Journal of Southeast University (English Edition) Vol. 26, No. 1, pp. 146 – 150 Mar. 2010 ISSN 1003—7985

Symmetries and conserved quantities of generalized Birkhoffian systems

Zhang Yi

(College of Civil Engineering, Suzhou University of Science and Technology, Suzhou 215011, China)

Abstract: Three kinds of symmetries and their corresponding conserved quantities of a generalized Birkhoffian system are studied. First, by using the invariance of the Pfaffian action under the infinitesimal transformations, the Noether theory of the generalized Birkhoffian system is established. Secondly, on the basis of the invariance of differential equations under infinitesimal transformations, the definition and the criterion of the Lie symmetry of the generalized Birkhoffian system are established, and the Hojman conserved quantity directly derived from the Lie symmetry of the system is given. Finally, by using the invariance that the dynamical functions in the differential equations of the motion of mechanical systems still satisfy the equations after undergoing the infinitesimal transformations, the definition and the criterion of the Mei symmetry of the generalized Birkhoffian system are presented, and the Mei conserved quantity directly derived from the Mei symmetry of the system is obtained. Some examples are given to illustrate the application of the results.

Key words: generalized Birkhoffian system; symmetry; conserved quantity

The Birkhoffian mechanics is the generalization of Hamiltonian mechanics. In 1927, Birkhoff[1] gave a new integral variational principle and a new form of the equations of motion in his famous works *Dynamical Systems*, which is more general than the Hamiltonian principle and Hamilton's equations. Santilli[2] generalized the equations and, at his suggestion, they are called Birkhoff's equations. In addition to the generalization of Galilei's relativity, Santilli studied Birkhoff's equations, the transformation theory of Birkhoff's equations and so on[2]. Since 1993, Mei et al. have constructed the theoretical framework of Birkhoffian dynamics[3-18], which include a series of contributions such as Birkhoff's equations and the Pfaff-Birkhoffian principle, the Birkhoffian mechanics of holonomic and nonholonomic systems, the integration theory of Birkhoffian mechanics, the inverse problem of Birkhoffian mechanics, the stability of motion of Birkhoffian mechanics, the prescription of algebra and geometry of Birkhoffian mechanics, and the theory of symmetry of Birkhoffian mechanics. Galiullin et al. [19-20] also made a study of the inverse problem of Birkhoffian system dynamics, the symmetry and the integral invariants of Birkhoffian systems and so on. Mei et al. [21] advanced a generalized Birkhoffian system in 1993, which is superior to traditional classical mechanical systems. In this paper, we study the symmetries and conserved quantities of generalized Birkhoffian systems, including the Noether symmetry and the Noether conserved quantity, the Lie symmetry and the Hojman conserved quantity, as well as the Mei symmetry and the Mei conserved quantity.

Received 2009-08-10.

Biography: Zhang Yi (1964—), male, doctor, professor, weidiezh@pub.sz.jsinfo.net.

Foundation items: The National Natural Science Foundation of China (No. 10972151), the Natural Science Foundation of Higher Education Institution of Jiangsu Province of China (No. 08KJB130002).

Citation: Zhang Yi. Symmetries and conserved quantities of generalized Birkhoffian systems[J]. Journal of Southeast University (English Edition), 2010, 26(1): 146 – 150.

1 Differential Equations of Motion of Generalized Birkhoffian Systems

The generalized Birkhoff's equation is[3]

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=-\Lambda_\mu \qquad \mu=1,2,\dots,2n \tag{1}$$

where $B=B(t, a)$ is the Birkhoffian; $R_\mu=R_\mu(t, a)$ are Birkhoff's functions. The arbitrary differentiable functions $\Lambda_\mu=\Lambda_\mu(t, a)$ are called additional items and

$$\Omega_{\mu\nu}=\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu} \tag{2}$$

is called the Birkhoff's tensor. A mechanical system whose motion is described by Eq. (1), or a physical system whose state is described by Eq. (1), is called a generalized Birkhoffian system. When $\Lambda_\mu\equiv 0$, the generalized Birkhoff's equation (1) becomes a Birkhoff's equation[2].

Suppose that Eq. (1) is nonsingular, i. e.,

$$\det(\Omega_{\mu\nu})\neq 0 \tag{3}$$

Then all of $\dot{a}^\mu$ can be solved by Eq. (1), and we obtain

$$\dot{a}^\mu=\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}-\Lambda_\nu\right) \qquad \mu=1,2,\dots,2n \tag{4}$$

where

$$\Omega^{\mu\nu}\Omega_{\nu\tau}=\delta_{\mu\tau} \tag{5}$$

Expanding Eq. (4), we obtain

$$\dot{a}^\mu=h_\mu(t, a) \qquad \mu=1,2,\dots,2n \tag{6}$$

2 Noether Theory of Generalized Birkhoffian Systems

The Noether symmetry of generalized Birkhoffian system is an invariance of the Pfaffian action under the infinitesimal transformations. The Pfaffian action is[2]

$$A=\int_{t_1}^{t_2}[R_\nu(t, a)\dot{a}^\nu-B(t, a)]\,\mathrm{d}t \tag{7}$$

Introduce the infinitesimal transformations of an r-parameter transformation group G_r with respect to time t and variable a as

$$t^* = t + \Delta t,\ a^{\mu *}(t^*) = a^\mu(t) + \Delta a^\mu \qquad \mu = 1, 2, \ldots, 2n \tag{8}$$

and their expanded form

$$t^* = t + \varepsilon_\alpha \xi_0^\alpha(t, a),\ a^{\mu *} = a^\mu + \varepsilon_\alpha \xi_\mu^\alpha(t, a)$$
$$\mu = 1, 2, \ldots, 2n;\ \alpha = 1, 2, \ldots, r \tag{9}$$

where ε_α are the infinitesimal parameters; ξ_0^α, ξ_μ^α are infinitesimal generators. We obtain[3]

$$\Delta A = \int_{t_1}^{t_2} \Big[(R_\mu \dot{a}^\mu - B) \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + \Big(\frac{\partial R_\mu}{\partial t}\dot{a}^\mu - \frac{\partial B}{\partial t} \Big)\Delta t + \Big(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} \Big)\Delta a^\mu + R_\mu \Delta \dot{a}^\mu \Big]\mathrm{d}t \tag{10}$$

According to the relationship between the simultaneous variation δ and the non-simultaneous variation Δ in calculation, we obtain[8]

$$\Delta a^\mu = \delta a^\mu + \dot{a}^\mu \Delta t \tag{11}$$

$$\Delta \dot{a}^\mu = \delta \dot{a}^\mu + \ddot{a}^\mu \Delta t \tag{12}$$

and notice that

$$\delta \dot{a}^\mu = \frac{\mathrm{d}}{\mathrm{d}t}\delta a^\mu \tag{13}$$

Then Eq. (10) can be written as

$$\Delta A = \int_{t_1}^{t_2} \varepsilon_\alpha \Big\{ \frac{\mathrm{d}}{\mathrm{d}t}(R_\mu \xi_\mu^\alpha - B\xi_0^\alpha) + \Big[\Big(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \Big)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} \Big]\bar{\xi}_\mu^\alpha \Big\} \mathrm{d}t \tag{14}$$

where $\bar{\xi}_\mu^\alpha = \xi_\mu^\alpha - \dot{a}^\mu \xi_0^\alpha$. Eqs. (10) and (14) are two elementary formulae of the variation of the Pfaffian action.

Definition 1 Suppose that the Pfaffian action (7) is a generalized quasi-invariant of the infinitesimal transformations. That is, for every transformation of (8), the following relationship

$$\Delta A = -\int_{t_1}^{t_2} \Big[\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G_{\mathrm{N}}) + \Lambda_\mu \delta a^\mu \Big]\mathrm{d}t \tag{15}$$

always holds, where $\Delta G_{\mathrm{N}} = \varepsilon_\alpha G_{\mathrm{N}}^\alpha(t, a)$. Then the invariance corresponding to the transformations (8) is called a Noether symmetry of the generalized Birkhoffian system.

From definition 1 and Eq. (14) or Eq. (10), we obtain the following criterion.

Criterion 1 For the infinitesimal transformations (8), if the following conditions

$$\Big(\frac{\partial R_\mu}{\partial t}\dot{a}^\mu - \frac{\partial B}{\partial t} \Big)\xi_0^\alpha + \Big(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} \Big)\xi_\mu^\alpha - B\dot{\xi}_0^\alpha + R_\mu \dot{\xi}_\mu^\alpha + \Lambda_\mu(\xi_\mu^\alpha - \dot{a}^\mu \xi_0^\alpha) = -\dot{G}_{\mathrm{N}}^\alpha \qquad \alpha = 1, 2, \ldots, r \tag{16}$$

are verified as being identical, then the invariance corresponding to the transformations is a Noether symmetry of the generalized Birkhoffian system.

From Eqs. (14) and (15), and using Eq. (1), we have the following proposition.

Proposition 1 For the generalized Birkhoffian system (1), if the infinitesimal transformations (8) correspond to the Noether symmetry, then the system has r linearly independent first integrals, such as

$$I_{\mathrm{N}}^\alpha = R_\mu \xi_\mu^\alpha - B\xi_0^\alpha + G_{\mathrm{N}}^\alpha = C^\alpha \qquad \alpha = 1, 2, \ldots, r \tag{17}$$

Proposition 1 can be called the Noether theorem of the generalized Birkhoffian system (1). Using this proposition, we can obtain a Noether conserved quantity (17) from a given Noether symmetry of the generalized Birkhoffian system.

Now we study the inverse theorem of the Noether theorem. Assuming that the generalized Birkhoffian system (1) has r linearly independent first integrals

$$I_{\mathrm{N}}^\alpha = I_{\mathrm{N}}^\alpha(t, a) = C^\alpha \qquad \alpha = 1, 2, \ldots, r \tag{18}$$

we try to determine the corresponding Noether symmetric transformations.

Expanding Eq. (16), making the coefficients of $\dot{a}^\mu$ and the terms excluding $\dot{a}^\mu$ equal to zero respectively, we have

$$\frac{\partial G_{\mathrm{N}}^\alpha}{\partial a^\mu} + R_\nu \frac{\partial \xi_\nu^\alpha}{\partial a^\mu} + \frac{\partial R_\mu}{\partial a^\nu}\xi_\nu^\alpha + \frac{\partial R_\mu}{\partial t}\xi_0^\alpha - B\frac{\partial \xi_0^\alpha}{\partial a^\mu} - \Lambda_\mu \xi_0^\alpha = 0$$
$$\mu = 1, 2, \ldots, 2n \tag{19}$$

$$\frac{\partial G_{\mathrm{N}}^\alpha}{\partial t} + R_\nu \frac{\partial \xi_\nu^\alpha}{\partial t} - \frac{\partial B}{\partial a^\nu}\xi_\nu^\alpha - B\frac{\partial \xi_0^\alpha}{\partial t} - \frac{\partial B}{\partial t}\xi_0^\alpha + \Lambda_\nu \xi_\nu^\alpha = 0 \tag{20}$$

Let the integrals (18) be equal to the Noether conserved quantities (17), i. e.,

$$R_\nu \xi_\nu^\alpha - B\xi_0^\alpha + G_{\mathrm{N}}^\alpha = I_{\mathrm{N}}^\alpha \tag{21}$$

By calculating the partial derivative of Eq. (21) with respect to a^μ and t, and comparing with Eqs. (19) and (20), respectively, we obtain

$$\Big(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \Big)\xi_\nu^\alpha - \Big(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} - \Lambda_\mu \Big)\xi_0^\alpha = \frac{\partial I_N^\alpha}{\partial a^\mu}$$
$$\mu = 1, 2, \ldots, 2n \tag{22}$$

$$\Big(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} - \Lambda_\nu \Big)\xi_\nu^\alpha = \frac{\partial I_{\mathrm{N}}^\alpha}{\partial t} \tag{23}$$

Hence, we have the following proposition.

Proposition 2 If the generalized Birkhoffian system (1) has r linearly independent first integrals (18), then the infinitesimal transformations determined by Eqs. (22), (23) and (21) must correspond to the Noether symmetry of the system.

Example 1 The Birkhoffian, the Birkhoff's functions and the additional item of a generalized Birkhoffian system are

$$\left. \begin{aligned} & B = \frac{1}{2}[(a^1)^2 + (a^3)^2 + (a^4)^2] \\ & R_1 = a^3,\ R_2 = a^4,\ R_3 = R_4 = 0 \\ & \Lambda_1 = a^1,\ \Lambda_2 = 0,\ \Lambda_3 = a^3,\ \Lambda_4 = a^1 - \frac{(a^3)^2}{a^1} \end{aligned} \right\} \tag{24}$$

Eq. (16) can be written as

$$-a^1\xi_1+(\dot a^1-a^3)\xi_3+(\dot a^2-a^4)\xi_4-\frac{1}{2}[(a^1)^2+(a^3)^2+(a^4)^2]\dot\xi_0+a^3\dot\xi_1+a^4\dot\xi_2+a^1(\xi_1-\dot a^1\xi_0)+a^3(\xi_3-\dot a^3\xi_0)+\left[a^1-\frac{(a^3)^2}{a^1}\right](\xi_4-\dot a^4\xi_0)+\dot G_{\mathrm{N}}=0 \tag{25}$$

Eq. (25) has the following solutions

$$\xi_0^1=0,\ \xi_1^1=0,\ \xi_2^1=1,\ \xi_3^1=0,\ \xi_4^1=0,\ G^1=0 \tag{26}$$

$$\left.\begin{array}{l}\xi_0^2=0,\ \xi_1^2=a^3,\ \xi_2^2=a^4,\ \xi_3^2=-a^1,\ \xi_4^2=0\\ G^2=\frac{1}{2}[(a^1)^2-(a^3)^2-(a^4)^2]\end{array}\right\} \tag{27}$$

$$\left.\begin{array}{l}\xi_0^3=0,\ \xi_1^3=-\frac{2a^3}{a^1}t,\ \xi_2^3=-t,\ \xi_3^3=-t-t\left(\frac{a^3}{a^1}\right)^2\\ \xi_4^3=-1,\ G^3=a^2+a^1t+\frac{(a^3)^2}{a^1}t\end{array}\right\} \tag{28}$$

According to criterion 1, the generators (26) to (28) all correspond to the Noether symmetry of the system. Using proposition 1, we obtain

$$I_{\mathrm{N}}^1=a^4=C^1 \tag{29}$$

$$I_{\mathrm{N}}^2=\frac{1}{2}[(a^1)^2+(a^3)^2+(a^4)^2]=C^2 \tag{30}$$

$$I_{\mathrm{N}}^3=a^2+a^1t-a^4t-\frac{(a^3)^2}{a^1}t=C^3 \tag{31}$$

where C^1, C^2 and C^3 are the integral constants.

Next, we study the inverse problem of the Noether symmetry. Assuming that the system has an integral (30), then Eqs. (22), (23) and (21) can be respectively expressed as

$$-\xi_3=a^1,\quad -\xi_4=0,\ \xi_1=a^3,\ \xi_2-\left[a^4-a^1+\frac{(a^3)^2}{a^1}\right]\xi_0=a^4 \tag{32a}$$

$$\left[a^4-a^1+\frac{(a^3)^2}{a^1}\right]\xi_4=0 \tag{32b}$$

$$a^3\xi_1+a^4\xi_2-\frac{1}{2}[(a^1)^2+(a^3)^2+(a^4)^2]\xi_0+G_{\mathrm{N}}=\frac{1}{2}[(a^1)^2+(a^3)^2+(a^4)^2] \tag{32c}$$

If we take

$$G_{\mathrm{N}}=\frac{1}{2}[(a^1)^2-(a^3)^2-(a^4)^2] \tag{33}$$

then we obtain

$$\xi_0=0,\ \xi_1=a^3,\ \xi_2=a^4,\ \xi_3=-a^1,\ \xi_4=0 \tag{34}$$

3 Lie Symmetry of Generalized Birkhoffian Systems

We introduce the infinitesimal transformation

$$t^*=t+\varepsilon\xi_0(t,\ a),\ a^{\mu *}=a^\mu+\varepsilon\xi_\mu(t,\ a)\qquad \mu=1,2,\ \ldots,\ 2n \tag{35}$$

and take the infinitesimal generator vector

$$\boldsymbol{X}^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_\mu\frac{\partial}{\partial a^\mu} \tag{36}$$

and its first extension vector

$$\boldsymbol{X}^{(1)}=\boldsymbol{X}^{(0)}+(\dot\xi_\mu-\dot a^\mu\dot\xi_0)\frac{\partial}{\partial\dot a^\mu} \tag{37}$$

By using the invariance of the ordinary differential equations under the infinitesimal transformation[22], the Lie symmetry of the generalized Birkhoffian system is an invariance of the generalized Birkhoff's equations under the infinitesimal transformation (35).

Definition 2 If and only if

$$\boldsymbol{X}^{(1)}\left[\dot a^\mu-\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}-\Lambda_\nu\right)\right]\equiv 0\qquad \mu=1,2,\ldots,2n \tag{38}$$

when

$$\dot a^\mu=\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}-\Lambda_\nu\right)$$

then the invariance corresponding to transformation (35) is called a Lie symmetry of the generalized Birkhoffian system.

From definition 2 and the vector (37), we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu-h_\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0=\boldsymbol{X}^{(0)}(h_\mu)\qquad \mu=1,2,\ldots,2n \tag{39}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+h_\mu\frac{\partial}{\partial a^\mu} \tag{40}$$

Eq. (39) is called the determining equation of the Lie symmetry of the generalized Birkhoffian system.

Criterion 2 If the generators ξ_0, ξ_μ of the infinitesimal transformation (35) satisfy the determining equation (39), then the corresponding invariance is called a Lie symmetry of the generalized Birkhoffian system.

Hojman[23] presented a conservation theorem in which the conserved quantity is constructed in terms of a special Lie symmetry transformation (with $\Delta t=0$) only, without using either Lagrangian or Hamiltonian structure. For the generalized Birkhoffian system, the general Lie symmetry transformation (with $\Delta t\neq 0$) of the system may also directly lead to a Hojman conserved quantity, and we have the following proposition.

Proposition 3 For the generalized Birkhoffian system (1), if the infinitesimal transformation (35) corresponds to a Lie symmetry of the system, and there exists some function $\lambda=\lambda(t,\ a)$ satisfying the following condition

$$\frac{\partial h_\mu}{\partial a^\mu}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\lambda=0 \tag{41}$$

then the Lie symmetry of the system can directly lead to a Hojman conserved quantity, such as

$$I_{\mathrm{H}}=\frac{1}{\lambda}\frac{\partial}{\partial t}(\lambda\xi_0)+\frac{1}{\lambda}\frac{\partial}{\partial a^\mu}(\lambda\xi_\mu)-\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0=C \tag{42}$$

Using Eq. (39) and Eq. (41), we can easily verify proposition 3.

Example 2 Consider a generalized Birkhoffian system, such as

$$B = t\,[(a^1)^2 + (a^2)^2],\ R_1 = ta^2,\ R_2 = -ta^1$$
$$\Lambda_1 = a^2,\ \Lambda_2 = -a^1 \tag{43}$$

Eq. (39) can be expressed as

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 - a^2\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \xi_2,\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 + a^1\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = -\xi_1 \tag{44}$$

Eq. (44) has a solution

$$\xi_0 = a^1\sin t,\ \xi_1 = -(a^2)^2\cos t,\ \xi_2 = a^1a^2\cos t \tag{45}$$

The condition (41) can be written as

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\lambda = 0 \tag{46}$$

Eq. (46) has a solution

$$\lambda = 1 \tag{47}$$

Substituting Eq. (45) and Eq. (47) into Eq. (42), we obtain

$$I_{\mathrm{H}} = a^1\cos t - a^2\sin t = C \tag{48}$$

The conserved quantity (48) is a Hojman conserved quantity directly derived from the Lie symmetry (45) of the system.

4 Mei Symmetry of Generalized Birkhoffian Systems

Mei et al. [18, 24] presented a new invariance of the Largrangian system which is called the form invariance. The form invariance or the Mei symmetry is different from the Noether symmetry, and also different from the Lie symmetry. For the generalized Birkhoffian system, the Mei symmetry can be defined as an invariance that the dynamical functions $B(t,\ a)$, $R_\mu(t,\ a)$, $\Lambda_\mu(t,\ a)$ appearing in the generalized Birkhoff's equation (1) still satisfy the original equations after undergoing the infinitesimal transformation (35).

Under the infinitesimal transformation (35), there are

$$\left.\begin{aligned} B^* &= B(t^*,\ a^*) = B(t,\ a) + \varepsilon\,\boldsymbol{X}^{(0)}(B) + O(\varepsilon^2)\\ R_\mu^* &= R_\mu(t^*,\ a^*) = R_\mu(t,\ a) + \varepsilon\,\boldsymbol{X}^{(0)}(R_\mu) + O(\varepsilon^2)\\ \Lambda_\mu^* &= \Lambda_\mu(t^*,\ a^*) = \Lambda_\mu(t,\ a) + \varepsilon\,\boldsymbol{X}^{(0)}(\Lambda_\mu) + O(\varepsilon^2)\end{aligned}\right\} \tag{49}$$

Hence we have the following definition.

Definition 3 Under the infinitesimal transformation (35), if the form of the generalized Birkhoff's equation (1) remains invariant, that is

$$\left(\frac{\partial R_\nu^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial t} = -\Lambda_\mu^*$$
$$\mu = 1,\ 2,\ \ldots,\ 2n \tag{50}$$

then the corresponding invariance is called a Mei symmetry of the generalized Birkhoffian system.

Substituting Eq. (49) into Eq. (50), ignoring ε^2 and higher-order infinitesimal terms, and using Eq. (1), we obtain

$$\left(\frac{\partial \boldsymbol{X}^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial \boldsymbol{X}^{(0)}(R_\mu)}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial \boldsymbol{X}^{(0)}(B)}{\partial a^\mu} - \frac{\partial \boldsymbol{X}^{(0)}(R_\mu)}{\partial t} =$$
$$-\boldsymbol{X}^{(0)}(\Lambda_\mu) \qquad \mu = 1, 2, \ldots, 2n \tag{51}$$

Eq. (51) is called the determining equation of the Mei symmetry for the generalized Birkhoffian system.

Criterion 3 If the generators ξ_0, ξ_μ of the infinitesimal transformation (35) satisfy the determining equation (51), then the corresponding invariance is called a Mei symmetry of the generalized Birkhoffian system.

For the generalized Birkhoffian system, the Mei symmetry can directly lead to a new conserved quantity which we call a Mei conserved quantity.

Proposition 4 For the generalized Birkhoffian system (1), if the infinitesimal transformation (35) corresponds to a Mei symmetry of the system, and there exists a gauge function $G_{\mathrm{M}} = G_{\mathrm{M}}(t,\ a)$ satisfying the structure equation,

$$[\boldsymbol{X}^{(0)}(R_\mu)\dot{a}^\mu - \boldsymbol{X}^{(0)}(B)]\dot{\xi}_0 + \boldsymbol{X}^{(1)}[\boldsymbol{X}^{(0)}(R_\mu)\dot{a}^\mu - \boldsymbol{X}^{(0)}(B)] +$$
$$\boldsymbol{X}^{(0)}(\Lambda_\mu)(\xi_\mu - \dot{a}^\mu\xi_0) = -\dot{G}_{\mathrm{M}} \tag{52}$$

then the Mei symmetry of the system can directly lead to a Mei conserved quantity, such as

$$I_{\mathrm{M}} = \boldsymbol{X}^{(0)}(R_\mu)\xi_\mu - \boldsymbol{X}^{(0)}(B)\xi_0 + G_{\mathrm{M}} = C \tag{53}$$

Using the determining equation (51) and the structure equation (52), we can easily obtain the Mei conserved quantity (53).

Example 3 The generalized Birkhoffian system is

$$\left.\begin{aligned} &B = \frac{1}{6}(a^1)^6 + \frac{1}{2}(a^2)^2,\ R_1 = a^2,\ R_2 = 0\\ &\Lambda_1 = -\frac{2}{t}a^2,\ \Lambda_2 = 0\end{aligned}\right\} \tag{54}$$

The differential equation (1) of motion can be written as

$$\dot{a}^1 = a^2,\ \dot{a}^2 = -(a^1)^5 - \frac{2}{t}a^2 \tag{55}$$

The determining equation (51) can be written as

$$\left.\begin{aligned} &\frac{\partial\xi_2}{\partial a^2}\dot{a}^2 + 5(a^1)^4\xi_1 + (a^1)^5\frac{\partial\xi_1}{\partial a^1} + a^2\frac{\partial\xi_2}{\partial a^1} + \frac{\partial\xi_2}{\partial t} =\\ &\quad\frac{2}{t^2}a^2\xi_0 - \frac{2}{t}\xi_2\\ &\frac{\partial\xi_2}{\partial a^2}\dot{a}^1 - (a^1)^5\frac{\partial\xi_1}{\partial a^2} - \xi_2 - a^2\frac{\partial\xi_2}{\partial a^2} = 0\end{aligned}\right\} \tag{56}$$

The structure equation (52) can be written as

$$-(a^1)^5\xi_1\dot{\xi}_0 + \boldsymbol{X}^{(1)}[\dot{a}^1\xi_2 - (a^1)^5\xi_1 - a^2\xi_2] +$$
$$\left(\frac{2}{t^2}a^2\xi_0 - \frac{2}{t}\xi_2\right)(\xi_1 - \dot{a}^1\xi_0) + \dot{G}_{\mathrm{M}} = 0 \tag{57}$$

Eq. (56) has a solution

$$\xi_0 = 0,\ \xi_1 = -2ta^1 - 3t^2a^2,\ \xi_2 = 3t^2(a^1)^5 \tag{58}$$

According to criterion 3, the generators (58) correspond to a Mei symmetry of the system. Substituting the generators (58) into Eq. (57), we can obtain

$$G_{\mathrm{M}} = 9t^4(a^1)^5 a^2 - 18t^2 a^1 a^2 - 18t^3(a^2)^2 \tag{59}$$

Substituting the generators (58) and Eq. (59) into Eq. (53), we obtain

$$I_{\mathrm{M}} = -6t^3(a^1)^6 - 18t^2 a^1 a^2 - 18t^3(a^2)^2 = C \tag{60}$$

The conserved quantity (60) is a Mei conserved quantity directly led by the Mei symmetry of the system.

References

[1] Birkhoff G D. *Dynamical systems* [M]. Providence, RI, USA: AMS College Publication, 1927.
[2] Santilli R M. *Foundations of theoretical mechanics* Ⅱ [M]. New York: Springer-Verlag, 1983.
[3] Mei F X. Noether theory of Birkhoffian system [J]. *Science in China Series A*: *Mathematic*, 1993, **36** (12): 1456 - 1547.
[4] Mei F X. Stability of equilibrium for the autonomous Birkhoff system [J]. *Chin Sci Bull*, 1993, **38**(10): 816 - 819.
[5] Mei F X, Lévesque E I. Generalized canonical realization and Birkhoff's realization of Chaplygin's nonholonomic system [J]. *Transactions of the CSME*, 1995, **19**(2): 59 - 73.
[6] Mei F X, Shi R C, Zhang Y F, et al. *Dynamics of Birkhoffian systems* [M]. Beijing: Beijing Institute of Technology Press, 1996. (in Chinese)
[7] Mei F X, Zhang Y F, Shang M. Lie symmetries and conserved quantities of Birkhoffian system [J]. *Mech Res Commun*, 1999, **26**(1): 7 - 12.
[8] Mei F X. *Applications of Lie groups and Lie algebras to constrained mechanical systems* [M]. Beijing: Science Press, 1999. (in Chinese)
[9] Zhu H P, Wu J K. Generalized canonical transformations and symplectic algorithm of autonomous Birkhoffian systems [J]. *Progr Nat Sci*, 1999, **9**(11): 820 - 828.
[10] Mei F X, Zhang Y F, Shi R C. Dynamics algebra and its application [J]. *Acta Mech*, 1999, **137**(3/4): 255 - 260.
[11] Mei F X. On the Birkhoffian mechanics [J]. *Int J Nonlinear Mech*, 2001, **36**(5): 817 - 834.
[12] Guo Y X, Luo S K, Shang M, et al. Birkhoffian formulations of nonholonomic constrained systems [J]. *Rep Math Phys*, 2001, **47**(3): 313 - 322.
[13] Zhang Y. Construction of the solution of variational equations for constrained Birkhoffian systems [J]. *Chin Phys*, 2002, **11**(5): 437 - 440.
[14] Zhang Y. A set of conserved quantities from Lie symmetries for Birkhoffian systems [J]. *Acta Phys Sinica*, 2002, **51** (3): 461 - 465. (in Chinese)
[15] Luo S K. First integrals and integral invariants of relativistic Birkhoffian systems [J]. *Commun Theor Phys*, 2003, **40** (2): 133 - 136.
[16] Chen X W. Closed orbits and limit cycles of second-order autonomous Birkhoff system [J]. *Chin Phys*, 2003, **12**(6): 586 - 589.
[17] Zhang Y, Fan C X, Ge W K. A new type of conserved quantities for Birkhoffian systems [J]. *Acta Phys Sinica*, 2004, **53**(11): 3644 - 3647. (in Chinese)
[18] Mei F X. *Symmetries and conserved quantities of constrained mechanical systems* [M]. Beijing: Beijing Institute of Technology Press, 2004. (in Chinese)
[19] Galiullin A S. *Analytical dynamics* [M]. Moscow: Nauka, 1989. (in Russian)
[20] Galiullin A S, Gafarov G G, Malaishka R P, et al. *Analytical dynamics of Helmholtz, Birkhoff and Nambu systems* [M]. Moscow: UFN, 1997. (in Russian)
[21] Mei F X, Zhang Y F, He G, et al. Fundamental framework of generalized Birkhoff system dynamics [J]. *Transactions of Beijing Institute of Technology*, 2007, **27** (12): 1035 - 1038. (in Chinese)
[22] Bluman G W, Kumei S. *Symmetries and differential equations* [M]. New York: Springer-Verlag, 1989.
[23] Hojman S A. A new conservation law constructed without using either Lagrangians or Hamiltonians [J]. *J Phys A*: *Math Gen*, 1992, **25**(7): L291 - L295.
[24] Mei F X. Form invariance of Lagrange system [J]. *J Beijing Institute of Technology*, 2000, **9**(2): 120 - 124.

广义 Birkhoff 系统的对称性与守恒量

张 毅

(苏州科技学院土木工程学院，苏州 215011)

摘要：研究了广义 Birkhoff 系统的 3 种对称性及其相应的守恒量. 首先，基于 Pfaffian 作用量在无限小变换下的不变性，建立了广义 Birkhoff 系统的 Noether 理论；其次，基于微分方程在无限小变换下的不变性，建立了广义 Birkhoff 系统的 Lie 对称性的定义和判据，给出了由系统的 Lie 对称性直接导致的 Hojman 守恒量；最后，基于力学系统运动微分方程中出现的动力学函数在经历无限小变换后仍然满足原来方程的一种不变性，建立了广义 Birkhoff 系统的 Mei 对称性的定义和判据，给出了由系统的 Mei 对称性直接导致的 Mei 守恒量. 举例说明了结果的应用.

关键词：广义 Birkhoff 系统；对称性；守恒量

中图分类号：O316