Journal of Southeast University (English Edition) Vol. 26, No. 3, pp. 502 – 504 Sept. 2010 ISSN 1003—7985

Biharmonic product maps between doubly warped product manifolds

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Abstract: The biharmonicity of the product map $\Phi_2 = \varphi \times \psi$ and the two generalized projections $\bar{\varphi}$ and $\bar{\psi}$ are analyzed. Some results are obtained, that is, Φ_2 is a proper biharmonic map if and only if b is a non-constant solution of $-\frac{1}{f^2}J_\varphi(\mathrm{d}\varphi(\mathrm{grad}(\ln b))) + \frac{n}{2}\mathrm{grad}\,|\mathrm{d}\varphi(\mathrm{grad}(\ln b))|^2 = 0$ and f is a non-constant solution of $-\frac{1}{b^2}J_\psi(\mathrm{d}\psi(\mathrm{grad}(\ln f))) + \frac{m}{2}\mathrm{grad}\,|\mathrm{d}\psi(\mathrm{grad}(\ln f))|^2 = 0$, and $\Phi_2 = \varphi \times \psi$ is a proper biharmonic map if and only if $\bar{\varphi}$ and $\bar{\psi}$ are proper biharmonic maps.

Key words: biharmonic map; product map; doubly warped product manifolds

As a natural generalization of harmonic maps, the biharmonic map introduced by Jiang[1-2], who followed the idea of Eells and Sampson[3], is a critical point of bienergy,

$$E_2 = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$$

where $\phi: (M, g) \to (N, h)$ is a smooth map between Riemannian manifolds and $\tau(\phi)$ is the first tension field $\mathrm{tr}_g \nabla \mathrm{d}\phi$.

The biharmonic map can also be characterized by the vanishing of the second tension field $\tau_2(\phi) = -J_\phi(\tau(\phi)) = -\Delta^\phi \tau(\phi) - \mathrm{tr}_g R^N(\mathrm{d}\phi, \tau(\phi))\mathrm{d}\phi = 0$, where J_ϕ is formally the Jacobi operator of ϕ, $\Delta^\phi = -\mathrm{tr}_g(\nabla^\phi \nabla^\phi - \nabla^\phi_{\nabla^N})$, and $R^N(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$, for $X, Y \in \Gamma(TN)$. Since any harmonic map is biharmonic, we are interested in non-harmonic maps which are called proper biharmonic. Proper biharmonic maps were extensively studied in the last decade[4-5].

In Ref. [6], the authors studied the biharmonic maps between warped product manifolds which were first defined by Bishop and O'Neil[7] and investigated the biharmonicity of the product map $1_B \times \psi: B \times_b F \to B \times F$ and of the projection $\bar{\pi}: B \times_b F \to F$. They also gave two new classes of proper biharmonic maps by using the product of harmonic maps and warping the metric in the domain or codomain. Perktas and Kilic[8] generalized the warped product manifolds into doubly warped product manifolds. They also investigated the biharmonicity of the product map $1_B \times \psi: {}_fB \times_b F \to B \times F$ and of the projection $\bar{\pi}: {}_fB \times_b F \to F$.

In this paper, we generalize some results in Refs. [6, 8] to the general case. We analyze the biharmonicity of the product map $\varphi \times \psi: {}_fB \times_b F \to B \times F$ and of the generalized projections $\bar{\varphi}: {}_fB \times_b F \to B$, $\bar{\psi}: {}_fB \times_b F \to F$. We obtain the result that the biharmonicity of the product map is equivalent to the biharmonicity of the two generalized projections.

1 Preliminaries

Let (B, g_B) and (F, g_F) be Riemannian manifolds of dimensions m and n, respectively, and $b: B \to (0, \infty)$, $f: F \to (0, \infty)$ be smooth functions. As a generalization of the warped product of two Riemannian manifolds, a doubly warped product of Riemannian manifolds B and F with warping functions b and f is a product manifold $B \times F$ with the metric tensor $g = f^2 g_B \oplus b^2 g_F$. We denote the doubly warped product of Riemannian manifolds B and F by ${}_fB \times_b F$. If $f = 1$, then ${}_1B \times_b F = B \times_b F$ becomes a warped product of B and F.

Let (B, g_B) and (F, g_F) be Riemannian manifolds with Levi-Civita connections ∇^B and ∇^F, respectively and let ∇ and $\bar{\nabla}$ denote the Levi-Civita connections of product manifold $B \times F$ and the doubly warped product manifold ${}_fB \times_b F$, respectively. The Levi-Civita connection of the doubly warped product manifold ${}_fB \times_b F$ is defined by

$$\begin{aligned}\bar{\nabla}_X Y = {} & \nabla_X Y + \frac{1}{2b^2}X_1(b^2)(0, Y_2) + \frac{1}{2b^2}Y_1(b^2)(0, X_2) + \\ & \frac{1}{2f^2}X_2(f^2)(Y_1, 0) + \frac{1}{2f^2}Y_2(f^2)(X_1, 0) - \\ & \frac{1}{2}g_B(X_1, Y_1)(0, \mathrm{grad} f^2) - \frac{1}{2}g_F(X_2, Y_2)(\mathrm{grad} b^2, 0)\end{aligned}$$

for $X, Y \in \Gamma(T(B \times F))$, where $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, $X_1, Y_1 \in \Gamma(TB)$ and $X_2, Y_2 \in \Gamma(TF)$. Let R and $\bar{R}$ denote the curvature tensors of $B \times F$ and ${}_fB \times_b F$, respectively.

2 Main Results of Product Maps

Let $\varphi: B \to B$ be a harmonic map and $\psi: F \to F$ be a harmonic map.

Case 1 We consider the following product map

$$\Phi_1 = \varphi \times \psi: B \times F \to B \times F$$
$$\Phi_1(x, y) = (\varphi(x), \psi(y)) \text{ for } (x, y) \in B \times F$$

We compute the tension field of Φ_1,

$$\tau(\Phi_1) = \mathrm{tr}_{g_0} \nabla \mathrm{d}\Phi_1 = \sum_{j=1}^m (\nabla \mathrm{d}\Phi_1)((B_j, 0), (B_j, 0)) +$$

Received 2009-08-26.
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Foundation item: The National Natural Science Foundation of China (No. 10971029).
Citation: Han Yingbo. Biharmonic product maps between doubly warped product manifolds[J]. Journal of Southeast University (English Edition), 2010, 26(3): 502 – 504.

$$\sum_{r=1}^{n}(\nabla \mathrm{d}\Phi_1)((0,F_r),(0,F_r)) = \sum_{j=1}^{m}\left\{\nabla_{(B_j,0)}\mathrm{d}\Phi_1((B_j,0)) - \mathrm{d}\Phi_1((\nabla^B_{B_j}B_j),0)\right\} + \sum_{r=1}^{n}\left\{\nabla_{(0,F_r)}\mathrm{d}\Phi_1((0,F_r)) - \mathrm{d}\Phi_1(0,(\nabla^F_{F_r}F_r))\right\} = (\tau(\varphi),\tau(\psi))$$

Since φ and ψ are harmonic maps, we know that Φ_1 is a harmonic map.

Case 2 We consider the following product map

$$\Phi_2 = \varphi\times\psi: {}_fB\times{}_bF\to B\times F$$
$$\Phi_2(x,y) = (\varphi(x),\psi(y)) \text{ for } (x,y)\in{}_fB\times{}_bF$$

For this map, we obtain the following results.

Theorem 1 Φ_2 is a proper biharmonic map if and only if b is a non-constant solution of

$$-\frac{1}{f^2}J_\varphi(\mathrm{d}\varphi(\mathrm{grad}(\ln b))) + \frac{n}{2}\mathrm{grad}\,|\mathrm{d}\varphi(\mathrm{grad}(\ln b))|^2 = 0 \tag{1}$$

and f is a non-constant solution of

$$-\frac{1}{b^2}J_\psi(\mathrm{d}\psi(\mathrm{grad}(\ln f))) + \frac{m}{2}\mathrm{grad}\,|\mathrm{d}\psi(\mathrm{grad}(\ln f))|^2 = 0 \tag{2}$$

Proof We first compute the tension field of Φ_2,

$$\tau(\Phi_2) = \mathrm{tr}_g(\nabla\mathrm{d}\Phi_2) = \sum_{j=1}^{m}(\nabla\mathrm{d}\Phi_2)\left(\frac{1}{f}(B_j,0),\frac{1}{f}(B_j,0)\right) + \sum_{r=1}^{n}(\nabla\mathrm{d}\Phi_2)\left(\frac{1}{b}(0,F_r),\frac{1}{b}(0,F_r)\right) = \frac{1}{f^2}\sum_{j=1}^{m}\{\nabla_{(B_j,0)}\mathrm{d}\Phi_2(B_j,0) - \mathrm{d}\Phi_2(\overline{\nabla}_{(B_j,0)}(B_j,0))\} + \frac{1}{b^2}\sum_{j=1}^{m}\{\nabla_{(0,F_r)}\mathrm{d}\Phi_2(0,F_r) - \mathrm{d}\Phi_2(\overline{\nabla}_{(0,F_r)}(0,F_r))\} = \frac{1}{f^2}(\tau(\varphi),0) + \frac{m}{2f^2}(0,\mathrm{d}\psi(\mathrm{grad}f^2)) + \frac{1}{b^2}(0,\tau(\psi)) + \frac{n}{2b^2}(\mathrm{d}\varphi(\mathrm{grad}b^2),0)$$

Since φ and ψ are harmonic maps, we obtain

$$\tau(\Phi_2) = \frac{m}{2f^2}(0,\mathrm{d}(\mathrm{grad}f^2)) + \frac{n}{2b^2}(\mathrm{d}\varphi(\mathrm{grad}b^2),0) = m(0,\mathrm{d}(\mathrm{grad}(\ln f))) + n(\mathrm{d}\varphi(\mathrm{grad}(\ln f)),0) \tag{3}$$

From Eq. (3), we know that Φ_2 is harmonic if and only if $\mathrm{grad}(\ln f)=0$ and $\mathrm{grad}(\ln b)=0$.

By a straightforward calculation,

$$-\Delta\tau(\Phi_2) = \mathrm{tr}_g(\nabla^2\tau(\Phi_2)) = \sum_{j=1}^{m}\left\{\nabla_{\frac{1}{f}(B_j,0)}\nabla_{\frac{1}{f}(B_j,0)}\tau(\Phi_2) - \nabla_{\overline{\nabla}_{\frac{1}{f}(B_j,0)}\frac{1}{f}(B_j,0)}\tau(\Phi_2)\right\} + \sum_{r=1}^{n}\left\{\nabla_{\frac{1}{b}(0,F_r)}\nabla_{\frac{1}{b}(0,F_r)}\tau(\Phi_2) - \nabla_{\overline{\nabla}_{\frac{1}{b}(0,F_r)}\frac{1}{b}(0,F_r)}\tau(\Phi_2)\right\} = \left(\frac{n}{f^2}\mathrm{tr}_{g_B}\nabla^2\mathrm{d}\varphi(\mathrm{grad}(\ln b)) + n^2\nabla^B_{\mathrm{d}\varphi(\mathrm{grad}(\ln b))}\mathrm{d}\varphi(\mathrm{grad}(\ln b)),0\right) + \left(0,\frac{m}{b^2}\mathrm{tr}_{g_F}\nabla^2\mathrm{d}\psi(\mathrm{grad}(\ln f)) + m^2\nabla^B_{\mathrm{d}\varphi(\mathrm{grad}(\ln b))}\mathrm{d}\psi(\mathrm{grad}(\ln f))\right)$$

Also by using the usual definition of the curvature tensor field of $B\times F$, we obtain the following equations:

$$\mathrm{tr}_gR(\mathrm{d}\Phi_2,\tau(\Phi_2))\mathrm{d}\Phi_2 = \sum_{j=1}^{m}R\left(\mathrm{d}\Phi_2\left(\frac{1}{f}(B_j,0)\right),\tau(\Phi_2)\right)\mathrm{d}\Phi_2\left(\frac{1}{f}(B_j,0)\right) + \sum_{r=1}^{n}R\left(\mathrm{d}\Phi_2\left(\frac{1}{b}(0,F_r)\right),\tau(\Phi_2)\right)\mathrm{d}\Phi_2\left(\frac{1}{b}(0,F_r)\right) = \frac{n}{f^2}(\mathrm{tr}_{g_B}R^B(\mathrm{d}\varphi,\mathrm{d}\varphi(\mathrm{grad}(\ln b)))\mathrm{d}\varphi,0) + \frac{m}{b^2}(0,\mathrm{tr}_{g_F}R^F(\mathrm{d}\psi,\mathrm{d}\psi(\mathrm{grad}(\ln f)))\mathrm{d}\psi)$$

So we have the bitension field of Φ_2,

$$\tau_2(\Phi_2) = -\Delta\tau(\Phi_2) - \mathrm{tr}_gR(\mathrm{d}\Phi_2,\tau(\Phi_2))\mathrm{d}\Phi_2 = \left(-\frac{n}{f^2}J_\varphi(\mathrm{d}\varphi(\mathrm{grad}(\ln b))) + \frac{n^2}{2}\mathrm{grad}\,|\mathrm{d}\varphi(\mathrm{grad}(\ln b))|^2,0\right) + \left(0,-\frac{m}{b^2}J_\psi(\mathrm{d}\psi(\mathrm{grad}(\ln f))) + \frac{m^2}{2}\mathrm{grad}\,|\mathrm{d}\psi(\mathrm{grad}(\ln f))|^2\right) \tag{4}$$

So we know that Φ_2 is properly biharmonic if and only if b is a non-constant solution of

$$-\frac{1}{f^2}J_\varphi(\mathrm{d}\varphi(\mathrm{grad}(\ln b))) + \frac{n}{2}\mathrm{grad}\,|\mathrm{d}\varphi(\mathrm{grad}(\ln b))|^2 = 0$$

and f is a non-constant solution of

$$-\frac{1}{b^2}J_\psi(\mathrm{d}\psi(\mathrm{grad}(\ln f))) + \frac{m}{2}\mathrm{grad}\,|\mathrm{d}\psi(\mathrm{grad}(\ln f))|^2 = 0$$

Remark 1 When $\varphi = I: B\to B$, we have theorem 5.1 in Ref. [8], and when $\psi = I: F\to F$, we obtain theorem 5.2 in Ref. [8].

Now we consider the following generalized projections:

$$\overline{\varphi}: {}_fB\times{}_bF\to B,\qquad \overline{\varphi}(x,y) = \varphi(x) \text{ for } (x,y)\in{}_fB\times{}_bF$$
$$\overline{\psi}: {}_fB\times{}_bF\to F,\qquad \overline{\psi}(x,y) = \psi(y) \text{ for } (x,y)\in{}_fB\times{}_bF$$

And we also investigate the biharmonicity of the two maps $\overline{\varphi}$ and $\overline{\psi}$.

Theorem 2 $\overline{\varphi}$ is a proper biharmonic map if and only if b is a non-constant solution of

$$-\frac{1}{f^2}J_\varphi(\mathrm{d}\varphi(\mathrm{grad}(\ln b))) + \frac{n}{2}\mathrm{grad}\,|\mathrm{d}\varphi(\mathrm{grad}(\ln b))|^2 = 0$$

Proof We first compute the tension tensor of $\overline{\varphi}$,

$$\tau(\overline{\varphi}) = \mathrm{tr}_g(\nabla\mathrm{d}\overline{\varphi}) = \sum_{j=1}^{m}\left\{(\nabla\mathrm{d}\overline{\varphi})\left(\frac{1}{f}(B_j,0),\frac{1}{f}(B_j,0)\right)\right\} + \sum_{r=1}^{n}\left\{(\nabla\mathrm{d}\overline{\varphi})\left(\frac{1}{b}(0,F_r),\frac{1}{f}(0,F_r)\right)\right\} = \frac{1}{f^2}\tau(\varphi)\circ\overline{\varphi} + n(\mathrm{d}\varphi(\mathrm{grad}(\ln b)))\circ\overline{\varphi}$$

Since φ is harmonic, we have $\tau(\overline{\varphi}) = n(\mathrm{d}\varphi(\mathrm{grad}(\ln b)))\circ\overline{\varphi}$.

By a straightforward calculation,

$$-\Delta\tau(\bar{\varphi}) = \mathrm{tr}_g(\nabla^2\tau(\bar{\varphi})) = \sum_{j=1}^{m}\left\{\nabla_{\frac{1}{f}(B_j,0)}\nabla_{\frac{1}{f}(B_j,0)}\tau(\bar{\varphi}) - \nabla_{\bar{\nabla}_{\frac{1}{f}(B_j,0)}\frac{1}{f}(B_j,0)}\tau(\bar{\varphi})\right\} + \sum_{r=1}^{n}\left\{\nabla_{\frac{1}{b}(0,F_r)}\nabla_{\frac{1}{b}(0,F_r)}\tau(\bar{\varphi}) - \nabla_{\bar{\nabla}_{\frac{1}{b}(0,F_r)}\frac{1}{b}(0,F_r)}\tau(\bar{\varphi})\right\} = \frac{n}{f^2}\mathrm{tr}_{g_B}\nabla^2 d\varphi(\mathrm{grad}(\ln b)) + \frac{n^2}{2}\mathrm{grad}\,|\mathrm{grad}(\ln b)|^2$$

and we also obtain

$$\mathrm{tr}_g R^B(d\bar{\varphi},\tau(\bar{\varphi}))d\bar{\varphi} = \sum_{j=1}^{m} R^B\left(d\bar{\varphi}\left(\frac{1}{f}(B_j,0)\right), n(d\varphi(\mathrm{grad}(\ln b)))\right)d\bar{\varphi}\left(\frac{1}{f}(B_j,0)\right) + \sum_{r=1}^{n} R^B\left(d\bar{\varphi}\left(\frac{1}{b}(0,F_r)\right), n(d\varphi(\mathrm{grad}(\ln b)))\right)d\bar{\varphi}\left(\frac{1}{b}(0,F_r)\right) = \frac{n}{f^2}\mathrm{tr}_{g_B} R^B(d\varphi, d\varphi(\mathrm{grad}(\ln b)))d\varphi$$

We have the bitension field of $\bar{\varphi}$,

$$\tau_2(\bar{\varphi}) = -\Delta\tau(\bar{\varphi}) - \mathrm{tr}_g R^B(d\bar{\varphi},\tau(\bar{\varphi}))d\bar{\varphi} = -\frac{n}{f^2}J_\varphi(d\varphi(\mathrm{grad}(\ln b))) + \frac{n^2}{2}\mathrm{grad}\,|\mathrm{grad}(\ln b)|^2 \tag{5}$$

From Eq. (5), we know that the theorem is true.

Similarly, we obtain the following equations for $\bar{\psi}$,

$$\tau(\bar{\psi}) = \frac{1}{b^2}\tau(\psi)\circ\bar{\varphi} + n(d\psi(\mathrm{grad}(\ln b)))\circ\bar{\psi}$$

and

$$\tau_2(\bar{\psi}) = -\frac{m}{b^2}J_\varphi(d\psi(\mathrm{grad}(\ln f))) + \frac{m^2}{2}\mathrm{grad}\,|\mathrm{grad}(\ln f)|^2 \tag{6}$$

So we obtain the following results.

Theorem 3 $\bar{\psi}$ is a proper biharmonic map if and only if f is a non-constant solution of

$$-\frac{1}{b^2}J_\psi(d\psi(\mathrm{grad}(\ln f))) + \frac{m}{2}\mathrm{grad}\,|d\psi(\mathrm{grad}(\ln f))|^2 = 0$$

From Eqs. (4) to (6), we obtain $\tau_2(\Phi_2) = (\tau_2(\bar{\varphi}), \tau_2(\bar{\psi}))$.

So we have the following theorem.

Theorem 4 Φ_2 is a proper biharmonic map if and only if $\bar{\varphi}$ and $\bar{\psi}$ are proper biharmonic maps.

Remark 2 When $\bar{\varphi} = \bar{\pi}_1 : {}_fB \times {}_bF \to B$, i. e., $\bar{\varphi}(x, y) = x$ for $(x, y) \in {}_fB \times {}_bF$, we have corollary 5. 1 in Ref. [8], and when $\bar{\psi} = \bar{\pi}_2 : {}_fB \times {}_bF \to F$, i. e., $\bar{\psi}(x, y) = y$ for $(x, y) \in {}_fB \times {}_bF$, we have corollary 5. 2 in Ref. [8].

Acknowledgement The author would like to thank professor Chao Xiaoli for his valuable suggestions and useful discussions.

References

[1] Jiang G Y. 2-harmonic isometric immersions between Riemannian manifolds [J]. *Chi Ann Math Ser A*, 1986, **7**(2): 130 − 144.
[2] Jiang G Y. 2-harmonic maps and their first and second variation formulas [J]. *Chi Ann Math Ser A*, 1986, **7**(4): 389 − 402.
[3] Eells J, Sampson J H. Harmonic maps of Riemannian manifolds [J]. *Amer J Math*, 1964, **86**(1): 109 − 160.
[4] Loubeau E, Oniciuc C. The index of biharmonic maps in spheres [J]. *Compos Math*, 2005, **141**(3): 729 − 745.
[5] Oniciuc C. On the second variation formula for biharmonic maps to a sphere [J]. *Publ Math Debrecen*, 2002, **61**(3): 613 − 622.
[6] Balmus A, Monaldo S, Oniciuc C. Biharmonic maps between warped product manifolds [J]. *J Geom Phys*, 2007, **57**(2): 449 − 466.
[7] Bishop R L, O'Neil B. Manifolds of negative curvature [J]. *Amer Math Soc*, 1969, **145**(1): 1 − 49.
[8] Perktas S Y, Kilic E. Biharmonic maps doubly warped product manifolds [J]. *Balk J of Geom and Its Appli*, 2010, **15**(2): 151 − 162.

双 warped 积流形之间 2-调和的积映射

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摘要:讨论了积映射 $\Phi_2 = \varphi\times\psi$ 和 2 个广义的投影 $\bar{\varphi}$ 和 $\bar{\psi}$ 的 2-调和性,得到了几个主要结论:$\Phi_2 = \varphi\times\psi$ 是恰当 2-调和映射的充分必要条件是函数 b,f 分别为方程 $-\frac{1}{f^2}J_\varphi(d\varphi(\mathrm{grad}(\ln b))) + \frac{n}{2}\mathrm{grad}\,|d\varphi(\mathrm{grad}(\ln b))|^2 = 0$, $-\frac{1}{b^2}J_\psi(d\psi(\mathrm{grad}(\ln f))) + \frac{m}{2}\mathrm{grad}\,|d\psi(\ln f)| = 0$ 的非常值解;$\Phi_2 = \varphi\times\psi$ 是恰当 2-调和映射的充分必要条件是 $\bar{\varphi}$ 和 $\bar{\psi}$ 都是恰当 2-调和映射.

关键词:2-调和映射; 积映射; 双 warped 积流形

中图分类号:O186. 1