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Real edge spans of distance two labelings of graphs

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Abstract: An $L(j, k)$-labeling of a graph G is an assignment of nonnegative integers to the vertices of G such that adjacent vertices receive integers which are at least j apart, and vertices at distance two receive integers which are at least k apart. Given an $L(j, k)$-labeling f of G, define the $L(j, k)$ edge span of f, $\beta_{j,k}(G,f) = \max\{|f(x)-f(y)|: \{x, y\} \in E(G)\}$. The $L(j, k)$ edge span of G, $\beta_{j,k}(G)$ is $\min \beta_{j,k}(G,f)$, where the minimum runs over all $L(j, k)$-labelings f of G. The real $L(j, k)$-labeling of a graph G is a generalization of the $L(j, k)$-labeling. It is an assignment of nonnegative real numbers to the vertices of G satisfying the same distance one and distance two conditions. The real $L(j, k)$ edge span of a graph G is defined accordingly, and is denoted by $\hat{\beta}_{j,k}(G)$. This paper investigates some properties of the $L(j, k)$ edge span and the real $L(j, k)$ edge span of graphs, and completely determines the edge spans of cycles and complete t-partite graphs.

Key words: $L(j, k)$-labeling; real $L(j, k)$-labeling; $L(j, k)$ edge span; real $L(j, k)$ edge span; frequency assignment

The $L(2, 1)$-labeling, first introduced by Griggs and Yeh[1], arose from a variation of the frequency assignment problem proposed by Hale[2]. For a given graph G, an $L(2, 1)$-labeling of G is defined as a function f from $V(G)$ to nonnegative integers such that $|f(u)-f(v)| \geqslant 2$ if u and v are adjacent, and $|f(u)-f(v)| \geqslant 1$ if u and v are of distance two. The span of f is the difference between the largest and the smallest labels assigned by f. The minimum span over all $L(2, 1)$-labelings of G, denoted by $\lambda(G)$, is called the $L(2, 1)$-labeling number of G. Any $L(2, 1)$-labeling of G with a span $\lambda(G)$ is called a λ-labeling of G.

Georges and Mauro[3] generalized the $L(2, 1)$-labeling problem. For any graph G and nonnegative integers j and k with $j \geqslant k$, a nonnegative integer assignment L to the vertices of G is called an $L(j, k)$-labeling if and only if

1) $|L(v)-L(w)| \geqslant j$ if v and w are adjacent;
2) $|L(v)-L(w)| \geqslant k$ if v and w are of distance two.

As before, the span of L is the difference between its largest and the smallest assigned labels. The minimum span over all $L(j, k)$-labelings of G, denoted by $\lambda_{j,k}(G)$, is called the $L(j, k)$-labeling number of G. Any $L(j, k)$-labeling of G with a span $\lambda_{j,k}(G)$ is called a $\lambda_{j,k}$-labeling of G.

In investigating the $L(j, k)$-labeling and $L(j, k)$ edge span of graphs, we usually suppose that the minimum labels of any $L(j, k)$-labelings of a graph G are zero. In this way, the $L(j, k)$-labeling number of a graph G is namely the maximum label of some $\lambda_{j,k}$-labeling of G.

Motivated by the frequency assignment problem, the $L(2, 1)$-labeling numbers of graphs have been studied extensively in the past decade[1, 4–6]. And, there are some papers about $L(d, 1)$-labeling numbers of graphs[7]. $L(j, k)$-labelings for $j \geqslant k$ were also investigated in Refs. [3, 8]. Recently, people have begun to study $L(j, k)$-labelings of graphs for $j \leqslant k$[9].

Suppose that f is an $L(j, k)$-labeling of a graph G. The $L(j, k)$ edge span of f, denoted by $\beta_{j,k}(G,f)$, equals the value $\max\{|f(x)-f(y)|: \{x, y\} \in E(G)\}$. The $L(j, k)$ edge span of a graph G, denoted by $\beta_{j,k}(G)$, is defined as $\min \beta_{j,k}(G,f)$, where the minimum runs over all $L(j, k)$-labelings f of G. Note that an $L(j, k)$-labeling of a graph G with the minimum edge span may not be a $\lambda_{j,k}$-labeling of G and a $\lambda_{j,k}$-labeling of G may not have the minimum edge span. For example, let $j > k$ and $v_1 v_2 v_3$ be a path P of length 2. The labeling of P assigning the labels $0, j, 2j$ to v_1, v_2 and v_3, respectively, obviously has the minimum $L(j, k)$ edge span j. But it is not a $\lambda_{j,k}$-labeling of P. The labeling of P assigning $j, 0, j+k$ to v_1, v_2 and v_3, respectively, is a $\lambda_{j,k}$-labeling of P. However, its edge span is greater than j.

It is obvious that $\beta_{j,k}(G) \leqslant \lambda_{j,k}(G)$ for any graph G. If G is a complete graph, then $\beta_{j,k}(G) = \lambda_{j,k}(G)$. However, for many graphs, the $L(j, k)$ edge spans might be far less than the $L(j, k)$-labeling numbers.

The $L(2, 1)$ edge span of a graph was first introduced by Yeh[10]. The author determined the $L(2, 1)$ edge spans of cycles, trees, complete t-partite graphs, triangular lattices and square lattices. Feng and Song[11] investigated the $L(d, 1)$ edge span of some graphs. Niu and Lin[12] gave $L(j, k)$ edge spans of trees and the Cartesian product of two paths.

The above mentioned $L(j, k)$-labelings are integral. In recent years, Griggs and Jin investigated real $L(j, k)$-labelings of some graphs[13–14]. A real $L(j, k)$-labeling of a graph G, where j and k are nonnegative real numbers, is a function f from $V(G)$ to nonnegative real numbers satisfying the following two conditions:

1) $|f(v)-f(w)| \geqslant j$ if v and w are adjacent;
2) $|f(v)-f(w)| \geqslant k$ if v and w are of distance two.

The span of f is the difference between its largest and smallest assigned labels. The minimum span over all real $L(j, k)$-labelings of G, denoted by $\hat{\lambda}_{j,k}(G)$, is called the real $L(j, k)$-labeling number of G. If the span of f is $\hat{\lambda}_{j,k}(G)$, then we say that f is a $\hat{\lambda}_{j,k}$-labeling of G. The real $L(j, k)$ edge span of f, denoted by $\hat{\beta}_{j,k}(G,f)$, equals the value $\max\{|f(x)-f(y)|: \{x, y\} \in E(G)\}$. The real $L(j, k)$ edge span of G, denoted by $\hat{\beta}_{j,k}(G)$, is $\min \hat{\beta}_{j,k}(G,f)$, where the minimum runs over all real $L(j, k)$-labelings f of G.

In general, we also suppose that the minimum labels of

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any real $L(j,k)$-labelings of a graph G are zero. If j and k are nonnegative integers, then $\hat{\lambda}_{j,k}(G) \leqslant \lambda_{j,k}(G)$ and $\hat{\beta}_{j,k}(G) \leqslant \beta_{j,k}(G)$ since any $L(j,k)$-labeling is also a real $L(j,k)$-labeling.

The following are some known results about edge spans of $L(j,k)$-labelings.

Theorem 1[10] Let C_n be a cycle of order n. Then $\beta_{2,1}(C_n)=3$ for $n \geqslant 4$ and $\beta_{2,1}(C_3)=4$.

Theorem 2[10] Let $K=K_{n_1,n_2,\ldots,n_t}$ be a complete t-partite graph, where $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_t$. Then

$$\beta_{2,1}(K)=\left\lceil \frac{n_1}{2} \right\rceil + n_2 + n_3 + \ldots + n_t + t - 2$$

Theorem 3[12] Let T be a tree with the maximum degree $\Delta \geqslant 2$. Let j and k be two nonnegative integers. Then

$$\beta_{j,k}(T)=\begin{cases} j+\left(\left\lceil \frac{\Delta}{2} \right\rceil -1\right)k & 2j \geqslant k \text{ or } 2j<k \text{ and } \Delta \text{ is odd} \\ \left\lceil \frac{\Delta-1}{2}k \right\rceil & 2j<k \text{ and } \Delta \text{ is even} \end{cases}$$

Theorem 4[12] Let m, n, j and k be nonnegative integers.

1) If $m, n \geqslant 3$, then

$$\beta_{j,k}(P_m \square P_n)=\begin{cases} j+k & 2j \geqslant k \\ \left\lceil \frac{3k}{2} \right\rceil & 2j<k \end{cases}$$

2) $\beta_{j,k}(P_2 \square P_n)=j+k$ for $n \geqslant 2$.

In this paper, integral $L(j,k)$-labeling and integral $L(j,k)$ edge span are namely called as $L(j,k)$-labeling and $L(j,k)$ edge span, respectively. Without falling into confusion, we sometimes say edge span instead of $L(j,k)$ edge span or real $L(j,k)$ edge span for short. This paper investigates some properties of the $L(j,k)$ edge span and the real $L(j,k)$ edge span of graphs, and completely determines the $L(j,k)$ edge spans and real $L(j,k)$ edge spans of cycles for $j \geqslant k$ and $j<k$, and of the complete t-partite graphs for $j \geqslant k$.

1 General Properties

It is obvious that the restriction of a real $L(j,k)$-labeling of G on an induced subgraph H of G is also a real $L(j,k)$-labeling of H. Furthermore, if $j \geqslant k$, then the restriction of a real $L(j,k)$-labeling of G on any subgraph H of G is also a real $L(j,k)$-labeling of H. These properties also hold for $L(j,k)$-labelings. However, these are not true if j is less than k and H is not an induced subgraph of G. Thus, we have the following two theorems.

Theorem 5 1) Let j and k be two nonnegative real numbers. Let H be a subgraph of G. If H is an induced subgraph of G or $j \geqslant k$, then

$$\hat{\beta}_{j,k}(H) \leqslant \min\{\hat{\beta}_{j,k}(G),\ \hat{\lambda}_{j,k}(H)\}$$

$$\hat{\lambda}_{j,k}(G) \geqslant \max\{\hat{\beta}_{j,k}(G),\ \hat{\lambda}_{j,k}(H)\}$$

2) Let j and k be two nonnegative integers. Let H be a subgraph of G. If H is an induced subgraph of G or $j \geqslant k$, then

$$\beta_{j,k}(H) \leqslant \min\{\beta_{j,k}(G),\ \lambda_{j,k}(H)\}$$

$$\lambda_{j,k}(G) \geqslant \max\{\beta_{j,k}(G),\ \lambda_{j,k}(H)\}$$

Theorem 6 If $j \geqslant k$, then

1) $\hat{\beta}_{j,k}(G) \geqslant \max\{\hat{\beta}_{j,k}(H) \mid H \text{ is a subgraph of } G\}$;

2) $\beta_{j,k}(G) \geqslant \max\{\beta_{j,k}(H) \mid H \text{ is a subgraph of } G\}$.

Theorem 7 For any two nonnegative integers j and k, $\beta_{j,k}(G)=\lceil \hat{\beta}_{j,k}(G) \rceil$.

Proof Suppose that f is a real $L(j,k)$-labeling of G and $\hat{\beta}_{j,k}(G)=\hat{\beta}_{j,k}(G,f)$. Define $f'(v)=\lfloor f(v) \rfloor$ for all $v \in V(G)$. For any two vertices u and v of G, if $f(u)-f(v) \geqslant j$, then $f'(u)-f'(v)=\lfloor f(u) \rfloor - \lfloor f(v) \rfloor \geqslant \lfloor f(u)-f(v) \rfloor \geqslant j$. Similarly, if $f(u)-f(v) \geqslant k$, then $f'(u)-f'(v) \geqslant k$. So f' is an $L(j,k)$-labeling of G. For any $\{u,v\} \in E(G)$, $|f'(u)-f'(v)| = \left| \lfloor f(u) \rfloor - \lfloor f(v) \rfloor \right| \leqslant \lceil |f(u)-f(v)| \rceil \leqslant \lceil \hat{\beta}_{j,k}(G) \rceil$. So, $\beta_{j,k}(G) \leqslant \lceil \hat{\beta}_{j,k}(G) \rceil$. On the other hand, $\beta_{j,k}(G) \geqslant \hat{\beta}_{j,k}(G)$ since any integral $L(j,k)$-labeling is also a real $L(j,k)$-labeling. Note that $\beta_{j,k}(G)$ is an integer. Thus $\beta_{j,k}(G)=\lceil \hat{\beta}_{j,k}(G) \rceil$.

For nonnegative integers j and k, if $\hat{\beta}_{j,k}(G)$ is determined, then $\beta_{j,k}(G)$ is also known. But the converse may not be true. And so, it is more meaningful to determine $\hat{\beta}_{j,k}(G)$ than $\beta_{j,k}(G)$.

By a slight modification of the proof of theorems 3 and 4 in Ref. [12], we can get the proof of the following two theorems. And theorems 3 and 4 can be regarded as the corollary of theorems 8 and 9.

Theorem 8 Let T be a tree with the maximum degree $\Delta \geqslant 2$. Let j and k be two nonnegative real numbers. Then

$$\hat{\beta}_{j,k}(T)=\begin{cases} j+\left(\left\lceil \frac{\Delta}{2} \right\rceil -1\right)k & 2j \geqslant k \text{ or } 2j<k \text{ and } \Delta \text{ is odd} \\ \frac{\Delta-1}{2}k & 2j<k \text{ and } \Delta \text{ is even} \end{cases}$$

Theorem 9 Let j and k be two nonnegative real numbers.

1) If $m, n \geqslant 3$, then

$$\hat{\beta}_{j,k}(P_m \square P_n)=\begin{cases} j+k & 2j \geqslant k \\ \frac{3k}{2} & 2j<k \end{cases}$$

2) $\hat{\beta}_{j,k}(P_2 \square P_n)=j+k$ for $n \geqslant 2$.

Theorem 10 Suppose that G is a graph with maximum degree $\Delta \geqslant 1$. Let j, k, j', k' be nonnegative real numbers.

① If $j \geqslant j'$ and $k \geqslant k'$, then $\hat{\beta}_{j,k}(G) \geqslant \hat{\beta}_{j',k'}(G)$;

② If $j \geqslant k$, then $\hat{\beta}_{j,k}(G) \geqslant j+\left(\left\lceil \frac{\Delta}{2} \right\rceil -1\right)k$.

Proof If $j \geqslant j'$ and $k \geqslant k'$, then a real $L(j,k)$-labeling of G is also a real $L(j',k')$-labeling of G. Thus ① holds.

Any graph G with the maximum degree Δ contains a subgraph $K_{1,\Delta}$. So we obtain $\hat{\beta}_{j,k}(G) \geqslant j+\left(\left\lceil \frac{\Delta}{2} \right\rceil -1\right)k$ by theorem 6 and theorem 8.

For an $L(j,k)$ edge span of a graph G, we can directly obtain the similar results of theorem 10 by theorem 7

Theorem 10 indicates that, the greater the restrictions of distance conditions, the larger the edge spans of a graph G. It also gives a general lower bound about real $L(j,k)$ edge span of a graph G in the case $j \geqslant k$. However, theorem 10 ② may not be true in the case $j < k$. For instance, the real $L(j,k)$ edge span of cycle C_3 is $2j$. If $j < k$, then $j + \left(\lceil \frac{\Delta}{2} \rceil - 1\right)k = j + k > 2j$ and it is not a lower bound.

What graphs on earth have the edge spans that obtain the lower bound of theorem 10 ②? It is a very interesting problem to characterize all graphs with their real $L(j,k)$ edge spans which are just the lower bound.

Theorem 11 ① For nonnegative real numbers j, k and positive real numbers c, $\hat{\beta}_{j,k}(G) = \hat{\beta}_{cj,ck}(G)/c$;

② For nonnegative integers j, k and positive integers c, $\beta_{j,k}(G) = \lceil \frac{\beta_{cj,ck}(G)}{c} \rceil$.

Proof 1) Suppose that f is a real $L(j,k)$-labeling of G and $\hat{\beta}_{j,k}(G) = \hat{\beta}_{j,k}(G,f)$. Define $f_1(v) = cf(v)$ for all $v \in V(G)$. It is evident that f_1 is a real $L(cj,ck)$-labeling of G. So $\hat{\beta}_{j,k}(G) = \hat{\beta}_{j,k}(G,f) = \frac{1}{c}\hat{\beta}_{cj,ck}(G,f_1) \geqslant \frac{1}{c}\hat{\beta}_{cj,ck}(G)$.

On the other hand, suppose that f is a real $L(cj,ck)$-labeling of G and $\hat{\beta}_{cj,ck}(G) = \hat{\beta}_{cj,ck}(G,f)$. Define $f_2(v) = f(v)/c$ for all $v \in V(G)$. It follows that f_2 is a real $L(j,k)$-labeling of G. So $\hat{\beta}_{j,k}(G) \leqslant \hat{\beta}_{j,k}(G,f_2) = \frac{\hat{\beta}_{cj,ck}(G,f)}{c} = \frac{\hat{\beta}_{cj,ck}(G)}{c}$. Thus, $\hat{\beta}_{j,k}(G) = \frac{\hat{\beta}_{cj,ck}(G)}{c}$.

2) Similar to 1), we can prove that $\beta_{j,k}(G) \geqslant \frac{1}{c}\beta_{cj,ck}(G)$.

Now suppose that f is an $L(cj,ck)$-labeling of G and $\beta_{cj,ck}(G) = \beta_{cj,ck}(G,f)$. Define $f'(v) = \lfloor \frac{f(v)}{c} \rfloor$ for all $v \in V(G)$. For any two vertices $u, v \in V(G)$, if $f(u) - f(v) \geqslant cj$, then $f'(u) - f'(v) = \lfloor \frac{f(u)}{c} \rfloor - \lfloor \frac{f(v)}{c} \rfloor \geqslant \lfloor \frac{f(u)-f(v)}{c} \rfloor \geqslant j$. Similarly, if $f(u) - f(v) \geqslant ck$, then $f'(u) - f'(v) \geqslant k$. So f' is an $L(j,k)$-labeling of G. For any $\{u,v\} \in E(G)$, $|f'(u) - f'(v)| = \left| \lfloor \frac{f(u)}{c} \rfloor - \lfloor \frac{f(v)}{c} \rfloor \right| \leqslant \lceil \left| \frac{f(u)-f(v)}{c} \right| \rceil \leqslant \lceil \frac{\beta_{cj,ck}(G)}{c} \rceil$. So $\beta_{j,k}(G) \leqslant \lceil \frac{\beta_{cj,ck}(G)}{c} \rceil$.

Hence, $\frac{\beta_{cj,ck}(G)}{c} \leqslant \beta_{j,k}(G) \leqslant \lceil \frac{\beta_{cj,ck}(G)}{c} \rceil$. Notice that $\beta_{j,k}(G)$ is an integer. It follows that $\beta_{j,k}(G) = \lceil \frac{\beta_{cj,ck}(G)}{c} \rceil$.

The following corollary follows from theorem 10 ① and theorem 11 ①.

Corollary 1 For real numbers j, k, j' and k',

1) If $\frac{j}{k} \geqslant \frac{j'}{k'}$, then $k'\hat{\beta}_{j,k}(G) \geqslant k\hat{\beta}_{j',k'}(G)$;

2) If $\frac{j}{k} \leqslant \frac{j'}{k'}$, then $j'\hat{\beta}_{j,k}(G) \geqslant j\hat{\beta}_{j',k'}(G)$;

3) If $j \geqslant 2k$, then $\hat{\beta}_{j,k}(G) \geqslant k\hat{\beta}_{2,1}(G)$;

4) If $j \leqslant 2k$, then $\hat{\beta}_{j,k}(G) \geqslant \frac{j}{2}\hat{\beta}_{2,1}(G)$.

Let j and k be two nonnegative real numbers. If k is positive, then, by theorem 11, we obtain $\hat{\beta}_{j,k}(G) = k\hat{\beta}_{x,1}(G)$ where $x = j/k$. Therefore, in this case, it suffices to determine $\hat{\beta}_{x,1}(G)$ for any nonnegative real number x.

2 Cycles

Let $C_n = v_1 v_2 \ldots v_n$ be a cycle on n vertices ($n \geqslant 3$). $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_n, v_1\}$ are n edges of C_n. In this section, we consider real $L(j,k)$ edge spans and $L(j,k)$ edge spans of C_n. Note that if one of the two numbers j and k is 0, then it is trivial to obtain $\hat{\beta}_{j,0}(C_n)$ and $\hat{\beta}_{0,k}(C_n)$. In fact,

$$\beta_{j,0}(C_n) = \begin{cases} 2j & n \text{ is odd} \\ j & n \text{ is even} \end{cases}$$

$$\beta_{0,k}(C_n) = \begin{cases} 0 & n = 3 \\ k & n \text{ is the multiple of } 4 \\ 2k & \text{others} \end{cases}$$

So we only need to consider the case that j and k are positive. In the following part of this paper, by investigating the $L(x,1)$ edge spans of C_n where x is a positive real number, we finally obtain the real $L(j,k)$ edge spans and $L(j,k)$ edge spans of C_n.

Proposition 1 $\hat{\beta}_{x,1}(C_n) \geqslant x + 1$.

Proof Let f be any real $L(x,1)$-labeling of C_n. Without loss of generality, suppose $f(v_1) = 0$. Then, due to the distance conditions, it is clear that $\max\{f(v_2), f(v_n)\} \geqslant x + 1$. Thus $\hat{\beta}_{j,k}(C_n) \geqslant x + 1$.

Proposition 2 For even number n, $\hat{\beta}_{x,1}(C_n) = x + 1$.

Proof If $x \geqslant 1/2$, it is straightforward to check that the labeling f defined below is a real $L(x,1)$-labeling of C_n with edge span $x + 1$.

$$f(v_i) = \begin{cases} (i-1)x & i = 1, 2, \ldots, \frac{n}{2} \\ (n-i+1)x + 1 & i = \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n \end{cases}$$

If $x \leqslant \frac{1}{2}$, define

$$f'(v_i) = \begin{cases} (i-1)x & i = 1, 2 \\ i - 2 & i = 3, 4, \ldots, \frac{n}{2} \\ \left(i - \frac{n}{2}\right)x + \frac{n}{2} - 1 & i = \frac{n}{2}+1, \frac{n}{2}+2 \\ x + n + 1 - i & i = \frac{n}{2}+3, \frac{n}{2}+4, \ldots, n \end{cases}$$

We can verify that f' is a real $L(j,k)$-labeling of C_n with edge span $x + 1$. Thus $\hat{\beta}_{x,1}(C_n) \leqslant x + 1$.

By proposition 1, $\hat{\beta}_{x,1}(C_n) = x + 1$ for the even number n.

Note that $\hat{\beta}_{x,1}(C_3) = 2x$ for any positive real number x. We next consider the edge spans of odd cycles C_{2t+1} for $t \geqslant 2$.

Proposition 3 If $x > 1$ and $t \geqslant \lceil x \rceil$ or if $0 < x \leqslant 1$ and $t \geqslant 3$, then $\hat{\beta}_{x,1}(C_{2t+1}) = x + 1$.

Proof By proposition 1, we only need to give the real $L(x,1)$-labelings of cycles with edge span $x + 1$.

Case 1 $x > 1$ and $t \geqslant \lceil x \rceil$.

1) $x>2$. If $t\geqslant\lfloor x\rfloor+1$, define

$$f_1(v_i)=\begin{cases}(i-1)x & i=1,2,\dots,t+2\\ (2t+3-i)x+t+2-i & i=t+3,\dots,t+\lfloor x\rfloor+1\\ (2t+2-i)x+1 & i=t+\lfloor x\rfloor+2,\dots,2t+1\end{cases}$$

This labeling is a real $L(x,1)$-labeling of C_{2t+1}, and its edge span is $\max\{x,x+1,2x-\lfloor x\rfloor\}=x+1$.

Otherwise, $t=\lfloor x\rfloor=\lceil x\rceil$, we may modify f_1 as follows:

$$f'_1(v_i)=\begin{cases}(i-1)x & i=1,2,\dots,t+2\\ (2t+3-i)x+t+2-i & i=t+3,t+4,\dots,2t+1\end{cases}$$

It is a real $L(x,1)$-labeling of C_{2t+1} with edge span $x+1$.

2) $1<x\leqslant 2$. Define

$$f_2(v_i)=\begin{cases}(i-1)x & i=1,2,\dots,t\\ (i-2)x+2 & i=t+1,t+2\\ (2t+2-i)x+1 & i=t+3,t+4,\dots,2t+1\end{cases}$$

This labeling is a real $L(x,1)$-labeling of C_{2t+1}, and its edge span is $\max\{x,x+1,2\}=x+1$.

Case 2 $0<x\leqslant 1$ and $t\geqslant 3$.

1) $0<x\leqslant 1/2$. Define

$$g_1(v_i)=\begin{cases}\dfrac{1+(-1)^i}{2}x+i & i=1,2,\dots,t-1\\ \dfrac{1+(-1)^i}{2}x+2t-1-i & i=t,t+1,\dots,2t-2\\ (2t-i)x & i=2t-1,2t\\ x+1 & i=2t+1\end{cases}$$

This labeling is a real $L(x,1)$-labeling of C_{2t+1}, and its edge span is $\max\{x+1,1-x,x,1\}=x+1$.

2) $1/2<x\leqslant 1$. Define

$$g_2(v_i)=\begin{cases}(i-1)x & i=1,2,\dots,t\\ (2t+1-i)x+1 & i=t+1,t+2,\dots,2t-1\\ (i-2t)x+1 & i=2t,2t+1\end{cases}$$

This labeling is a real $L(x,1)$-labeling of C_{2t+1}, and its edge span is $\max\{x,x+1,2x\}=x+1$.

Proposition 4 If $1\leqslant t\leqslant\lfloor x\rfloor$, then $\hat\beta_{x,1}(C_{2t+1})=x+x/t$.

Proof If $t=1\leqslant\lfloor x\rfloor$, then $\hat\beta_{x,1}(C_{2t+1})=\hat\beta_{x,1}(C_3)=2x=x+x/t$.

Now we prove that if $2\leqslant t\leqslant\lfloor x\rfloor$, then $\hat\beta_{x,1}(C_{2t+1})=x+x/t$. Define

$$f(v_i)=\Big[\frac{1+(-1)^i}{2}+\frac{2i-1+(-1)^i}{4t}\Big]x \qquad i=1,2,\dots,2t+1$$

It is a real $L(x,1)$-labeling with edge span $x+x/t$. So $\hat\beta_{x,1}(C_{2t+1})\leqslant x+x/t$.

On the other hand, we prove that $\hat\beta_{x,1}(C_{2t+1})\geqslant x+x/t$. Suppose that f is a real $L(x,1)$-labeling of C_{2t+1} with edge span $\hat\beta_{x,1}(C_{2t+1},f)<x+x/t$. Without loss of generality, suppose $f(v_1)=0$. Since the edge span of f is less than $x+x/t$, the labels of the two adjacent vertices of v_1, $f(v_2)$ and $f(v_{2t+1})$, must belong to the set $(x,x+x/t)$. Let $l_i=f(v_{i+1})-f(v_i)$, $i=1,2,\dots,2t$, then we can obtain

$$x\leqslant|l_i|<x+\frac{x}{t}$$

and

$$x\leqslant\sum_{i=1}^{2t}l_i=f(v_{2t}+1)<x+\frac{x}{t}$$

Let m_1 be the number of l_i which are positive, and let $m_2=2t-m_1$ be the number of l_i which are negative. If $m_1>m_2$, then $m_1\geqslant t+1, m_2\leqslant t-1$, and $f(v_{2t+1})=\sum_{i=1}^{2t}l_i=\sum_{l_i>0}l_i-\sum_{l_i<0}|l_i|>m_1x-m_2\left(x+\frac{x}{t}\right)\geqslant x+\frac{x}{t}$. This is a contradiction. If $m_1<m_2$, then $m_1\leqslant t-1, m_2\geqslant t+1$, and $f(v_{2t+1})=\sum_{i=1}^{2t}l_i=\sum_{l_i>0}l_i-\sum_{l_i<0}|l_i|<m_1\left(x+\frac{x}{t}\right)-m_2x<0$, another contradiction. If $m_1=m_2=t$, then $f(v_{2t+1})=\sum_{i=1}^{2t}l_i=\sum_{l_i>0}l_i-\sum_{l_i<0}|l_i|<m_1\left(x+\frac{x}{t}\right)-m_2x=(t+1)x-tx=x$. We again obtain a contradiction. So $\hat\beta_{x,1}(C_{2t+1})\geqslant x+\frac{x}{t}$. Thus $\hat\beta_{x,1}(C_{2t+1})=x+\frac{x}{t}$.

Proposition 5
$$\hat\beta_{x,1}(C_5)=\begin{cases}\dfrac{3}{2} & x\leqslant\dfrac{1}{2}\\ x+1 & \dfrac{1}{2}\leqslant x\leqslant 2\\ \dfrac{3x}{2} & x\geqslant 2\end{cases}$$

Proof From proposition 3 we know that, if $1<x\leqslant 2$, then $\hat\beta_{x,1}(C_5)=x+1$. By proposition 4, if $x\geqslant 2$, then $\hat\beta_{x,1}(C_5)=3x/2$. If $x=1$, then the $L(x,1)$-labeling f of C_5 defined as $f(v_1)=0, f(v_2)=1, f(v_3)=3, f(v_4)=4$, and $f(v_5)=2$ has an edge span $x+1$. Thus $\hat\beta_{1,1}(C_5)=2=x+1$. So

$$\hat\beta_{x,1}(C_5)=\begin{cases}x+1 & 1\leqslant x\leqslant 2\\ \dfrac{3x}{2} & x\geqslant 2\end{cases}$$

Note that C_5^c, the complement of C_5, is also a cycle. And if the vertices u and v are adjacent in C_5, then they are at distance two in C_5^c; if the vertices u and v are at distance two in C_5, then they are adjacent in C_5^c. Hence

$$\hat\beta_{x,1}(C_5)=\hat\beta_{1,x}(C_5^c)=x\hat\beta_{\frac{1}{x},1}(C_5^c)=x\hat\beta_{\frac{1}{x},1}(C_5)$$

By the above conclusion, we obtain

$$\hat\beta_{\frac{1}{x},1}(C_5)=\begin{cases}\dfrac{1}{x}+1 & \dfrac{1}{2}\leqslant x\leqslant 1\\ \dfrac{3}{2x} & x\leqslant\dfrac{1}{2}\end{cases}$$

So

$$\hat\beta_{x,1}(C_5)=\begin{cases}x+1 & \dfrac{1}{2}\leqslant x\leqslant 1\\ \dfrac{3}{2} & x\leqslant\dfrac{1}{2}\end{cases}$$

This completes the proof.

We summarize the above several propositions as the following theorem which gives all the $L(x,1)$ edge spans of C_n $(x>0)$.

Theorem 12 1) $\hat{\beta}_{x,1}(C_n)=x+1$ if one of the following cases holds: ① n is even; ② n is odd, $0<x\leqslant 1$ and $n\geqslant 7$; ③ n is odd, $x>1$ and $n\geqslant 2\lceil x\rceil+1$.

2) $\hat{\beta}_{x,1}(C_n)=x+x/t$ if n is odd, $x>1$ and $3\leqslant n\leqslant 2\lfloor x\rfloor+1$.

3) $$\hat{\beta}_{x,1}(C_5)=\begin{cases}\frac{3}{2} & x\leqslant\frac{1}{2}\\ x+1 & \frac{1}{2}\leqslant x\leqslant 2\\ \frac{3x}{2} & x\geqslant 2\end{cases}$$

By theorems 7, 11 and 12, we can directly obtain the real $L(j,k)$ edge spans and $L(j,k)$ edge spans of cycles. The following corollary is about the $L(j,k)$ edge spans of cycles.

Corollary 2 Suppose that j and k are two positive integers, then

1) $\beta_{j,k}(C_n)=j+k$ if one of the following cases holds: ① n is even; ② n is odd, $0<j\leqslant k$ and $n\geqslant 7$; ③ n is odd, $j>k$ and $n\geqslant 2\lceil \frac{j}{k}\rceil+1$.

2) $\beta_{j,k}(C_n)=j+\lceil\frac{2j}{n-1}\rceil$ if n is odd, $j>k$ and $3\leqslant n\leqslant 2\lfloor\frac{j}{k}\rfloor+1$.

3) $$\beta_{j,k}(C_5)=\begin{cases}k+\lceil\frac{k}{2}\rceil & j\leqslant\frac{k}{2}\\ j+k & \frac{k}{2}\leqslant j\leqslant 2k\\ j+\lceil\frac{j}{2}\rceil & j\geqslant 2k\end{cases}$$

3 Complete t-Partite Graphs

Theorem 13 Let $K=K_{n_1,n_2,\ldots,n_t}$ be a complete t-partite graph, where $n_1\geqslant n_2\geqslant\ldots\geqslant n_t$. Let j and k be two positive real numbers with $j\geqslant k$. Then $\hat{\beta}_{j,k}(K)=(t-1)j+\left(\lceil\frac{n_1}{2}\rceil+\sum_{i=2}^{t}n_i-t\right)k$.

Proof Let $\hat{\beta}=(t-1)j+\left(\lceil\frac{n_1}{2}\rceil+\sum_{i=2}^{t}n_i-t\right)k$. Suppose that $V(K)=V_1\cup V_2\cup\ldots\cup V_t\cup V_{t+1}$, where $V_1\cup V_{t+1}$, V_2, …, V_t are t partite sets of K, and $|V_1|=\lceil\frac{n_1}{2}\rceil$, $|V_{t+1}|=\lfloor\frac{n_1}{2}\rfloor$, $|V_i|=n_i$, $i=2,3,\ldots,t$. Label the vertices in each $V_i(i=1,2,\ldots,t+1)$ with the real numbers which are consecutive multiples of k such that the minimum label in V_1 is 0, and the minimum label of $V_i(i=2,3,\ldots,t+1)$ is the maximum label of V_{i-1} plus the real number j. It is straightforward to check that this labeling is a real $L(j,k)$-labeling of K and its edge span is $\hat{\beta}$. Hence $\hat{\beta}_{j,k}(K)\leqslant\hat{\beta}$.

On the other hand, suppose that f is a real $L(j,k)$-labeling with $\hat{\beta}_{j,k}(K,f)=\hat{\beta}_{j,k}(K)$. We prove that $\hat{\beta}_{j,k}(K,f)\geqslant\hat{\beta}$. Let $v\in V_r$, $u\in V_s$ and $f(v)=0$, $f(u)=M=\max_{w\in V(K)}f(w)$. Note that

$$\hat{\lambda}_{j,k}(K)=(t-1)j+\sum_{i=1}^{t}(n_i-1)k$$

and

$$M\geqslant\hat{\lambda}_{j,k}(K)=(t-1)j+\sum_{i=1}^{t}(n_i-1)k\geqslant\hat{\beta}$$

If $r\neq s$, then $\hat{\beta}_{j,k}(K,f)=M\geqslant\hat{\beta}$. So, we can assume that $r=s$. In this case, $M\geqslant\hat{\lambda}_{j,k}(K)+j=tj+\sum_{i=1}^{t}(n_i-1)k$. Let a and b be the maximum and minimum values of f on all vertices not in V_r. Then

$$a-b\geqslant\hat{\lambda}_{j,k}(K)-j-(n_r-1)k=(t-1)j+\sum_{i=1}^{t}(n_i-1)k-j-(n_r-1)k\geqslant(t-2)j+\sum_{i=2}^{t}(n_i-1)k$$

and

$$(a-0)+(M-b)\geqslant 2(t-1)j+2\sum_{i=2}^{t}(n_i-1)k+(n_1-1)k$$

Thus, either $a-0\geqslant\hat{\beta}$ or $M-b\geqslant\hat{\beta}$, which implies that $\hat{\beta}_{j,k}(K,f)\geqslant\hat{\beta}$. So, $\hat{\beta}_{j,k}(K,f)=\hat{\beta}$.

By theorem 7, the $L(j,k)$ edge spans of complete t-partite graphs where j and k are two positive integers with $j\geqslant k$ are the same as theorem 13.

Suppose that one of two real numbers j and k is zero, then it is very trivial that $\hat{\beta}_{j,0}(K_{n_1,n_2,\ldots,n_t})=\hat{\lambda}_{j,0}(K_{n_1,n_2,\ldots,n_t})=(t-1)j$, and $\hat{\lambda}_{0,k}(K_{n_1,n_2,\ldots,n_t})=(n_1-1)k$, $\hat{\beta}_{0,k}(K_{n_1,n_2,\ldots,n_t})=\left(\frac{n_1+n_2}{2}-1\right)k$.

For $j<k$, it seems not easy to determine the real $L(j,k)$-labeling numbers and the real $L(j,k)$ edge spans of the complete t-partite graphs.

References

[1] Griggs J R, Yeh R K. Labeling graphs with a condition at distance two [J]. *SIAM J Discrete Math*, 1992, **5**(4): 586–595.

[2] Hale W K. Frequency assignment: theorem and applications [J]. *Proceedings of the IEEE*, 1980, **68**(12): 1497–1514.

[3] Georges J P, Mauro D W. Generalized vertex labelings with a condition at distance two [J]. *Congr Numer*, 1995, **109**: 141–159.

[4] Chang G J, Kuo D. The $L(2,1)$-labeling on graphs [J]. *SIAM J Discrete Math*, 1996, **9**(2): 309–316.

[5] Georges J P, Mauro D W, Whittlesey M. Relating path covering to vertex labelings with a condition at distance two [J]. *Discrete Math*, 1994, **135**(1/2/3): 103–111.

[6] Whittlesey M, Georges J P, Mauro D W. On the λ-number of Q_n and related graphs [J]. *SIAM J Discrete Math*, 1995, **8**(4): 499–506.

[7] Chang G J, Ke W T, Kuo D, et al. On $L(d,1)$-labelings of graphs [J]. *Discrete Math*, 2000, **220**(1/2/3): 57–66.

[8] Georges J P, Mauro D W, Stein M I. Labeling products of complete graphs with a condition at distance two [J]. *SIAM J Discrete Math*, 2000, **14**(1): 28–35.

[9] Jin X T, Yeh R K. Graph distance-dependent labeling related to code assignment in computer networks [J]. *Naval Research Logistics*, 2005, **52**(2): 159–164.

[10] Yeh R K. The edge span of distance two labelings of graphs [J]. *Taiwanese J Math*, 2000, **4**(3): 397 − 405.
[11] Feng Guizhen, Song Zengmin. Edge span of $L(d, 1)$-labeling on some graphs [J]. *Journal of Southeast University: English Edition*, 2005, **21**(1): 111 − 114.
[12] Niu Qingjie, Lin Wensong, Song Zengmin. $L(s, t)$ edge spans of trees and product of two paths [J]. *Journal of Southeast University: English Edition*, 2007, **23**(4): 639 − 642.
[13] Griggs J R, Jin X T. Real number graph labelings with distance conditions [J]. *SIAM J Discrete Math*, 2006, **20**(2): 302 − 327.
[14] Griggs J R, Jin X T. Real number channel assignments for lattices [J]. *SIAM J Discrete Math*, 2008, **22**(3): 996 − 1021.

图的实值距离二边跨度

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摘要： 图 G 的 $L(j,k)$ 标号是图的顶点集到非负整数集的一个映射，使得相邻顶点所对应的整数相差至少为 j，距离为 2 的顶点所对应的整数相差至少为 k. 对于图 G 的一个 $L(j,k)$ 标号 f，定义其 $L(j,k)$ 边跨度为 $\beta_{j,k}(G,f)=\max\{|f(x)-f(y)|:\{x,y\}\in E(G)\}$. 图 G 的 $L(j,k)$ 边跨度定义为 $\beta_{j,k}(G)$，它是 G 的所有 $L(j,k)$ 标号 f 的 $L(j,k)$ 边跨度中最小的. 图 G 的实值 $L(j,k)$ 标号是整数 $L(j,k)$ 标号的推广，是满足相应的距离一条件和距离二条件的从顶点集到实数集的一个映射. 图 G 的实值 $L(j,k)$ 标号的边跨度记为 $\hat{\beta}_{j,k}(G)$. 研究了图的实值 $L(j,k)$ 边跨度和整数 $L(j,k)$ 边跨度的若干性质，完全确定了所有圈以及完全 t-部图的边跨度.

关键词： $L(j,k)$ 标号；实值 $L(j,k)$ 标号；$L(j,k)$ 边跨度；实值 $L(j,k)$ 边跨度；频道分配
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