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Precise large deviation result for heavy-tailed random sums and applications to risk theory

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Abstract: The differences between two sequences of nonnegative independent and identically distributed random variables with sub-exponential tails and the random index are studied. The random index is a strictly stationary renewal counting process generated by some negatively associated random variables. Using a revised large deviation result of partial sums, the elementary renewal theorem and the central limit theorem of negatively associated random variables, a precise large deviation result is derived for the random sums. The result is applied to the customer-arrival-based insurance risk model. Some uniform asymptotics for the ruin probabilities of an insurance company are obtained as the number of customers or the time tends to infinity.

Key words: precise large deviation; random sum; sub-exponential distribution; renewal counting process; customer-arrival-based insurance risk model

1 Main Result

Precise large deviations describe some results on the asymptotics for some rare event probabilities, which is an important branch of applied probability. There are many applications such as insurance risk and queueing theory among others in this area in the last decade. Much attention has been paid to heavy-tailed random sums. Embrechts et al. [1] pointed out that random sums are the bread and butter of insurance and mathematics. Some earlier works on large deviations for random sums can be found in Refs. [2 – 3].

Throughout this paper, let $\{X_k, k\geqslant 1\}$ be a sequence of independent and identically distributed nonnegative random variables with common distribution B satisfying its tailed distribution $\overline{B}(x)=1-B(x)>0$ for all $x\geqslant 0$, $\{Y_k, k\geqslant 1\}$ is also a sequence of independently and identically distributed nonnegative random variables. Denote the differences by $Z_k = X_k - Y_k (k\geqslant 1)$ with distribution F and finite mean $\mu<0$. Denote the n-th partial sum (non-random sum) by $S_n = \sum_{k=1}^{n} Z_k (n\geqslant 1)$. Let $\{N(t), t\geqslant 0\}$ be a strictly stationary renewal counting process generated by nonnegative negatively associated random variables $\{T_k, k\geqslant 1\}$, that is

$$N(t) = \sup\left\{n: \sum_{k=0}^{n} T_k \leqslant t\right\}$$

satisfying $ET_1=\lambda^{-1}>0$, $0<\operatorname{var}T_1=\sigma^2<\infty$, and

$$0 < \operatorname{var}T_1 + 2\sum_{n=2}^{\infty}\operatorname{cov}(T_1, T_n) < \infty \tag{1}$$

Negative association is one of the reasonable dependence structures in practice, which was introduced in Ref. [4]. Random variables are called negatively associated (NA), if for each $k\geqslant 2$ and any disjoint subsets A_1, A_2 of $\{1, 2, \dots, k\}$,

$$\operatorname{cov}(f_1(\xi_i, i\in A_1), f_2(\xi_j, j\in A_2))\leqslant 0$$

where f_1 and f_2 are any two nondecreasing functions such that the covariance exists. Assume that $\{X_k, k\geqslant 1\}$, $\{Y_k, k\geqslant 1\}$ and $\{N(t), t\geqslant 0\}$ are mutually independent. In this paper, we are interested in the precise large deviations for the random sums $S_{N(t)} = \sum_{k=1}^{N(t)} Z_k$ under the assumption that the distribution B is heavy-tailed. An important class of heavy-tailed distributions is the sub-exponential distribution class $\mathscr{S}$. By definition, a distribution V on $(-\infty, \infty)$ belongs to the class $\mathscr{S}$, if $\lim_{x\to\infty}\overline{V^{*2}}(x)/\overline{V}(x)=2$, where V^{*2} denotes the second Stieltjes convolution.

Kluppelberg et al. [2, 5–6] studied the precise large deviations for random sums in some subclasses of the class $\mathscr{S}$. Recently, Baltrunas et al. [7] derived a precise large deviation result for nonnegative random variables. This result is required to be reconsidered since it used an important result of partial sums only for the random variables with a negative mean (see Ref. [8], theorem 4.1); however, its result is for nonnegative random variables. In this paper, we establish a precise large deviation for the random sums of the differences between two sequences of nonnegative random variables, where the random index $N(t)$ is a renewal counting process generated by nonnegative NA random variables.

Hereafter, all the limit relationships hold for t tending to ∞ unless stated otherwise. For two positive functions $a(t)$ and $b(t)$, we write $a(t)\sim b(t)$ if $\lim a(t)/b(t)=1$ and $a(t)=o(b(t))$ if $\lim a(t)/b(t)=0$. Furthermore, for two positive bivariate functions $a(t, x)$ and $b(t, x)$, we write $a(t, x)\sim b(t, x)$ uniformly for all x in a nonempty set Δ, if

$$\lim_{t\to\infty}\sup_{x\in\Delta}\left|\frac{a(t, x)}{b(t, x)}-1\right| = 0$$

Asymptotic formulae that hold with such a uniformity feature are usually of higher theoretical and practical interest.

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The indicator function of an event A is denoted by I_A.

To formulate our main results, we need to introduce some notations and assumptions. Let $Q(u) = -\log\bar{B}(u), u\geqslant 0$ be the hazard function of the distribution B. Assume that there exists a nonnegative function q such that $Q(u) = Q(0) + \int_0^u q(v)\,\mathrm{d}v,\ u\geqslant 0$, which is called the hazard rate of B. Denote the hazard ratio index by $r\equiv\lim\sup tq(t)/Q(t)$. We need the following essential conditions for our purpose.

Condition 1 Assume that Y has a finite second moment, and the distribution B is absolutely continuous and satisfies 1) $r<1/2$; 2) $\lim\inf tq(t)>2$; 3) $\kappa\equiv\sup\{k: EX^k<\infty\}>\alpha(r)$, where

$$\alpha(r)=\begin{cases}2 & \text{if } r=0 \text{ or } \kappa=\infty\\ \dfrac{2+\sqrt{2}}{1-r} & \text{if } 0<r<1 \text{ and } \kappa<\infty\end{cases}$$

Condition 1 is a modification of condition B in Ref. [8], which plays an important role in proving the precise large deviations result for partial sums (see lemma 1 below). Condition 1 is a mild one, which can be satisfied by many common distributions such as the Weibull distribution $\bar{B}(x) = \exp\{-cx^\tau\}, x\geqslant 0, c>0,\ 0<\tau<1/2$, where $r=\tau<1/2$, $\lim tq(t)=\infty>2$ and $\kappa=\infty>\alpha(r)=2$.

The following theorem is our main result.

Theorem 1 Assume that condition 1 and formula (1) hold, then for any $\gamma>0$ and any $p>(1-r)^{-1}>1$,

$$\lim_{t\to\infty}\sup_{x\geqslant\gamma t^p}\left|\frac{P(S_{N(t)}-\mu\lambda t>x)}{\lambda t\bar{B}(x)}-1\right|=0 \tag{2}$$

The proof of theorem 1 will be given in section 2. In section 3, we obtain some applications in the customer-arrival-based insurance risk model (CIRM).

2 Proof of Theorem 1

In this sequel, the constant C always represents a positive constant, which may vary from different places. Before proving theorem 1, we require some lemmas.

We first give an important precise large deviation for partial sums, which is originally due to Ref. [8]. However, there exist some gaps in the proof of theorem 4.1 in Ref. [8], where it missed the summand $P(Z_1-\mu>y)$; the inequality, on page 252 in the last line, is incorrect; and the severe problem is that the inequality to estimate I_{11}, on page 253 line 14, is incorrect. So, we modify it as follows. One can refer to Ref. [9] for the detailed proof.

Lemma 1 Assume that condition 1 holds, then

$$\lim_{n\to\infty}\sup_{x\geqslant t_n}\left|\frac{P(S_n-n\mu>x)}{n\bar{B}(x)}-1\right|=0$$

holds for any sequence $\{t_n,\ n\geqslant 1\}$ satisfying $\lim_{n\to\infty}\sqrt{n}\cdot\sup_{u\geqslant t_n}Q(u)/u=0$.

The following lemma describes the relationships among the hazard ratio index, the class $\mathscr{S}$ and the hazard function (see Refs. [7-8]).

Lemma 2 If $r<1$, then 1) $B\in\mathscr{S}$; 2) $Q(u)/u$ decreases for sufficiently large u; 3) For any $\varepsilon>0$, there exist positive u_ε and c_ε such that $Q(u)\leqslant c_\varepsilon u^{r+\varepsilon}$ for $u\geqslant u_\varepsilon$.

As for the NA renewal counting process $N(t)$, Refs. [10-11] obtained the elementary renewal theorem and the central limit theorem, respectively.

Lemma 3 Let the NA counting process $N(t)$ be the same as in section 1. Then 1) $N(t)/(\lambda t)\xrightarrow{\text{a. s.}}1$ and $EN(t)/(\lambda t)\to 1$; 2) If formula (1) is satisfied, then $(N(t)-\lambda t)/(\sigma\sqrt{\lambda^3 t})\xrightarrow{d}N(0,1)$.

In order to prove theorem 1, we divide $P(S_{N(t)}-\mu\lambda t>x)$ into three parts

$$P(S_{N(t)}-\mu\lambda t>x)=\sum_{n=1}^{\infty}P(S_n-\mu\lambda t>x)P(N(t)=n)=\left(\sum_{|n-\lambda t|\leqslant\varepsilon(t)\lambda t}+\sum_{n<(1-\varepsilon(t))\lambda t}+\sum_{n>(1+\varepsilon(t))\lambda t}\right)P(S_n-\mu\lambda t>x)P(N(t)=n)\equiv I_1+I_2+I_3 \tag{3}$$

where $\varepsilon(t)$ is some positive function satisfying $\varepsilon(t)\to 0$ and $t\varepsilon^2(t)\to\infty$. We provide a series of lemmas below to prove theorem 1.

Lemma 4 Assume that condition 1 and formula (1) hold. Let $\varepsilon(t)$ be any positive function satisfying $\varepsilon(t)\to 0$ and $t\varepsilon^2(t)\to\infty$. Then for any $\gamma>0$ and $p>(1-r)^{-1}$, $I_2=o(\lambda t\bar{B}(x))$ holds uniformly for all $x\geqslant\gamma t^p$.

Proof For any $\varepsilon>0$, by lemma 1, there exists a sufficiently large integer $n_0>0$, such that for all $u\geqslant t_n$ and $n\geqslant n_0$,

$$\left|\frac{P(S_n-n\mu>u)}{n\bar{B}(u)}-1\right|\leqslant\varepsilon \tag{4}$$

Now we divide I_2 into two parts,

$$I_2=\left(\sum_{n\leqslant n_0}+\sum_{n_0<n\leqslant(1-\varepsilon(t))\lambda t}\right)P(S_n-\mu\lambda t>x)P(N(t)=n)\equiv I_{21}+I_{22} \tag{5}$$

We first estimate I_{21}. By $r<1$ and lemma 2, we obtain $B\in\mathscr{S}$. It holds from lemma 4.3(b) in Ref. [8] that $\bar{F}(t)\sim\bar{B}(t)$, and so $F\in\mathscr{S}$. By $p>(1-r)^{-1}>1$, we choose $\varepsilon>0$ small enough such that $r+\varepsilon<1/2$ and $p(r+\varepsilon-1)+1<0$. Hence, by lemma 2, there exists some constant $x_0\in(x+\mu\lambda t, x)$ such that

$$1\leqslant\frac{\bar{B}(x+\mu\lambda t)}{\bar{B}(x)}=\exp\left\{\int_{x+\mu\lambda t}^{x}q(u)\,\mathrm{d}u\right\}\leqslant \exp\left\{|\mu|\lambda t(r+\varepsilon)\frac{Q(x+\mu\lambda t)}{x+\mu\lambda t}\right\}\leqslant\exp\left\{Ct\,\frac{Q(\gamma t^p+\mu\lambda t)}{\gamma t^p+\mu\lambda t}\right\}=\exp\{Ct(\gamma t^p+\mu\lambda t)^{r+\varepsilon-1}\}\to 1 \tag{6}$$

By $F\in\mathscr{S}$, $\bar{F}(t)\sim\bar{B}(t)$ and formula (6), we obtain that

$$I_{21}\sim\bar{F}(x+\mu\lambda t)\sum_{n\leqslant n_0}nP(N(t)=n)\sim\bar{B}(x+\mu\lambda t)EN(t)I_{\{N(t)\leqslant n_0\}}\leqslant n_0\bar{B}(x)=o(\lambda t\bar{B}(x)) \tag{7}$$

holds uniformly for all $x\geqslant\gamma t^p$.

We finally estimate I_{22}. By $t\varepsilon^2(t)\to\infty$ and lemma 3, we obtain

$$\lim P(|N(t)-\lambda t|>\varepsilon(t)\lambda t)=\lim P\left(\left|\frac{N(t)-\lambda t}{\sigma\sqrt{\lambda^3 t}}\right|>\sqrt{\frac{t\varepsilon^2(t)}{\lambda\sigma^2}}\right)=0 \tag{8}$$

Note that $x+\mu\lambda t-n\mu\geqslant(\gamma t^p+\mu\lambda t)+n|\mu|\geqslant t_n$. Thus, by formulae (4), (6) and (8), we obtain that

$$I_{22}\leqslant(1+\varepsilon)\sum_{n_0<n\leqslant(1-\varepsilon(t))\lambda t}n\bar{B}(x+\mu\lambda t-n\mu)P(N(t)=n)\leqslant (1+\varepsilon)(1-\varepsilon(t))\lambda t\bar{B}(x)P(n_0<N(t)\leqslant(1-\varepsilon(t))\lambda t)=o(\lambda t\bar{B}(x)) \quad (9)$$

holds uniformly for all $x\geqslant\gamma t^p$. Therefore, the desired result follows from formulae (7) and (9).

Lemma 5 Assume that condition 1 and formula (1) hold. Let $\varepsilon(t)$ be any positive function satisfying $\varepsilon(t)\to0$ and $t\varepsilon^2(t)\to\infty$. Then for any $\tilde{\gamma}>|\mu|$, $I_3=o(\lambda t\bar{B}(x))$ holds uniformly for all $x\geqslant\tilde{\gamma}t$.

Proof By lemma 3 and the dominant convergence theorem, we obtain $\lim(\lambda t)^{-1}E(N(t)I_{\{N(t)\leqslant(1+\delta)\lambda t\}})=1$ for any $\delta>0$, which implies

$$E(N(t)I_{\{N(t)>(1+\delta)\lambda t\}})=o(\lambda t) \quad (10)$$

Since $x+\mu\lambda t-n\mu\geqslant(\tilde{\gamma}+\mu)\lambda t+n|\mu|>t_n$, by lemma 1, formulae (10) and (8), we obtain that

$$I_3\sim\sum_{n>(1+\varepsilon(t))\lambda t}n\bar{B}(x+\mu\lambda t-n\mu)P(N(t)=n)\leqslant \bar{B}(x)(EN(t)I_{\{N(t)>(1+\delta)\lambda t\}}+EN(t)I_{\{(1+\varepsilon(t))\lambda t<N(t)\leqslant(1+\delta)\lambda t\}})\leqslant o(\lambda t\bar{B}(x))+(1+\delta)\lambda t\bar{B}(x)P((1+\varepsilon(t))\lambda t<N(t)\leqslant(1+\delta)\lambda t)=o(\lambda t\bar{B}(x))$$

holds uniformly for all $x\geqslant\tilde{\gamma}t$.

Lemma 6 Assume that condition 1 and formula (1) hold. Let $\varepsilon(t)=c_1\log t/\sqrt{t}$ with $c_1>0$. Then for any $\tilde{\gamma}>0$, $I_1\sim\lambda t\bar{B}(x)$ holds uniformly for all $x\geqslant\tilde{\gamma}t$.

Proof By lemma 1, uniformly for all $|n-\lambda t|\leqslant\varepsilon(t)\lambda t$ and $x\geqslant\tilde{\gamma}t$, we have

$$P(S_n-\mu\lambda t>x)\sim n\bar{B}(x+\mu\lambda t-n\mu) \quad (11)$$

If $(n-\lambda t)\mu\geqslant0$, then, analogously to the proof of (6), for any $\varepsilon>0$ satisfying $r+\varepsilon<1/2$, by lemma 2, for sufficiently large t, we obtain

$$1\leqslant\frac{\bar{B}(x+\mu\lambda t-n\mu)}{\bar{B}(x)}\leqslant \exp\left\{(r+\varepsilon)c_1|\mu|\lambda\sqrt{t}\log t\,\frac{Q(\gamma\lambda t/2)}{\gamma\lambda t/2}\right\}\leqslant \exp\{Ct^{r+\varepsilon-1/2}\log t\}\to1 \quad (12)$$

If $(n-\lambda t)\mu<0$, the proof of (12) is similar. Hence, by formulae (11), (12), (8) and the dominant convergence theorem, we obtain that

$$I_1\sim\bar{B}(x)\sum_{|n-\lambda t|\leqslant\varepsilon(t)\lambda t}nP(N(t)=n)\sim \lambda t\bar{B}(x)P(|N(t)-\lambda t|\leqslant\varepsilon(t)\lambda t)\sim\lambda t\bar{B}(x)$$

holds uniformly for all $x\geqslant\tilde{\gamma}t$.

Notice that by $p>(1-r)^{-1}>1$, for any $\gamma>0$, lemmas 5 and 6 also hold uniformly for all $x\geqslant\gamma t^p$. Therefore, theorem 1 follows from formula (3), lemmas 4, 5 and 6.

3 Application to Customer-Arrival-Based Insurance Risk Model

In this section, we start by introducing the CIRM, which satisfies the following requirements:

Assumption 1 The customer-inter-arrival times $\{T_k, k\geqslant1\}$ form a sequence of strictly stationary nonnegative NA random variables. Denote the customer-arrival counting process by $N(t)=\sup\{n\geqslant0:\sigma_n\leqslant t\}$, where $\sigma_n=\sum_{k=1}^{n}T_k$ represents the arrival time of the n-th customer, by convention $\sigma_0=0$. Assume that $ET_1=\lambda^{-1}<\infty$, $0<\mathrm{var}T_1<\infty$.

Assumption 2 At the time σ_n, the n-th customer purchases an insurance policy. Assume that an insurance period lasts τ. Then during an insurance period τ, the insurance company has a potential risk of payment.

Assumption 3 The potential claims $\{X_k, k\geqslant1\}$, independent of $\{T_k, k\geqslant1\}$, are nonnegative independent and identically distributed random variables with common distribution B and finite mean μ_B. The price of an insurance policy is $(1+\rho)\mu_B$, where the positive constant ρ is interpreted as a relative safety loading. The net loss of the n-th customer is $X_n-(1+\rho)\mu_B$.

Denote the risk reserve process up to time $t\geqslant0$ by $R(x,t)=x-W(t)$, where x is the initial capital reserve and the claim surplus process $W(t)$ is defined as

$$W(t)=\sum_{k=1}^{N(t)}(X_k-(1+\rho)\mu_B) \qquad t\geqslant0$$

In the discrete case, the claim surplus process can be rewritten as

$$W_n=\sum_{k=1}^{n}(X_k-(1+\rho)\mu_B) \qquad n\geqslant1$$

This model is introduced in Ref. [12], which investigated the independent case. In the independence structure, one can also refer to Ref. [13] for some recent works. Clearly, lemma 1 and theorem 1 lead to some precise large deviation results for the processes W_n and $W(t)$ in the CIRM.

Theorem 2 In the CIRM, 1) Assume that condition 1 holds, then for any $\gamma>0$

$$\lim_{n\to\infty}\sup_{x\geqslant\gamma n}\left|\frac{P(W_n>x)}{n\bar{B}(x+\rho n\mu_B)}-1\right|=0$$

2) Assume that condition 1 and formula (1) hold, then for any $\gamma>0$ and $p>(1-r)^{-1}>1$

$$\lim_{t\to\infty}\sup_{x\geqslant\gamma t^p}\left|\frac{P(W(t)>x)}{\lambda t\bar{B}(x+\rho\lambda\mu_B t)}-1\right|=0$$

We notice that the ruin probability of an insurance company occurring at the time t can be represented as $P(R(x,t)<0)=P(W(t)>x)$. Hence, theorem 2 describes the uniform asymptotics for the ruin probabilities of an insurance company as the number of customers tends to infinity and the time tends to infinity, respectively.

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重尾随机和的精致大偏差及其在风险理论中的应用

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摘要:对2列非负带有次指数分布的独立同分布随机变量的差,以及随机脚标为负相协随机变量生成的严平稳更新记数过程进行了探讨.利用修正的随机变量部分和的精致大偏差结果及关于负相协随机变量的基本更新定理和中心极限定理,得到了随机变量列差的随机和的精致大偏差.考虑了基于顾客来到过程的保险风险模型,利用随机和的精致大偏差结果,得到了当顾客数或者时间趋于无穷时,保险公司破产概率的一致渐近性.

关键词:精致大偏差;随机和;次指数分布;更新记数过程;基于顾客来到过程的保险风险模型
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