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Circular $L(j, k)$-labeling numbers of trees and products of graphs

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Abstract: Let j, k and m be three positive integers, a circular m-$L(j, k)$-labeling of a graph G is a mapping f: $V(G) \rightarrow \{0, 1, \ldots, m-1\}$ such that $|f(u) - f(v)|_m \geqslant j$ if u and v are adjacent, and $|f(u) - f(v)|_m \geqslant k$ if u and v are at distance two, where $|a-b|_m = \min\{|a-b|, m - |a-b|\}$. The minimum m such that there exists a circular m-$L(j, k)$-labeling of G is called the circular $L(j, k)$-labeling number of G and is denoted by $\sigma_{j,k}(G)$. For any two positive integers j and k with $j \leqslant k$, the circular $L(j, k)$-labeling numbers of trees, the Cartesian product and the direct product of two complete graphs are determined.

Key words: circular $L(j, k)$-labeling number; tree; Cartesian product of graphs; direct product of graphs

For two positive integers j and k, an $L(j, k)$-labeling f of a graph G is an assignment of integers to the vertices of G such that $|f(u) - f(v)| \geqslant j$ if $uv \in E(G)$, and $|f(u) - f(v)| \geqslant k$ if u and v are distance two apart. Then the span of f is the difference between the maximum and the minimum integers assigned by f. The $L(j, k)$-labeling number of G, denoted by $\lambda_{j,k}(G)$, is the minimum span over all $L(j, k)$-labelings of G.

Motivated by the channel assignment problem introduced by Hale[1], Griggs and Yeh[2] first proposed and studied the $L(2, 1)$-labeling of a graph. Since then the $L(2, 1)$-labeling numbers of graphs and $L(j, k)$-labeling numbers of graphs for $j \geqslant k$ have been investigated in many papers, please refer to the survey in Ref. [3]. Recently, people have begun to study the $L(j, k)$-labeling numbers of graphs for $j \leqslant k$[4–6].

If we choose the "cyclic channel distance" instead of the absolute difference in the above definition, then we obtain the circular $L(j, k)$-labeling of a graph.

For positive integers j, k and m, a circular m-$L(j, k)$-labeling of a graph G is a function f: $V(G) \rightarrow \{0, 1, \ldots, m-1\}$ such that $|f(u) - f(v)|_m \geqslant j$ if u and v are adjacent, and $|f(u) - f(v)|_m \geqslant k$ if u and v are at distance two, where $|a-b|_m = \min\{|a-b|, m - |a-b|\}$. The minimum m such that there exits a circular m-$L(j, k)$-labeling of G is called the circular $L(j, k)$-labeling number of G and is denoted by $\sigma_{j,k}(G)$.

Heuvel et al.[7] first investigated the circular $L(j, k)$-labeling of a graph, where the circular $L(j, k)$-labeling numbers of triangular lattices and square lattices for any two positive integers j and k with $j \geqslant k$ were determined. Liu[8] related the circular $L(2, 1)$-labeling number of a graph G to the path covering number of its complement. She also determined circular $L(2, 1)$-labeling numbers of cycles, the Cartesian product of two complete graphs $K_m \square K_n$, and proved that $\sigma_{2,1}(T) = \Delta + 3$ for any tree T with the maximum degree Δ. Wu and Yeh[9] showed that $\sigma_{j,1}(T) = 2j + \Delta - 1$ for any tree T with the maximum degree Δ. In Refs. [10 – 11], it was proved that, for $j \geqslant k$, $\sigma_{j,k}(T) = 2j + (\Delta - 1)k$ for any tree T with the maximum degree Δ. The circular $L(j, k)$-labeling numbers of cycles for $j \geqslant k$ were completely determined in Ref. [10]. In Ref. [12], the circular $L(j, k)$-labeling numbers of the Cartesian product of two complete graphs and the direct product of two complete graphs for $j \geqslant k$ were determined. Recently, the circular $L(2, 1)$-labeling numbers of the Cartesian products of three complete graphs were obtained in Ref. [13].

Let j and k be any two nonnegative integers. For any graph G, it is easy to see that

$$\lambda_{j,k}(G) + 1 \leqslant \sigma_{j,k}(G) \leqslant \lambda_{j,k}(G) + \max\{j, k\}$$

We would like to point out that even when $j = 2$ and $k = 1$ it is not easy to determine whether $\sigma_{2,1}(G)$ equals $\lambda_{2,1}(G) + 1$ or $\lambda_{2,1}(G) + 2$ provided that $\lambda_{2,1}(G)$ is known.

The following two lemmas are not difficult to see.

Lemma 1 Let j and k be two positive integers with $j \geqslant k$. Suppose that G is a graph and H is a subgraph of G. Then $\sigma_{j,k}(G) \geqslant \sigma_{j,k}(H)$.

Lemma 2 Let j and k be two positive integers with $j \leqslant k$. Suppose that G is a graph and H is an induced subgraph of G. Then $\sigma_{j,k}(G) \geqslant \sigma_{j,k}(H)$.

Note that if H is not an induced subgraph, then lemma 2 is not true. For example, $K_{1,3}$ is a subgraph but not an induced subgraph of K_4 and we obtain $\sigma_{1,2}(K_{1,3}) = 6 > 4 = \sigma_{1,2}(K_4)$.

This paper completely determines the circular $L(j, k)$-labeling numbers of trees, the Cartesian product of two complete graphs, and the direct product of two complete graphs for $j \leqslant k$.

For a positive real number r, let $S(r)$ denote the circle obtained from the interval $[0, r]$ by identifying 0 and r into a single point. For a positive integer m, denote by $[m]$ the set of integers $0, 1, \ldots, m-1$. The circular distance of two points p and q on $S(m)$ is defined as $\min\{|p-q|, m - |p-q|\}$. A set of labels in $[m]$ is said to be (m, k)-circular separated if the circular distance on $S(m)$ of any two labels from the set is greater than or equal to k.

1 $\sigma_{j,k}$-Numbers of Trees for $j \leqslant k$

In this section, we determine the circular $L(j, k)$-labeling number of any tree for any two positive integers j and k with $j \leqslant k$. For $j \geqslant k$, Leese et al.[10–11] proved the following result.

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Theorem 1 Let T be a tree with the maximum degree Δ, and j and k be any two nonnegative integers with $j \geqslant k$; and then $\sigma_{j,k}(T) = 2j + (\Delta - 1)k$.

Our result is as follows.

Theorem 2 Let T be a tree with the maximum degree Δ ($\geqslant 2$). For any two positive integers j and k with $j \leqslant k$, we have $\sigma_{j,k}(T) = \max\{k, 2j\} + (\Delta - 1)k$.

Proof Let $m = \max\{k, 2j\} + (\Delta - 1)k$. It is not difficult to prove that $\sigma_{j,k}(K_{1,\Delta}) = m$. Since $K_{1,\Delta}$ is an induced subgraph of T, by lemma 2, we obtain $\sigma_{j,k}(T) \geqslant \sigma_{j,k}(K_{1,\Delta}) = m$.

To prove the theorem, it suffices to construct a circular m-$L(j, k)$-labeling of T.

Let v_0 be a vertex of the maximum degree of T. We view v_0 as the root of T. For any other vertex v of T, let $l(v)$ denote the distance from v to v_0. For two adjacent vertices u and v, if $l(u) < l(v)$ then we call u the father of v and v the son of u. Note that every vertex v has a unique father, we denote the father of v by $F(v)$. For $i = 0, 1, 2, \ldots$, denote by V_i the set of vertices at distance i from v_0.

We deal with two cases.

Case 1 $k \leqslant 2j$

In this case, $m = \max\{k, 2j\} + (\Delta - 1)k = 2j + (\Delta - 1)k$. For any color $x \in [0, m-1]$, let

$$A(x) = \{x + j + tk:\ t = 0, 1, \ldots, \Delta - 1\}$$

where +'s are taken modulo m. Clearly $A(x)$ is a (m, k)-circular separated set of colors in $[0, m-1]$.

We construct a circular m-$L(j, k)$-labeling f of T as follows. First the vertex v_0 is colored by the color 0 and then all the neighbors of v_0 are colored by different colors from $A(0)$. Suppose that all the vertices in V_i have been colored ($i \geqslant 1$). Then we color all the sons of each vertex in V_i. A vertex v is chosen in V_i. Its sons are colored by different colors from $A(f(v)) \setminus \{f(F(v))\}$. It is possible since we have $\Delta - 1$ colors available and v has at most $\Delta - 1$ sons. After all vertices in V_i are considered, all the vertices in V_{i+1} will be colored. Continuing in this way, one can color all vertices of T.

Let v be any vertex with $l(v) \geqslant 1$. By the definition of $A(f(v))$, $f(v)$ is (m, j)-circular separated from all colors of the sons of v. Since $f(v) = f(F(v)) + j + tk \pmod m$ for some $0 \leqslant t \leqslant \Delta - 1$, $f(F(v)) = f(v) + j + (\Delta - 1 - t)k \pmod m$. It follows that $f(F(v)) \in A(f(v))$. Therefore, all neighbors of v receive colors that are (m, k)-circular separated. This proves that f is a circular m-$L(j, k)$-labeling of T.

Case 2 $k \geqslant 2j$

In this case, $m = \max\{k, 2j\} + (\Delta - 1)k = \Delta k$. For any color $x \in [0, m-1]$, let

$$B(x) = \{x + j + tk:\ t = 0, 1, \ldots, \Delta - 1\}$$

and

$$C(x) = \{x - j + sk:\ s = 1, 2, \ldots, \Delta\}$$

where +'s and −'s are taken modulo m. Clearly $B(x)$ and $C(x)$ are (m, k)-circular separated.

We construct a circular m-$L(j, k)$-labeling f of T as follows. First the vertex v_0 is colored by the color 0 and then all the neighbors of v_0 are colored by different colors from $B(0)$. Suppose that all the vertices in V_i have been colored ($i \geqslant 1$). A vertex v is chosen in V_i. All its sons are colored by different colors from $B(f(v)) \setminus \{f(F(v))\}$ if i is even and from $C(f(v)) \setminus \{f(F(v))\}$ if i is odd. This is possible since we have $\Delta - 1$ colors available and v has at most $\Delta - 1$ sons. After all the vertices in V_i are considered, all the vertices in V_{i+1} will be colored. Continuing in this way, one can color all the vertices of T.

Let v be any vertex with $l(v) = i \geqslant 1$. By the definitions of $B(f(v))$ and $C(f(v))$, $f(v)$ is (m, j)-circular separated from all colors of the sons of v. If i is even then $f(v) = f(F(v)) - j + sk \pmod m$ for some $1 \leqslant s \leqslant \Delta$, and so $f(F(v)) = f(v) + j + (\Delta - s)k \pmod m$. It follows that $f(F(v)) \in B(f(v))$. If i is odd then $f(v) = f(F(v)) + j + tk \pmod m$ for some $0 \leqslant t \leqslant \Delta - 1$, and so $f(F(v)) = f(v) - j + (\Delta - t)k \pmod m$. It follows that $f(F(v)) \in C(f(v))$. Therefore, all the neighbors of v receive colors that are (m, k)-circular separated. This proves that f is a circular m-$L(j, k)$-labeling of T.

2 $\sigma_{j,k}$-Numbers of $K_m \times K_n$ and $K_m \square K_n$ for $j \leqslant k$

Given two graphs G and H, the Cartesian product of G and H is the graph $G \square H$ with vertex set $V(G) \times V(H)$ in which two vertices (x, y) and (x', y') are adjacent if $x = x'$ and $yy' \in E(H)$ or $y = y'$ and $xx' \in E(G)$. The direct product of G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ in which two vertices (x, y) and (x', y') are adjacent if and only if $xx' \in E(G)$ and $yy' \in E(H)$.

Theorem 3[14] Let j, k, m and n be positive integers with $2 \leqslant n < m$ and $j \geqslant k$. Then

$$\lambda_{j,k}(K_m \square K_n) = \begin{cases} (m-1)j + (n-1)k & j/k > n \\ (mn-1)k & j/k \leqslant n \end{cases}$$

Theorem 4[14] Let j, k, and n be positive integers with $2 \leqslant n$ and $j \geqslant k$. Then

$$\lambda_{j,k}(K_n \square K_n) = \begin{cases} (n-1)j + (2n-2)k & j/k > n-1 \\ (n^2-1)k & j/k \leqslant n-1 \end{cases}$$

Theorem 5[12] Let j, k, and m be positive integers with $m \geqslant 2$ and $j \geqslant k$. Then

$$\sigma_{j,k}(K_m \times K_2) = \begin{cases} mj & j/k \leqslant 2 \\ (2m-4)k + 2j & j/k \geqslant 2 \end{cases}$$

Theorem 6[12] Let j, k, m and n be positive integers with $3 \leqslant n \leqslant m$ and $j \geqslant k$. Then

$$\sigma_{j,k}(K_m \times K_n) = \begin{cases} mnk & j/k \leqslant 2 \\ (mn-2n)k + nj & j/k > 2 \end{cases}$$

Theorem 7[12] Let j, k, m and n be positive integers with $2 \leqslant n < m$ and $j \geqslant k$. Then

$$\sigma_{j,k}(K_m \square K_n) = \begin{cases} mnk & j/k \leqslant n \\ mj & j/k > n \end{cases}$$

Theorem 8[12] Let j, k, and n be positive integers with $2 \leqslant n < m$ and $j \geqslant k$. Then

$$\sigma_{j,k}(K_n \square K_n) = \begin{cases} n^2 k & j/k \leqslant n-1 \\ nj + nk & j/k > n-1 \end{cases}$$

It is worth pointing out that the graph $K_m \times K_n$ is exactly the complement of $K_m \square K_n$.

Throughout this paper, the vertices of the graph $K_m \times K_n$ and $K_m \square K_n$ are displayed as a matrix with m rows and n columns. For $0 \leqslant s \leqslant m-1$ and $0 \leqslant t \leqslant n-1$, let $v_{s,t}$ be the vertex on the s-th row and the t-th column.

Let m and n be two integers with $m \geqslant n \geqslant 2$. Suppose that $K_m \square K_n$ and $K_m \times K_n$ have the same vertex set $\{v_{s,t} \mid s=0, 1, \ldots, m-1, t=0, 1, \ldots, n-1\}$. Then it is not difficult to see that the complement of $K_m \square K_n$ is $K_m \times K_n$, and vice versa. Furthermore, if $n \geqslant 3$ then any two vertices at distance two in $K_m \square K_n$ are adjacent in $K_m \times K_n$ and any two adjacent vertices in $K_m \square K_n$ are at distance two in $K_m \times K_n$. Let σ, j, k be positive integers. Therefore, any circular σ-$L(j, k)$-labeling of $K_m \square K_n$ is a circular σ-$L(j, k)$-labeling of $K_m \times K_n$, and vice versa.

Theorem 9 Let j, k, m, and n be positive integers with $2 \leqslant n \leqslant m$ and $j \leqslant k$. Then

$$\sigma_{j,k}(K_m \square K_n) = \begin{cases} mnj & k/j \leqslant 2 \\ (mn-2n)j+nk & k/j > 2 \end{cases}$$

Proof The case $n \geqslant 3$ follows from theorem 6 and the above discussion. Thus we only need to prove the case $n = 2$.

We first assume that $k/j \leqslant 2$. Clearly the diameter of $K_m \square K_2$ is 2. Let $\sigma = \sigma_{j,k}(K_m \square K_2)$ and let f be a circular σ-$L(j, k)$-labeling of $K_m \square K_2$. Then any two vertices must receive labels that are (σ, j)-circular separated and so $\sigma_{j,k}(K_m \square K_2) \geqslant 2mj$. On the other hand, define a mapping g: $V(K_m \square K_2) \to [2mj]$ as

$$g(v_{s,t}) = \begin{cases} sj & 0 \leqslant s \leqslant m-1;\ t=0 \\ (2m-s-1)j & 0 \leqslant s \leqslant m-1;\ t=1 \end{cases}$$

It is easy to check that g is a circular $2mj$-$L(j, k)$-labeling of $K_m \square K_2$. Thus $\sigma_{j,k}(K_m \square K_2) = 2mj$ if $k/j \leqslant 2$.

We now deal with the case $k/j > 2$. Define a mapping h: $V(K_m \square K_2) \to [(2m-4)j+2k]$ as

$$h(v_{s,t}) = \begin{cases} sj & 0 \leqslant s \leqslant m-1;\ t=0 \\ k+(2m-s-3)j & 0 \leqslant s \leqslant m-1;\ t=1 \end{cases}$$

It is not difficult to check that h is a circular $((2m-4)j+2k)$-$L(j, k)$-labeling of $K_m \square K_2$. Thus $\sigma_{j,k}(K_m \square K_2) \leqslant (2m-4)j+2k$ if $k/j > 2$.

To prove the opposite inequality, let $\sigma = \sigma_{j,k}(K_m \square K_2)$ and let f be a circular σ-$L(j, k)$-labeling of $K_m \square K_2$. We shall prove that $\sigma \geqslant (2m-4)j+2k$. Clearly this holds for $m = 2$. So we assume that $m \geqslant 3$.

Since $K_m \square K_2$ is of diameter 2, any two vertices should receive labels that are (σ, j)-circular separated. Without loss of generality, we may assume that the label 0 is always used. The label sequence $F = f_0, f_1, \ldots, f_{2m-1}$ with $f_0 = 0$ denotes the labels used by f from 0 in the order in a clockwise direction on the circular $S(\sigma)$. We refer to vertex labeled by f_i as $v(f_i)$.

In this proof, all +'s and −'s in subscripts will be taken modulo $2m$. We first make an observation.

Observation Let $i \in [2m]$. Suppose that $v(f_i)$ and $v(f_{i+1})$ are in different columns. If $|f_i - f_{i+1}|_\sigma < k$, then $v(f_i)$ and $v(f_{i+1})$ are in the same row. It follows that $|f_i - f_{i+2}|_\sigma \geqslant k$ or $|f_{i+1} - f_{i+2}|_\sigma \geqslant k$. Symmetrically $|f_{i-1} - f_{i+1}|_\sigma \geqslant k$ or $|f_{i-1} - f_i|_\sigma \geqslant k$.

Let $f_s, f_{s+1}, \ldots, f_t$ be a consecutive subsequence of F such that $v(f_s), v(f_{s+1}), \ldots, v(f_t)$ are in the same column and $v(f_{s-1}), v(f_{t+1})$ are in the other column. Clearly $s-2 \neq t+1 \pmod{2m}$ since we have assumed $m \geqslant 3$. Now it follows from the above observation that $|f_t - f_{t+1}|_\sigma \geqslant k$, or $|f_t - f_{t+2}|_\sigma \geqslant k$, or $|f_{t+1} - f_{t+2}|_\sigma \geqslant k$; and $|f_{s-1} - f_s|_\sigma \geqslant k$, or $|f_{s-2} - f_s|_\sigma \geqslant k$, or $|f_{s-2} - f_{s-1}|_\sigma \geqslant k$. This together with $|f_i - f_{i+1}|_\sigma \geqslant j$ implies that $\sigma \geqslant (2m-4)j+2k$.

Theorem 10 Let j, k, m, and n be positive integers with $2 \leqslant n \leqslant m$ and $j \leqslant k$. Then $\sigma_{j,k}(K_2 \times K_2) = 2j$ and $\sigma_{j,k}(K_m \times K_2) = mk$ for $m \geqslant 3$. If $2 < n < m$, then

$$\sigma_{j,k}(K_m \times K_n) = \begin{cases} mnj & k/j \leqslant n \\ mk & k/j > n \end{cases}$$

And if $n > 2$, then

$$\sigma_{j,k}(K_n \times K_n) = \begin{cases} n^2 j & k/j \leqslant n-1 \\ nk+nj & k/j > n-1 \end{cases}$$

Proof It is obvious that $\sigma_{j,k}(K_2 \times K_2) = 2j$. For the case $n = 2$ and $m \geqslant 3$, we define a mapping f: $V(K_m \times K_2) \to [mk]$ as $f(v_{s,t}) = sk$ for $s = 0, 1, \ldots, m-1$ and $t = 0, 1$. It is easy to see that f is a circular mk-$L(j, k)$-labeling of $K_m \times K_2$. Thus $\sigma_{j,k}(K_m \times K_2) \leqslant mk$ for $m \geqslant 3$. On the other hand, since vertices in $\{v_{0,0}, v_{0,1}, \ldots, v_{0,m-1}\}$ are pairwise at distance two, we conclude that $\sigma_{j,k}(K_m \times K_2) \geqslant mk$. Therefore, $\sigma_{j,k}(K_m \times K_2) = mk$. The case $n > 2$ follows from theorems 7 and 8 and the discussion between theorem 8 and theorem 9.

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树和乘积图的圆 $L(j, k)$-标号数

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摘要： 设 j, k 和 m 是 3 个正整数. 给定一个图 G. 设 $f: V(G) \to \{0, 1, \cdots, m-1\}$ 是一个映射. 如果对图 G 的任意一对相邻顶点 u 和 v 都有 $|f(u)-f(v)|_m \geqslant j$，对任意一对距离为二的顶点都有 $|f(u)-f(v)|_m \geqslant k$，其中 $|a-b|_m = \min\{|a-b|, m-|a-b|\}$，则称 f 是图 G 的一个圆 m-$L(j, k)$-标号. 使得图 G 有圆 m-$L(j, k)$-标号的最小的正整数 m 称为图 G 的圆 $L(j, k)$-标号数，记为 $\sigma_{j,k}(G)$. 对任意 2 个满足 $j \leqslant k$ 的正整数，确定了树以及 2 个完全图的笛卡尔乘积图和直积图的圆 $L(j, k)$-标号数.

关键词： 圆 $L(j, k)$-标号数；树；笛卡尔乘积图；直积图
中图分类号： O157. 5