde{\Gamma} = \{z \in \Gamma: \|z^-\| \leqslant \gamma\|z^+\|\}$. By a similar argument with the claim of Ref. [13], there exists $\varepsilon_1 > 0$ such that $\forall u \in \tilde{\Gamma}$, $\operatorname{meas}\{t \in [0, r]: u(t) \geqslant \varepsilon_1\} \geqslant \varepsilon_1$.

Now for $z = z^+ + z^- \in \tilde{\Gamma}$, set $\Omega_z = \{t \in [0, r]: |z(t)| \geqslant \varepsilon_1\}$. By H2), for a constant $M = \|A\|/\varepsilon_1^3 > 0$, there is an $L_1 > 0$ such that $H(t, z) \geqslant M|z|^2$, $\forall |z| > L_1$ uniformly in t. Choosing $s_1 \geqslant L_1/\varepsilon_1$, for $s \geqslant s_1$, $H(t, sz(t)) \geqslant M|sz(t)|^2 \geqslant Ms^2\varepsilon_1^2$, $\forall t \in \Omega_z$. Then one has

$$\begin{aligned}\varphi_{E_0}(sz) &= \frac{1}{2}s^2\langle Az^+, z^+\rangle + \frac{1}{2}s^2\langle Az^-, z^-\rangle - \int_0^{2\pi} H(t, sz)\,dt \leqslant \frac{1}{2}\|A\|s^2 - \int_{\Omega_z} H(t, sz)\,dt \leqslant \\ &\frac{1}{2}\|A\|s^2 - Ms^2\varepsilon_1^2 \operatorname{meas}(\Omega_z) \leqslant \\ &\frac{1}{2}\|A\|s^2 - Ms^2\varepsilon_1^3 = -\frac{1}{2}\|A\|s^2 < 0\end{aligned}$$

Therefore, $\varphi_{E_0}(sz) \leqslant 0$ for any $z \in \Gamma$ and $s \geqslant s_1$; i. e., $\varphi_{E_0}|_{\partial Q} \leqslant 0$. Then ② of 3) holds.

Lemma 5 S and ∂Q are linking.

The proof is similar to the theorem of Ref. [12]. Here we omit the details. Now it remains to verify that φ_{E_0} satisfies (PS)-condition.

Lemma 6 Under the assumptions of lemma 1, φ_{E_0} satisfies (PS)-condition.

Proof Suppose that $|\varphi_{E_0}(z_m)| \leqslant M$, $\varphi'_{E_0}(z_m) \to 0$ as $m \to \infty$. By a standard argument, we only need to show that $\{z_m\}$ is bounded. Then $\{z_m\}$ has a convergent subsequence. If $\{z_m\}$ is not bounded, then pass to a subsequence if necessary, $\|z_m\| \to +\infty$ as $m \to \infty$.

By H2), there are two constants $M_1, c_7 > 0$ such that $H(t, z) \geqslant c_7|z|^2$ for $|z| > M_1$. Then one has

$$2\varphi_{E_0}(z_m) - \langle\varphi'_{E_0}(z_m), z_m\rangle = \int_0^{2\pi}(z_m H_z(t, z_m) - 2H(t, z_m))\,dt \geqslant \int_0^{2\pi}(\mu - 2)H(t, z_m)\,dt \geqslant c_7(\mu - 2)\int_0^{2\pi}|z_m|^2\,dt$$

This yields

$$\frac{\int_0^{2\pi}|z_m|^2\,dt}{\|z_m\|} \to 0 \qquad m \to \infty \tag{12}$$

Write $\kappa = 1/2(\lambda - 1)$. By H3), there is a constant $c_8 > 0$ such that $|H_z(t, z)|^\kappa \leqslant c_4^\kappa|z|^{\lambda\kappa} + c_8$, $\forall t \in [0, r] \times \mathbf{R}^{2N}$. Therefore,

$$\begin{aligned}\int_0^{2\pi}|H_z(t, z_m)|^\kappa\,dt &\leqslant \int_0^{2\pi}(c_2^\kappa|z_m|^{\lambda\kappa} + c_8)\,dt \leqslant \\ &c_9\left(\int_0^{2\pi}\|z_m\|^2 dt\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\|z_m\|^{2(\kappa\lambda-1)}\,dt\right)^{\frac{1}{2}} + c_{10} \leqslant \\ &c_{11}\left(\int_0^{2\pi}\|z_m\|^2\,dt\right)^{\frac{1}{2}}\|z_m\|^{\kappa\lambda-1} + c_{12}\end{aligned}$$

This inequality and (12) imply that

$$\left(\frac{\left(\int_0^{2\pi}|H_z(t, z_m)|^\kappa\,dt\right)^{\frac{1}{\kappa}}}{\|z_m\|^\kappa}\right)^{\frac{1}{\kappa}} \leqslant \frac{c_{11}\left(\int_0^{2\pi}\|z_m\|^2\,dt\right)^{\frac{1}{2}}}{\|z_m\|^{\frac{1}{2}}} \cdot \frac{\|z_m\|^{\kappa\lambda-1}}{\|z_m\|^{\kappa-1/2}} + \frac{c_{12}}{\|z_m\|^\kappa} \to 0$$

as $m\to\infty$, since $\kappa>1$. Let $z_m=z_m^+ + z_m^- \in E_0^+\oplus E_0^-$. We have

$$\langle \varphi'_{E_0}(z_m), z_m^-\rangle = \langle Az_m^-, z_m^-\rangle - \int_0^{2\pi} z_m^- H_z(t,z_m)\,\mathrm{d}t \geqslant$$
$$\langle Az_m^-, z_m^-\rangle - \int_0^{2\pi} |z_m^-|\,|H_z(t,z_m)|\,\mathrm{d}t \geqslant$$
$$\langle Az_m^-, z_m^-\rangle - \left(\int_0^{2\pi} |H_z(t,z_m)|^{\kappa}\mathrm{d}t\right)^{\frac{1}{\kappa}} C_\kappa \|z_m^-\|$$

where $C_\kappa>0$ is a constant independent of m.

By the above inequality, one has

$$\frac{\langle Az_m^-, z_m^-\rangle}{\|z_m\|\|z_m^-\|} \leqslant \frac{\|\varphi'_{E_0}(z_m)\|\|z_m^-\|}{\|z_m\|\|z_m^-\|} + \int_0^{2\pi} \frac{(|H_z(t,z_m)|^{\kappa}\mathrm{d}t)^{\frac{1}{\kappa}}}{\|z_m\|}\frac{C_\kappa\|z_m^-\|}{\|z_m^-\|} \to 0$$

as $m\to\infty$. This yields

$$\frac{\|z_m^-\|}{\|z_m\|}\to 0 \qquad m\to\infty \tag{13}$$

Similarly, we have

$$\frac{\|z_m^+\|}{\|z_m\|}\to 0 \qquad m\to\infty \tag{14}$$

Thus, it follows from (13) and (14) that

$$1=\frac{\|z_m\|}{\|z_m\|}\leqslant\frac{\|z_m^-\|+\|z_m^+\|}{\|z_m\|}\to 0 \qquad m\to\infty$$

a contradiction. Hence, $\{z_m\}$is bounded.

Now we are ready to prove theorem 1.

Proof of Theorem 1 It is obvious that theorem 1 holds from lemma 1 to lemma 6 and theorem 2.

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具有超线性性质的非自治微分方程的周期解

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摘要： 研究了具有形式 $x'(t) = -\sum_{k=1}^{n-1} f(t, x(t-kr))$ 的非自治时滞微分方程周期解的存在性，其中 $r>0$ 是一个给定的常数，$f\in \mathbf{C}(\mathbf{R}\times\mathbf{R},\mathbf{R})$ 对变量 x 是奇的，对变量 t 是 r-周期的，而且在原点和无穷远处满足超线性性质. 首先将此方程转化成一个与之等价的哈密顿系统，然后研究了哈密顿系统的周期解的存在性. 哈密顿系统的周期解由一个定义在 Hilbert 空间上的变分泛函 $\varphi(z)$ 的临界点获得，即使得 $\varphi'(z)=0$ 的点. 运用临界点理论中的一个环绕定理，得到此变分泛函的临界点的存在性. 从而建立哈密顿系统以及与之等价的时滞微分方程的周期解的存在性定理.

关键词： 周期解；时滞方程；哈密顿系统；环绕定理

中图分类号： O175