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π-quasitriangular group-cograded multiplier Hopf algebras

Yang Tao Wang Shuanhong

(Department of Mathematics, Southeast University, Nanjing 211189, China)

Abstract: Let G be a group and $\langle A, B\rangle$ be a pair of multiplier Hopf algebras, where B is regular G-cograded. Let π be a crossing action of G on B, $D^{\pi} = A^{\mathrm{cop}} \propto \tilde{B} = \oplus_{p\in G} D_p^{\pi}$ with $D_p^{\pi} = A^{\mathrm{cop}} \propto \tilde{B}_p$ is the Drinfeld double of the pair $\langle A, B\rangle$, and then the deformation $\tilde{D}^{\pi}$ becomes a multiplier Hopf algebra. $B\otimes A$ can be considered as a subalgebra of $M(D^{\pi}\otimes D^{\pi})$, the image of element $b\otimes a$ in $B\otimes A$ is $(1 \propto b)\otimes(a\propto 1)$ in $M(D^{\pi}\otimes D^{\pi})$. Let $W = \sum_{\alpha} W_{\alpha} \in M(B\otimes A)$ be a π-canonical multiplier for the pair $\langle A, B\rangle$ with $W_{\alpha} \in M(B_{\alpha}\otimes A)$ for all $\alpha\in G$. The image of W in $M(D^{\pi}\otimes D^{\pi})$ is a π-quasitriangular structure over D^{π}.

Key words: multiplier Hopf algebra; group-cograded; Drinfeld double; quasitriangular

Throughout this paper, we will consider associative algebras over the complex number field C, with or without identity, but with a non-degenerate product. For algebra A with a non-degenerate product, it is possible to construct the multiplier algebra $M(A)$. $M(A)$ is an algebra with identity such that A sits in $M(A)$ as an essential two-sided ideal. The multiplier algebra $M(A)$ can also be characterized as the largest algebra with identity containing A as an essential ideal. If A already has an identity, the product is obviously non-degenerate and $M(A) = A$. For more details about the concept of the multiplier algebra of an algebra, we can refer to Ref. [1].

In 1994, Van Daele[1] introduced a much larger class of multiplier Hopf algebras, generalizing the ordinary Hopf algebras. Multiplier Hopf algebra A is an associative algebra over C, with or without identity, but with a non-degenerate product and with an algebraic homomorphism coproduct Δ such that the linear maps T_1, $T_2 \in \mathrm{End}(A\otimes A)$ defined by $T_1(a\otimes b) = \Delta(a)(1\otimes b)$, $T_2(a\otimes b) = (a\otimes 1)\Delta(b)$ for all a, $b\in A$, are bijective. If also Δ satisfies $\Delta(A)(A\otimes 1)\subseteq A\otimes A$ and $(1\otimes A)\Delta(A)\subseteq A\otimes A$, then (A, Δ) is called regular, and if (A, Δ) has a non-zero integral, (A, Δ) is said to be an algebraic quantum group. Multiplier Hopf algebras have become a power tool to treat the dual theory of infinite dimensional co-Frobenius Hopf algebras[2].

Recently, the concept of group-cograded multiplier Hopf algebra was introduced in Ref. [3] as a generalization of Hopf group-coalgebras introduced in Ref. [4]. Let (A, Δ) be a multiplier Hopf algebra and G a group. Assume that there is a family of (non-trivial) subalgebras $(A_p)_{p\in G}$ of A so that 1) $A = \oplus_{p\in G} A_p$ with $A_pA_q = 0$ whenever p, $q\in G$ and $p\neq q$; and 2) $\Delta(A_{pq})(1\otimes A_q) = A_p\otimes A_q$ and $(A_p\otimes 1)\Delta(A_{pq}) = A_p\otimes A_q$ for all p, $q\in G$. Then (A, Δ) is called a G-cograded multiplier Hopf algebra. The theory of group-cograded multiplier Hopf algebras was further developed in Refs. [5 – 7]. In particular in Ref. [6], the authors studied quasitriangular group-cograded multiplier Hopf algebras in the following sense: a G-cograded multiplier Hopf algebra with a crossing action ξ is called a π-quasitriangular if there is a multiplier $R = \sum_{\alpha, \beta\in G} R_{\alpha, \beta}$ with $R_{\alpha, \beta} \in M(A_{\alpha}\otimes A_{\beta})$ so that $(\xi_p\otimes\xi_p)(R) = R$, $(\tilde{\Delta}\otimes i)(R) = R_{13}R_{23}$, $(i\otimes\Delta)(R) = R_{13}R_{12}$ and $R\Delta(a) = (\tilde{\Delta})^{\mathrm{cop}}(a)R$ for all $p\in G$ and $a\in A$, where $\tilde{\Delta}(a)(1\otimes a') = (\xi_{q^{-1}}\otimes i)(\Delta(a)(1\otimes a'))$ for all $a\in A$ and $a'\in A_q$.

1 Deformation of the Product by a π-Twisting Element

We first give the definition of a π-twisting element as follows.

Definition 1 Let A be G-cograded. A π-invertible multiplier element $R = \sum_{p, q\in G} R_{p, q}$ with $R_{p, q} \in M(A_p\otimes A_q)$ is a π-twisting element if

1) $\sum_{\alpha, \beta, \gamma\in G} (R_{\beta, \gamma})_{23}((i_{\alpha}\otimes\Delta_{\beta, \gamma})(R_{\alpha, \beta\gamma})) = \sum_{\alpha, \beta, \gamma\in G} (R_{\alpha, \beta})_{12}((\Delta_{\alpha, \beta}\otimes i_{\gamma})(R_{\alpha\beta, \gamma}))$;

2) For all $p\in G$, $(\varepsilon\otimes i_p)(R_{e, p}) = 1_{A_p} = (i_p\otimes\varepsilon)(R_{p, e})$.

Example 1 Let A be a π-quasitriangular G-cograded multiplier Hopf algebra, and then R is a π-twisting element. Indeed,

$$\sum_{\alpha, \beta, \gamma\in G} (R_{\beta, \gamma})_{23}((i_{\alpha}\otimes\Delta_{\beta, \gamma})(R_{\alpha, \beta\gamma})) = \sum_{\alpha, \beta, \gamma\in G} (1_{\alpha}\otimes R_{\beta, \gamma})((i_{\alpha}\otimes\Delta_{\beta, \gamma})(R_{\alpha, \beta\gamma})) =$$
$$\sum_{\alpha, \beta, \gamma\in G} (i_{\alpha}\otimes(\tilde{\Delta}_{\beta\gamma\beta^{-1}, \beta})^{\mathrm{cop}})(R_{\alpha, \beta\gamma})(1_{\alpha}\otimes R_{\beta, \gamma}) = \sum_{\alpha, \beta, \gamma\in G} (i_{\alpha}\otimes i_{\beta}\otimes\pi_{\beta^{-1}})((i_{\alpha}\otimes\Delta^{\mathrm{cop}}_{\beta\gamma\beta^{-1}, \beta})R_{\alpha, \beta\gamma})(1_{\alpha}\otimes R_{\beta, \gamma}) =$$

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Biographies: Yang Tao (1984—), male, graduate; Wang Shuanhong (corresponding author), male, doctor, professor, shuanhwang2002@yahoo.com.

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$$\sum_{\alpha,\beta,\gamma\in G}(i_\alpha\otimes i_\beta\otimes\pi_{\beta^{-1}})((R_{\alpha,\beta})_{12}(R_{\alpha,\beta\gamma\beta^{-1}})_{13})(R_{\beta,\gamma})_{23} = \sum_{\alpha,\beta,\gamma\in G}(R_{\alpha,\beta})_{12}((i_\beta\otimes\pi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})_{13})(R_{\beta,\gamma})_{23} = \sum_{\alpha,\beta,\gamma\in G}(R_{\alpha,\beta})_{12}((\Delta_{\alpha,\beta}\otimes i_\gamma)(R_{\alpha\beta,\gamma}))$$

We now suppose that $\langle A, B\rangle$ is a pair of multiplier Hopf algebras so that B is G-cograded with a π-twisting element R. By Ref. [8], we have a new multiplication on A by using the above π-twisting element R as follows:

$$m_R(a\otimes a')=m_A((a\otimes a')\lhd R)$$

This new algebra with a non-degenerate product will be denoted by A_R. Then we have the following propositions.

Proposition 1 With the notation above, we obtain

$$\langle a\cdot a', b\rangle=\langle a\otimes a', R_{\alpha,\beta}\Delta_{\alpha,\beta}(b)\rangle$$

for all $a\in A_\alpha$, $a'\in A_\beta$ and $b\in A_{\alpha\beta}$. And A_R is a G-graded algebra and $1_{A_e}\in M(A_e)$ remains the unit in $M(A_R)$.

Proof It follows from Refs. [7-8] that A_R is a G-graded algebra.

To see that $1_{A_e}\in M(A_e)$ remains the unit in $M(A_R)$, we use that $\langle 1_{A_e}, b_e\rangle=\varepsilon(b_e)$ for all $b_e\in B_e$. In fact, for all b, $b'\in B_p$, $p\in G$ and $a\in A_p$ with $b'\rhd a=a$, we obtain $\langle 1_{A_e}\cdot a, b\rangle=\langle 1_{A_e}\otimes a, R_{e,p}\Delta_{e,p}(b)(1\otimes b')\rangle=\langle a, (\varepsilon\otimes i)(R_{e,p}\Delta_{e,p}(b)\cdot(1\otimes b'))\rangle=\langle a, bb'\rangle=\langle a, b\rangle$

As a consequence, $1_{A_e}\cdot a=a$ for all $a\in A_p$ and $p\in G$. Similarly, $a\cdot 1_{A_e}=a$ for all $a\in A_p$ and $p\in G$.

Remark 1 Definition 1 yields that

$$\sum_{\alpha,\beta,\gamma\in G}((i_\alpha\otimes\Delta_{\beta,\gamma})(R^{-1}_{\alpha,\beta\gamma}))(R^{-1}_{\beta,\gamma})_{23} = \sum_{\alpha,\beta\in G}((\Delta_{\alpha,\beta}\otimes i_\gamma)(R^{-1}_{\alpha\beta,\gamma}))(R^{-1}_{\alpha,\beta})_{12}$$

The π-twisting element $R^{-1}=\sum_{p,q\in G}R^{-1}_{p,q}$ can be used to deform the product of A to obtain $m_{R^{-1}}(a\otimes a')=m_A(R^{-1}\rhd(a\otimes a'))$ for all a, $a'\in A$. The algebra $A_{R^{-1}}$ is also a G-graded algebra and has properties similar to those of A_R.

Proposition 2 Let $\langle A, B\rangle$ be a pair of multiplier Hopf algebra so that B is G-cograded. Let $R=\sum_{p,q\in G}R_{p,q}$ with $R_{p,q}\in M(A_p\otimes A_q)$ be a π-twisting element in the sense of definition 1, and set $A_R=(A, m_R)$. Then

1) A_R is again a left B-module algebra for action $B\rhd A$ associated to the pair $\langle A, B\rangle$ so that $b\rhd(a\cdot a')=m_R(\Delta_{\alpha,\beta}(b)\rhd(a\otimes a'))$ for all $a\in A_\alpha$, $a'\in A_\beta$ and $b\in A_{\alpha\beta}$;

2) If furthermore $R=\sum_{p,q\in G}R_{p,q}$ is a generalized R-matrix providing $B=\oplus_{p\in G}B_p$ with a π-quasitriangular structure, then we have ① A_R is also a right B^{cop}-module algebra so that $(a\cdot a')\lhd b=m_R((a\otimes a')\lhd(\tilde{\Delta}_{\alpha\beta\alpha^{-1},\alpha})^{\mathrm{cop}}(b))$ for all $a\in A_\alpha$, $a'\in A_\beta$ and $b\in A_{\alpha\beta}$; ② The non-commutativity in A_R is controlled as

$$m_R(a\otimes a')=m_R((\pi'_\alpha\otimes i)(\sigma_{\alpha,\beta}(R_{\alpha,\beta})\rhd(a'\otimes a)\lhd(\pi_{\alpha^{-1}}\otimes i)(R^{-1}_{\alpha\beta\alpha^{-1},\alpha}))) = m_R(R^{-1}_{\beta,\beta^{-1}\alpha\beta}\rhd(i\otimes\pi'_{\beta^{-1}})(a'\otimes a)\lhd(i\otimes\pi_{\beta^{-1}})(\sigma_{\alpha,\beta}R_{\alpha,\beta}))$$

where $\sigma_{\alpha,\beta}$ denotes the flip map on $B_\alpha\otimes B_\beta$, extended to $M(B_\alpha\otimes B_\beta)$.

2 π-Twisting Elements Based on the Drinfeld Double

Let $\langle A, B\rangle$ be a multiplier Hopf algebra pair so that B is G-cograded. Let π be an admissible action of G on B. Put the G-cograded Drinfeld double $D^\pi=\oplus_{\alpha\in G}D^\pi_\alpha$ with $D^\pi_\alpha=A^{\mathrm{cop}}\propto B_\alpha$. In general, we consider the situation when D^π is π-quasitriangular.

By Ref. [7], for any $p\in G$, we can define the linear map π'_p on A by the formula $\langle\pi'_p(a), b\rangle=\langle a, \pi_{p^{-1}}(b)\rangle$ for all $a\in A$ and $b\in B$. Clearly π'_p is a linear isomorphism such that $(\pi'_p)^{-1}=\pi'_{p^{-1}}$. From the definition of π'_p, it easily follows that π'_p is an algebraic isomorphism on A, $\Delta(\pi'_p(a))=(\pi'_p\otimes\pi'_p)\Delta(a)$ and $\pi'_{pq}=\pi'_p\pi'_q$ for all p, $q\in G$.

We start with a canonical multiplier W in $M(A\otimes B)$ for the pair $\langle A, B\rangle$ defined in Ref. [4]. Let $D^\pi=A^{\mathrm{cop}}\propto\tilde{B}$ denote the Drinfeld double of the pair $\langle A, B\rangle$ with B being G-cograded. We prove that the embedding of W in $M(D^\pi\otimes D^\pi)=\prod_{\alpha,\beta}M(D^\pi_\alpha\otimes D^\pi_\beta)$ is a generalized R-matrix for D^π.

Definition 2 Let $\langle A, B\rangle$ be a multiplier Hopf algebra pair so that B is G-cograded. An invertible multiplier $W=\sum_\alpha W_\alpha$ in $M(B\otimes A)$ with $W_\alpha\in M(B_\alpha\otimes A)$ is called a π-canonical multiplier if

1) $(\pi_p\otimes\pi'_p)(W)=W$ for all $p\in G$;

2) $\langle W_\alpha, b\otimes a\rangle=\langle a_\alpha, b_\alpha\rangle$, for all $a\in A$ and $b\in B$, where a_α is the α-th component of a.

Observe that we use the extension of the non-degenerate bilinear from $\langle B\otimes A, B\otimes A\rangle$ to $\langle M(B\otimes A), B\otimes A\rangle$. If there is a π-canonical multiplier in $M(B\otimes A)$, then it is unique.

Example 2 1) In the case that B is G-cograded with finite-dimensional components B_α for all $\alpha\in G$, consider the dual

G-graded Hopf algebra B' and the natural pair $\langle B', B\rangle$. The canonical multiplier in $M(B\otimes B')$ is given by $W=\sum_{\alpha,i} e_{\alpha i}\otimes f_{\alpha i}$, where $\{e_{\alpha i}\}$ and $\{f_{\alpha i}\}$ are dual bases of B_α and B'_α.

2) Consider the pair $\langle k(G), k[G]\rangle$, where $K[G]$ is the group algebra for group G and $K(G)$ is the multiplier Hopf algebra of functions with finite support on G. The canonical multiplier $W\in M(k[G]\otimes k(G))$ is given by $W=\sum_{g\in G} W_g=\sum_{g\in G} u_g\otimes\delta_g$.

Proposition 3 Take the notation as above. For an invertible multiplier $W\in M(B\otimes A)$ with $W_\alpha\in M(B_\alpha\otimes A)$ for all $\alpha\in G$, the following are equivalent:

1) $\langle W_\alpha, b\otimes a\rangle=\langle a_\alpha, b_\alpha\rangle$ for all $a\in A$ and $b\in B$;

2) $(\langle a,\cdot\rangle\otimes i_A)(W_p)=a_p$ for all $a\in A$;

3) $(i_{B_p}\otimes\langle\cdot, b\rangle)(W_p)=b_p$ for all $b\in B$.

Proof For all $a\in A$, $b\in B$, then there exist $a'\in A$ and $b'\in B$ such that $b=a'\triangleright b$ and $a=b'\triangleright a$, $\langle a, (i_{B_p}\otimes\langle\cdot, b\rangle)(W_p)\rangle=\langle a, (i_{B_p}\otimes\langle\cdot, b\rangle)(W_p))b'\rangle=\langle a, (i_{B_p}\otimes\langle\cdot, b\rangle)(W_p(b'\otimes a')\rangle=\langle W_p(b'\otimes a'), b\otimes a\rangle=\langle W_p, b\otimes a\rangle=\langle a_p, b_p\rangle=\langle a, b_p\rangle$.

By using this equation, the equivalence between 1) and 3) is clear. The one between 1) and 2) can be proven in a similar way.

Proposition 4 Let $W=\sum_\alpha W_\alpha\in M(B\otimes A)$ be a π-canonical multiplier for the pair $\langle A, B\rangle$ with $W_\alpha\in M(B_\alpha\otimes A)$ for all $\alpha\in G$. Then we obtain that W is a π-copairing in the following sense:

1) $(\pi_p\otimes\pi'_p)(W)=W$ for all $p\in G$.

2) $(\Delta_{B_{\alpha,\beta}}\otimes i_A)(W_{\alpha\beta})=W_\alpha^{13}W_\beta^{23}$ for all $\alpha, \beta\in G$ in $M(B_\alpha\otimes B_\beta\otimes A)$; $(i_{B_\alpha}\otimes\Delta_A)(W_\alpha)=W_\alpha^{12}W_\alpha^{13}$ for all $\alpha\in G$ in $M(B_\alpha\otimes A\otimes A)$.

3) $(i_{B_p}\otimes\varepsilon_{A_p})(W_p)=1_{B_p}$ in $M(B_p)$ for any $p\in G$; $(\varepsilon_{B_e}\otimes i_A)(W_e)=1_{A_e}$ in $M(A_e)$.

Proof 1) It is obvious; 2) For any $\alpha, \beta\in G$, we note that since $\Delta_{B_{\alpha,\beta}}\otimes i_A$ is a non-degenerate homomorphism on $B_{\alpha\beta}\otimes A$, it extends $B_{\alpha,\beta}\otimes A$ to $M(B_{\alpha\beta}\otimes A)$ in a natural way. For a multiplier $M\in M(B_\alpha\otimes B_\beta\otimes A)$ and $a_\alpha\in A_\alpha$, $a'_\beta\in A_\beta$, one can define the multiplier $(\langle a\otimes a',\cdot\rangle\otimes i_A)(M)\in M(A)$ in a similar way as introduced in Ref. [4].

Now, we compute

$$(\langle a_\alpha\otimes a'_\beta,\cdot\rangle\otimes i_A)((\Delta_{B_{\alpha,\beta}}\otimes i_A)(W_{\alpha\beta}))=(\langle a_\alpha a'_\beta,\cdot\rangle\otimes i_A)(W_{\alpha\beta})=a_\alpha a'_\beta$$

$$(\langle a_\alpha\otimes a'_\beta,\cdot\rangle\otimes i_A)(W_\alpha^{12}W_\beta^{13})=(\langle a_\alpha,\cdot\rangle\otimes i_A)W_\alpha(i_A\otimes\langle a'_\beta,\cdot\rangle)W_\beta=a_\alpha a'_\beta$$

Then $(\Delta_{B_{\alpha,\beta}}\otimes i_A)(W_{\alpha\beta})=W_\alpha^{13}W_\beta^{23}$ for all $\alpha, \beta\in G$ in $M(B_\alpha\otimes B_\beta\otimes A)$ follows.

For $(i_{B_\alpha}\otimes\Delta_A)(W_\alpha)=W_\alpha^{12}W_\alpha^{13}$ for all $\alpha\in G$ in $M(B_\alpha\otimes A\otimes A)$, $W_\alpha\in M(B_\alpha\otimes A)$, and then $(i_{B_\alpha}\otimes\Delta_A)(W_\alpha)\in M(B_\alpha\otimes A\otimes A)$. For all $b, b'\in B$,

$$(i_{B_\alpha}\otimes\langle\cdot, b\otimes b'\rangle)(i_{B_\alpha}\otimes\Delta_A)(W_\alpha)=(i_{B_\alpha}\otimes\langle\cdot, bb'\rangle)(W_\alpha)=(bb')_\alpha=b_\alpha b'_\alpha$$

$$(i_{B_\alpha}\otimes\langle\cdot, b\otimes b'\rangle)(W_\alpha^{12}W_\alpha^{13})=(i_{B_\alpha}\otimes\langle\cdot, b\rangle)(W_\alpha)(i_{B_\alpha}\otimes\langle\cdot, b'\rangle)(W_\alpha)=b_\alpha b'_\alpha$$

3) From 2) we obtain that $(\Delta_{B_{e,p}}\otimes i_A)(W_p)=W_e^{13}W_p^{23}$. If we apply the extension of the non-degenerate homomorphism $\varepsilon_{B_e}\otimes i_B\otimes i_A$ on both sides of this equation, we obtain that $(\varepsilon_{B_e}\otimes i_A)W_e=1_{A_e}$. The proof of the other counitary property is similar.

We note that $D^\pi=A^{\mathrm{cop}}\propto\tilde{B}$ with the commultiplication $\bar{\Delta}(a\propto b)=\Delta^{\mathrm{cop}}(a)\tilde{\Delta}(b)$. We now look how a π-canonical multiplier $W\in M(B\otimes A)$ with $W_\alpha\in M(B_\alpha\otimes A)$ for all $\alpha\in G$ relates to the product of D^π. Recall that the non-degenerate algebraic embeddings of A and B_p in $M(D_p^\pi)$ give rise to non-degenerate algebraic embeddings of $A\otimes A$ in $M(A\otimes D_p^\pi)$, of $B_p\otimes B_p$ in $M(D_p^\pi\otimes B_p)$ and of $B_p\otimes A$ in $M(D_p^\pi\otimes A)$, respectively, and $M(B_p\otimes D_p^\pi)$ for any $p\in G$. Now, we have the following proposition.

Proposition 5 Take the notations as above. Then we have that

1) $W_\alpha\Delta_A^{\mathrm{cop}}(a)=((i\otimes\pi'_\alpha)\Delta_A(a))W_\alpha$ in $M(D^\pi\otimes A)$;

2) $W\tilde{\Delta}_{qp^{-1},p}(b_q)=[(i\otimes\pi_{p^{-1}q})(\tilde{\Delta}_{p,p^{-1}q})^{\mathrm{cop}}(b)]W=(\Delta_{p,p^{-1}q})^{\mathrm{cop}}(b)W$ in $M(B\otimes D^\pi)$, for all $p, q\in G$, $a\in A$ and $b\in B_q$.

A sketch of the proof following Ref. [8], we only prove part 1). Part 2) can be verified in a similar way. For part 1), it suffices to show that in the $M(D^\pi)$,

$$(i_D\otimes\langle\cdot, b_p\rangle)(W_\alpha\Delta_A^{\mathrm{cop}}(a_q)((x\propto y)\otimes 1_{M(A)}))=(i_D\otimes\langle\cdot, b_p\rangle)((((i\otimes\pi'_\alpha)\Delta_A(a_q))W_\alpha((x\propto y)\otimes 1_{M(A)}))$$

for all $b_p\in B_p$, $\forall p\in G$, $a_q\in A_q$, $\forall q\in G$ and $x\in A$, $y\in B$.

Theorem 1 Let $\langle A, B\rangle$ be a pair of multiplier Hopf algebras and assume that B is a regular G-cograded multiplier Hopf algebra. Let π be a crossing action of G on B. $D^\pi=A^{\mathrm{cop}}\propto\tilde{B}$ is the Drinfeld double [7], then the deformation $\tilde{D}^\pi$ becomes a multiplier Hopf algebra, with the multiplication, the coproduct, the counit and the antipode in the following way:

● $(a\propto b)(a'\propto b')=(m_A\otimes m_B)(i_A\otimes T\otimes i_B)(a\otimes b\otimes a'\otimes b')$, where $T(b\otimes a')=\sum(\pi_{p^{-1}}(b_{(1)})\triangleright a'\triangleleft S^{-1}(b_{(3)}))\otimes b_{(2)}$

for all $a' \in A$ and $b \in B_p$;

- $\tilde{\bar{\Delta}}_{\alpha,\beta}(a \propto b_{\alpha\beta}) = ((\pi'_\beta \otimes \pi_\beta) \otimes i_D)\bar{\Delta}_{\alpha,\beta}(a \propto b_{\alpha\beta}) := \tilde{\Delta}^{cop}(a)\Delta_{\alpha,\beta}(b_{\alpha\beta})$ for all $a \in A$ and $b_{\alpha\beta} \in B_{\alpha\beta}$;
- $\tilde{\bar{\varepsilon}} = \bar{\varepsilon}$;
- $\tilde{\bar{S}}(a \propto b_p) = T(S(b) \otimes \pi'_p(S^{-1}(a)))$ for all $a \in A$ and $b_p \in B_p$.

Proof First we check the coassociativity of the new coproduct: $(\tilde{\bar{\Delta}}_{\alpha,\beta} \otimes i)\tilde{\bar{\Delta}}_{\alpha\beta,\gamma} = (i \otimes \tilde{\bar{\Delta}}_{\beta,\gamma})\tilde{\bar{\Delta}}_{\alpha,\beta\gamma}$. Indeed,

$$(\tilde{\bar{\Delta}}_{\alpha,\beta} \otimes i)\tilde{\bar{\Delta}}_{\alpha\beta,\gamma}(a \propto b_{\alpha\beta\gamma}) = (\tilde{\bar{\Delta}}_{\alpha,\beta} \otimes i)((\pi_\gamma \otimes i)\Delta^{cop}(a)\Delta(b)) = (((\pi_\beta \otimes i)\Delta^{cop})\pi_\gamma \otimes i)\Delta^{cop}(a)(\Delta \otimes i)\Delta(b) = (\pi_{\beta\gamma} \otimes i \otimes \pi_\gamma \otimes i \otimes i \otimes i)((\Delta^{cop} \otimes i)\Delta^{cop}(a)(\Delta \otimes i)\Delta(b))$$

$$(i \otimes \tilde{\bar{\Delta}}_{\beta,\gamma})\tilde{\bar{\Delta}}_{\alpha,\beta\gamma}(a \propto b_{\alpha\beta\gamma}) = (i \otimes \tilde{\bar{\Delta}}_{\beta,\gamma})((\pi_{\beta\gamma} \otimes i)\Delta^{cop}(a)\Delta(b)) = (\pi_{\beta\gamma} \otimes (\pi_\gamma \otimes i)\Delta^{cop})\Delta^{cop}(a)(i \otimes \Delta)\Delta(b) = (\pi_{\beta\gamma} \otimes i \otimes \pi_\gamma \otimes i \otimes i \otimes i)((i \otimes \Delta^{cop})\Delta^{cop}(a)(i \otimes \Delta)\Delta(b))$$

It is easy to obtain $(\tilde{\bar{\varepsilon}} \otimes i)\tilde{\bar{\Delta}}_{e,\alpha}(a \propto b_\alpha) = a \propto b_\alpha$ and $(i \otimes \tilde{\bar{\varepsilon}})\tilde{\bar{\Delta}}_{\alpha,e}(a \propto b_\alpha) = a \propto b_\alpha$.

The next is to rectify the antipode.

$$\tilde{\bar{S}}(a \propto b_p) = T(S(b) \otimes \pi'_p(S^{-1}(a))) = \langle \pi'_p(S^{-1}(a))_{(1)},\ S^{-1}(S(b)_{(3)}) \rangle \langle \pi'_p(S^{-1}(a))_{(3)},\ \pi_p(S(b)_{(1)}) \rangle \cdot (\pi'_p(S^{-1}(a))_{(2)} \propto S(b)_{(2)}) = \langle \pi'_p(S^{-1}(a_{(3)})),\ S^{-1}(S(b_{(1)})) \rangle \langle \pi'_p(S^{-1}(a_{(1)})),\ \pi_p(S(b_{(3)})) \rangle \cdot (\pi'_p(S^{-1}(a_{(2)})) \propto S(b_{(2)})) = \langle \pi'_p(S^{-1}(a_{(3)})),\ b_{(1)} \rangle \langle a_{(1)},\ b_{(3)} \rangle (\pi'_p(S^{-1}(a_{(2)})) \propto S(b_{(2)}))$$

It is not difficult to show that $\tilde{\bar{S}}$ defined as above is an antimorphism.

Finally, we can obtain that $m(\tilde{\bar{S}} \otimes i)\tilde{\bar{\Delta}}_{p,p^{-1}}(a \propto b_e) = \varepsilon(a)\varepsilon(b_e)(1_{M(A)} \propto 1_{M(B_p^{-1})})$ and $m(i \otimes \tilde{\bar{S}})\tilde{\bar{\Delta}}_{p,p^{-1}}(a \propto b_e) = \varepsilon(a)\varepsilon(b_e) \cdot (1_{M(A)} \propto 1_{M(B_p)})$.

Recall that $B \otimes A$ can be considered as a subalgebra of $M(D^\pi \otimes D^\pi)$. The elements $b \otimes a$ are now denoted as $(1 \propto b) \otimes (a \propto 1)$. In the theorem below we consider the π-canonical multiplier $W \in M(B \otimes A)$ as a multiplier in $M(D^\pi \otimes D^\pi)$ and we prove that W provides a π-quasitriangular structure. A similar conclusion is also given briefly in Ref. [6], but we do not need the additional assumption in this paper.

Theorem 2 Let $W = \sum_\alpha W_\alpha \in M(B \otimes A)$ be a π-canonical multiplier for the pair $\langle A, B \rangle$ with $W_\alpha \in M(B_\alpha \otimes A)$ for all $\alpha \in G$. Let $D^\pi = A^{cop} \propto \tilde{B} = \oplus_p D_p^\pi$ with $D_p^\pi = A^{cop} \propto \tilde{B}_p$ be the Drinfeld double of the pair $\langle A, B \rangle$. The image of W in $M(D^\pi \otimes D^\pi)$ is a π-quasitriangular structure over D^π.

Proof We still denote the image of W in $M(D^\pi \otimes D^\pi)$ by the symbol W. Since W is invertible in $M(B \otimes A)$, it is also invertible in $M(D^\pi \otimes D^\pi)$. For all $a \in A$, $b' \in B$,

1) Let $\eta_p = \pi'_p \otimes \pi_p$. Then it is obvious that $(\eta_p \otimes \eta_p)(W) = W$.

2) Recall that the coproduct $\bar{\Delta}_D$ is given by the formula $\bar{\Delta}_D(a \propto b) = \Delta_A^{cop}(a)\tilde{\Delta}_B(b)$ in $M(D^\pi \otimes D^\pi)$. Hence, we obtain in M$(D^\pi \otimes D^\pi \otimes D^\pi)$ that

$$(\tilde{\bar{\Delta}}_{\alpha,\beta} \otimes i_D)((1 \propto b_{\alpha\beta}) \otimes (a \propto 1)) = \tilde{\Delta}_A^{cop}(1)\Delta_{\alpha,\beta}(b_{\alpha\beta}) \otimes (a \propto 1) = (\Delta_{\alpha,\beta} \otimes i)(b_{\alpha\beta} \otimes a)$$

$$(i_D \otimes \bar{\Delta}_D)((1 \propto b) \otimes (a \propto 1)) = (1 \propto b) \otimes \Delta_A^{cop}(a)\tilde{\Delta}_B(1) = (i_B \otimes \Delta_A^{cop})(b \otimes a)$$

From the above two formulae we have

$$(\tilde{\bar{\Delta}}_{\alpha,\beta} \otimes i_D)(W_{\alpha\beta}) = (\Delta_{\alpha,\beta} \otimes i_A)(W_{\alpha\beta}) = W_\alpha^{13}\ W_\beta^{23}, \qquad (i_D \otimes \bar{\Delta}_D)(W_\alpha) = (i_B \otimes \Delta_A^{cop})(W) = W_\alpha^{13}\ W_\alpha^{12}$$

3) We have that, for any $a \in A$ and $b \in B$,

$$W[\bar{\Delta}_{\alpha,\beta}(a \propto b_{\alpha\beta})] = W\Delta_A^{cop}(a)\tilde{\Delta}_{\alpha,\beta}(b_{\alpha\beta}) = \sum_\gamma W_\gamma \Delta_A^{cop}(a)\tilde{\Delta}_{\alpha,\beta}(b_{\alpha\beta}) = \sum_\gamma ((i \otimes \pi'_\gamma)\Delta_A(a))W_\gamma \tilde{\Delta}_{\alpha,\beta}(b_{\alpha\beta}) = ((i \otimes \pi'_{\beta^{-1}\alpha\beta})\Delta_A(a)(i \otimes \pi_{\beta^{-1}\alpha\beta})(\tilde{\Delta}_{\beta,\beta^{-1}\alpha\beta})^{cop}(b))W_{\beta^{-1}\alpha\beta} = (i \otimes (\pi'_{\beta^{-1}\alpha\beta} \otimes \pi_{\beta^{-1}\alpha\beta}))(\Delta_A(a)(\tilde{\Delta}_{\beta,\beta^{-1}\alpha\beta})^{cop}(b))W_{\beta^{-1}\alpha\beta} = (i \otimes (\pi'_{\beta^{-1}\alpha\beta} \otimes \pi_{\beta^{-1}\alpha\beta}))(\bar{\Delta}_{\beta,\beta^{-1}\alpha\beta})^{cop}(a \propto b)\Big)W = ([((\pi'_{\beta^{-1}\alpha\beta} \otimes \pi_{\beta^{-1}\alpha\beta}) \otimes i)(\bar{\Delta}_{\beta,\beta^{-1}\alpha\beta})]^{cop}(a \propto b))W = ((\tilde{\bar{\Delta}}_{\beta,\beta^{-1}\alpha\beta})^{cop}(a \propto b))W$$

Example 3 Let $A = \oplus_{p \in G} A_P$ be a finite type Hopf group-coalgebra, the reduced dual multiplier Hopf algebra which is given by the Hopf algebra $A^* = \oplus_{p \in G}(A_P)'$. Let $\{f_{pi}\} \subset (A_P)'$ and $\{e_{pi}\} \subset A_P$ be dual basis. $W = \sum_{\alpha,i} e_{\alpha i} \otimes f_{\alpha i}$ is a π-canonical multiplier for the pair $\langle A^*, A \rangle$. Let $D^\pi = (A^*)^{cop} \propto \tilde{A} = \oplus_p D_p^\pi$ with $D_p^\pi = (A^*)^{cop} \propto \tilde{A}_p$ be the Drinfeld double of the pair $\langle A^*, A \rangle$. The image of W in $M(D^\pi \otimes D^\pi)$ is a π-quasitriangular structure over D^π [6].

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π-拟三角群余分次乘子 Hopf 代数

杨　涛　　王栓宏

（东南大学数学系，南京 211189）

摘要：设 G 是一个群，$\langle A, B\rangle$ 是乘子 Hopf 代数对，其中 B 为正则的 G-余分次乘子 Hopf 代数．设 π 是群 G 在 B 上的交叉作用，$D^{\pi}=A^{\mathrm{cop}}\propto\widetilde{B}=\oplus_{p\in G}D_{p}^{\pi}$，$D_{p}^{\pi}=A^{\mathrm{cop}}\propto\widetilde{B}_{p}$，是关于乘子 Hopf 代数对 $\langle A, B\rangle$ 的 Drinfeld 偶，则 Drinfeld 偶 D^{π} 的变形 $\widetilde{D}^{\pi}$ 也是乘子 Hopf 代数．$B\otimes A$ 可以看作是 $M(D^{\pi}\otimes D^{\pi})$ 的子代数，$B\otimes A$ 中的元素 $b\otimes a$ 在 $M(D^{\pi}\otimes D^{\pi})$ 中的像是 $(1\propto b)\otimes(a\propto 1)$．设 $W=\sum_{\alpha}W_{\alpha}\in M(B\otimes A)$ 是一个关于乘子 Hopf 代数对 $\langle A, B\rangle$ 的 π-典范乘子，其中对任意的 $\alpha\in G$，$W_{\alpha}\in M(B_{\alpha}\otimes A)$，则 W 在 $M(D^{\pi}\otimes D^{\pi})$ 中的像是 D^{π} 上的一个 π-拟三角结构．

关键词：乘子 Hopf 代数；群余分次；Drinfeld 偶；拟三角

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