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Finite element analysis of spatial curved beam in large deformation

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Abstract: For the purpose of carrying out the large deformation finite element analysis of spatial curved beams, the total Lagrangian (TL) and the updated Lagrangian (UL) incremental formulations for arbitrary spatial curved beam elements are established with displacement vector interpolation, which is improved from component interpolation of the straight beam displacement. A strategy of replacing the actual curve with the isoparametric curve is used to expand the applications of the UL formulation. The examples indicate that the process of establishing the curved beam element is correct, and the accuracy with the curved beam element is obviously higher than that with the straight beam element. Generally, the same level of computational accuracy can be achieved with 1/5 as many curved beam elements as otherwise with straight beam elements.

Key words: spatial curved beams; total Lagrangian incremental formulation; updated Lagrangian incremental formulation; geometrical nonlinearity; isoparametric curve

The spatial curved beam is a common form of structures and components in engineering practice, such as curve bridges, arch bridges, and so on. An intuitionistic analysis would suggest that the mechanical behavior of curved beams must be far more complex than that of straight beams due to the effect of considerable initial curvature. Ojalvo et al.[1−4] pointed out that the geometric relationships of spatial curved beams are characterized by complex coupling. In terms of the equilibrium for curved beams, Vlasov et al.[5−7] also found the wordy equilibrium relationships of various internal forces on micro sections of curved beams. Moreover, the equilibrium equations established by the above scholars are only suitable for planar curved beams. Analytical research on spatial curved beams appears to be stagnant.

As a result of these difficulties, the finite element study for the spatial curved beams is still in an exploratory stage. In order to meet computational requirements in engineering practice, the curved beam is replaced by a series of straight beams to approximately carry out the structural analysis at the cost of computational efficiency.

In this paper, based on the studies of the geometrical relationships of arbitrary spatial curved beams, the equilibrium equation, and the nonlinear virtual work equation, we establish finite element formulations for spatial curved beams in large deformation, and test the validation of the formulations and the computational accuracy of the element by applying several examples.

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1 Displacement Interpolation

The coordinate system of the curved beam section under the Kirchhoff hypothesis is described in Fig. 1. Where $\boldsymbol{e}_i$ are three orthogonal unit base vectors of the fixed Cartesian coordinate system C_{f}; $\tilde{\boldsymbol{e}}_i$ are three orthogonal unit covariant base vectors of the centroidal principal axis coordinate system $\tilde{C}_{\mathrm{R}}$; $\tilde{\boldsymbol{e}}_i$ change along with the beam axes, but remain identical and orthogonal; and x^1 is the curve length between the beginning of the beam and the centroid of section. $\tilde{\boldsymbol{e}}_i$ change along with coordinates x^1, and their derivatives with respect to x^1 follow the rule:

$$\tilde{\boldsymbol{e}}_{i,1} = \tilde{K}_i^j \tilde{\boldsymbol{e}}_j, \quad \tilde{\boldsymbol{\kappa}} = \tilde{K}_i^j = \begin{bmatrix} 0 & \tilde{\kappa}_3 & -\tilde{\kappa}_2 \\ -\tilde{\kappa}_3 & 0 & \tilde{\kappa}_1 \\ \tilde{\kappa}_2 & -\tilde{\kappa}_1 & 0 \end{bmatrix} \tag{1}$$

where $\tilde{\kappa}_1 = \tilde{\upsilon} + \alpha_{,1}$, $\tilde{\kappa}_2 = \tilde{\kappa}\sin\alpha$, $\tilde{\kappa}_3 = \tilde{\kappa}\cos\alpha$; $\tilde{\boldsymbol{e}}_{i,1}$ is the differential coefficient of variable $\tilde{\boldsymbol{e}}_i$ with respect to x^1; $\tilde{\kappa}$ is the initial curvature; $\tilde{\upsilon}$ is the initial torsion; and α is the section's initial angle of torsion.

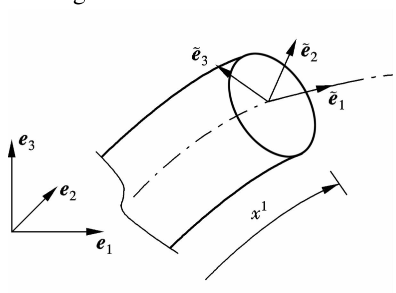

Fig. 1 Geometrical description for spatial curved beam

1.1 Displacement field

Due to the uncoupling of deformations, the field displacement components of the straight beam element can be obtained by the components interpolation of the nodal point displacement. However, the curved beam does not have such advantages, and the interpolation of the displacement vector needs to be carried out, taking coupling of curved beam deformation into consideration. Imitating the straight beam element, the cubic Hermitian polynomials are used for displacement interpolation, while the Lagrangian polynomials are used for torsion angle interpolation. Their specific expressions are

$$H_1 = 1 - 3\xi\xi + 2\xi\xi\xi, \quad H_2 = (\xi - 2\xi\xi + \xi\xi\xi)L$$

$$H_3 = 3\xi\xi - 2\xi\xi\xi, \quad H_4 = (\xi\xi\xi - \xi\xi)L$$

$$L_1 = 1 - \xi, \quad L_2 = \xi, \quad \xi = x^1/L$$

The specific expressions for the displacement field are

$$(u^i\tilde{\boldsymbol{e}}_i)\big|_{x^1} = H_1(u^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=0} + H_2(u^i\tilde{\boldsymbol{e}}_i)_{,1}\big|_{x^1=0} + H_3(u^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=L} + H_4(u^i\tilde{\boldsymbol{e}}_i)_{,1}\big|_{x^1=L}$$

$$(\theta^i\tilde{\boldsymbol{e}}_i)\big|_{x^1} = L_1(\theta^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=0} + L_2(\theta^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=L}$$

where $(u^i\tilde{\boldsymbol{e}}_i)\big|_{x^1}$ is the displacement vector for section x^1 and $u^i(i=1,2,3)$ are the displacement components in $\tilde{\boldsymbol{e}}_i$ direction. $(u^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=0}$, $(u^i\tilde{\boldsymbol{e}}_i)_{,1}\big|_{x^1=0}$, $(u^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=L}$ and $(u^i\tilde{\boldsymbol{e}}_i)_{,1}\big|_{x^1=L}$ represent the displacement vectors and their derivative vectors of the starting point and the ending point of the beam, respectively. $(\theta^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=0}$ and $(\theta^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=L}$ stand for the rotational angle vectors of the starting point and the ending point of the beam, and $\theta^i(i=1,2,3)$ are the rotational angle components in $\tilde{\boldsymbol{e}}_i$ direction. L is the total curve length.

Rewriting $(u^i\tilde{\boldsymbol{e}}_i)_{,1}$ and considering Eq. (1), the following equation can be obtained,

$$(u^i\tilde{\boldsymbol{e}}_i)_{,1} = (u^i_{,1} + u^j\tilde{K}^i_j)\tilde{\boldsymbol{e}}_i = \omega^i\tilde{\mathrm{e}}_i$$

$$\omega^2 = \theta^3,\ \omega^3 = -\theta^2$$

Therefore,

$$\left.\begin{aligned}(u^i\tilde{\boldsymbol{e}}_i)\big|_{x^1} &= H_1(u^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=0} + H_2(\omega^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=0} + H_3(u^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=L} + H_4(\omega^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=L}\\ (\theta^i\tilde{\boldsymbol{e}}_i)\big|_{x^1} &= L_1(\theta^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=0} + L_2(\theta^i\tilde{\boldsymbol{e}}_i)\big|_{x^1=L}\end{aligned}\right\} \tag{2}$$

Eq. (2) is formally consistent with the straight beam displacement interpolation formula, but it is vectorial.

Obviously, Eq. (2) is not applicable due to the fact that it contains $(\tilde{\boldsymbol{e}}_i)\big|_{x^1=0}$ and $(\tilde{\boldsymbol{e}}_i)\big|_{x^1=L}$, which should be eliminated by means of conversion, indicating $(\tilde{\boldsymbol{e}}_i)\big|_{x^1=0}$ and $(\tilde{\boldsymbol{e}}_i)\big|_{x^1=L}$ with $\boldsymbol{e}_j$, and then with $\tilde{\boldsymbol{e}}_i$.

$$\left.\begin{aligned}(\tilde{\boldsymbol{e}}_i)\big|_{x^1=0} &= {}^0\tilde{\Lambda}^j_i\boldsymbol{e}_j = {}^0\tilde{\Lambda}^j_i({}^{x^1}\tilde{\Lambda}^j_i)^{-1}(\tilde{\boldsymbol{e}}_i)\big|_{x^1}\\ (\tilde{\boldsymbol{e}}_i)\big|_{x^1=L} &= {}^L\tilde{\Lambda}^j_i\,\boldsymbol{e}_j = {}^L\tilde{\Lambda}^j_i({}^{x^1}\tilde{\Lambda}^j_i)^{-1}(\tilde{\boldsymbol{e}}_i)\big|_{x^1}\end{aligned}\right\} \tag{3}$$

where ${}^{x^1}\tilde{\Lambda}^j_i$ is the relational matrix between $(\tilde{\boldsymbol{e}}_i)\big|_{x^1}$ and $\boldsymbol{e}_j$ on x^1 section.

$$(\tilde{\boldsymbol{e}}_i)\big|_{x^1} = {}^{x^1}\tilde{\Lambda}^j_i\,\boldsymbol{e}_j,\quad \boldsymbol{e}_j = ({}^{x^1}\tilde{\Lambda}^j_i)^{-1}(\tilde{\boldsymbol{e}}_i)\big|_{x^1} \tag{4}$$

${}^{x^1}\tilde{\Lambda}^j_i$ is determined by the specific curve, and the expression ${}^{x^1}\tilde{\Lambda}^j_i$ of the isoperimetric curve will be deduced later in this paper. ${}^{x^1}\tilde{\Lambda}^j_i$ has the following property regardless of the kinds of curves:

$${}^{x^1}\tilde{\Lambda}^j_{i,1} = \tilde{K}^k_i\,{}^{x^1}\tilde{\Lambda}^j_k$$

because

$${}^{x^1}\tilde{\Lambda}^j_{i,1}\,\boldsymbol{e}_j = \tilde{\boldsymbol{e}}_{i,1} = \tilde{K}^k_i\,\tilde{\boldsymbol{e}}_k = \tilde{K}^k_i\,{}^{x^1}\tilde{\Lambda}^j_k\,\boldsymbol{e}_j$$

After conversion, Eq. (2) can be written as

$$(u^i\tilde{\boldsymbol{e}}_i)\big|_{x^1} = (H_1u^{k0}_1\tilde{\Lambda}^j_k + H_2\omega^{k0}_1\tilde{\Lambda}^j_k + H_3u^{kL}_2\tilde{\Lambda}^j_k + H_4\omega^{kL}_2\tilde{\Lambda}^j_k)({}^{x^1}\tilde{\Lambda}^j_i)^{\mathrm{T}}(\tilde{\boldsymbol{e}}_i)\big|_{x^1}$$

$$(\theta^i\tilde{\boldsymbol{e}}_i)\big|_{x^1} = (L_1\theta^{k0}_1\tilde{\Lambda}^j_k + L_2\theta^{k0}_2\tilde{\Lambda}^j_k)({}^{x^1}\tilde{\Lambda}^j_i)^{\mathrm{T}}(\tilde{\boldsymbol{e}}_i)\big|_{x^1}$$

By eliminating $(\tilde{\boldsymbol{e}}_i)\big|_{x^1}$ and considering the angle of torsion interpolation only for the rotational angle, we obtain

$$\left.\begin{aligned}u^i &= H_1u^{k0}_1\tilde{\Lambda}^j_k({}^{x^1}\tilde{\Lambda}^j_i)^{\mathrm{T}} + H_2\omega^{k0}_1\tilde{\Lambda}^j_k({}^{x^1}\tilde{\Lambda}^j_i)^{\mathrm{T}} + H_3u^{kL}_2\tilde{\Lambda}^j_k({}^{x^1}\tilde{\Lambda}^j_i)^{\mathrm{T}} + H_4\omega^{kL}_2\tilde{\Lambda}^j_k({}^{x^1}\tilde{\Lambda}^j_i)^{\mathrm{T}}\\ \theta^1 &= L_1\theta^{k0}_1\tilde{\Lambda}^j_k({}^{x^1}\tilde{\Lambda}^j_1)^{\mathrm{T}} + L_2\theta^{k0}_2\tilde{\Lambda}^j_k({}^{x^1}\tilde{\Lambda}^j_1)^{\mathrm{T}}\end{aligned}\right\} \tag{5}$$

Eq. (5) is the tensor expression, which is not suitable for coding. The corresponding matrix expression is given. The displacement column matrix of x^1 section is written as

$$\boldsymbol{u}_{\mathrm{S}} = [\boldsymbol{u}^{\mathrm{T}}\quad \theta^1]^{\mathrm{T}},\ \boldsymbol{u} = [u^1\quad u^2\quad u^3]^{\mathrm{T}}$$

The purpose of interpolation is to indicate $\boldsymbol{u}_{\mathrm{S}}$ with the element nodal point displacement matrix $\boldsymbol{u}_{\mathrm{e}}$. Based on the inferential result above, $\boldsymbol{u}_{\mathrm{S}}$ can be written as

$$\boldsymbol{u}_{\mathrm{S}} = \boldsymbol{N}\boldsymbol{\Lambda}_{\mathrm{S}}\boldsymbol{\Lambda}_{\mathrm{T}}\boldsymbol{T}_{\omega}\boldsymbol{u}_{\mathrm{e}} \tag{6}$$

$$\boldsymbol{u}_{\mathrm{e}} = [\boldsymbol{u}^{\mathrm{T}}_1\quad \boldsymbol{\theta}^{\mathrm{T}}_1\quad u^1_{1,1}\quad \boldsymbol{u}^{\mathrm{T}}_2\quad \boldsymbol{\theta}^{\mathrm{T}}_2\quad u^1_{2,1}]^{\mathrm{T}}$$

$$\boldsymbol{N} = \begin{bmatrix} N_1 & 0 & N_2 & N_3 & 0 & N_4\\ 0 & L_1 & 0 & 0 & L_2 & 0\end{bmatrix}$$

$$\boldsymbol{N}_i = \mathrm{diag}(N_i\quad N_i\quad N_i)$$

$$\boldsymbol{\Lambda}_{\mathrm{S}} = \mathrm{diag}({}^{x^1}\tilde{\boldsymbol{\Lambda}}\quad {}^{x^1}\tilde{\boldsymbol{\Lambda}}_1\quad {}^{x^1}\tilde{\boldsymbol{\Lambda}}\quad {}^{x^1}\tilde{\boldsymbol{\Lambda}}\quad {}^{x^1}\tilde{\boldsymbol{\Lambda}}_1\quad {}^{x^1}\tilde{\boldsymbol{\Lambda}})$$

$$\boldsymbol{\Lambda}_{\mathrm{T}} = \begin{bmatrix} {}^0\tilde{\boldsymbol{\Lambda}}^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & {}^0\tilde{\boldsymbol{\Lambda}}^{*\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & {}^0\tilde{\boldsymbol{\Lambda}}^{\#\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & {}^L\tilde{\boldsymbol{\Lambda}}^{\mathrm{T}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & {}^L\tilde{\boldsymbol{\Lambda}}^{*\mathrm{T}}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & {}^L\tilde{\boldsymbol{\Lambda}}^{\#\mathrm{T}}\end{bmatrix}$$

$$\boldsymbol{T}_{\omega} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{\omega}_0 & \boldsymbol{I}^* & \boldsymbol{I}^{\#} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\omega}_{\mathrm{L}} & \boldsymbol{I}^* & \boldsymbol{I}^{\#}\end{bmatrix}$$

$${}^{x^1}\tilde{\boldsymbol{\Lambda}}^* = \begin{bmatrix} {}^{x^1}\tilde{\Lambda}^1_1 & 0 & {}^{x^1}\tilde{\Lambda}^3_1 & -{}^{x^1}\tilde{\Lambda}^2_1\\ {}^{x^1}\tilde{\Lambda}^1_2 & 0 & {}^{x^1}\tilde{\Lambda}^3_2 & -{}^{x^1}\tilde{\Lambda}^2_2\\ {}^{x^1}\tilde{\Lambda}^1_3 & 0 & {}^{x^1}\tilde{\Lambda}^3_3 & -{}^{x^1}\tilde{\Lambda}^2_3\end{bmatrix}$$

$${}^{x^1}\tilde{\boldsymbol{\Lambda}}^{\#} = \begin{bmatrix} 0 & {}^{x^1}\tilde{\Lambda}^1_1 & {}^{x^1}\tilde{\Lambda}^2_1 & {}^{x^1}\tilde{\Lambda}^3_1\\ 0 & {}^{x^1}\tilde{\Lambda}^1_2 & {}^{x^1}\tilde{\Lambda}^2_2 & {}^{x^1}\tilde{\Lambda}^3_2\\ 0 & {}^{x^1}\tilde{\Lambda}^1_3 & {}^{x^1}\tilde{\Lambda}^2_3 & {}^{x^1}\tilde{\Lambda}^3_3\end{bmatrix}$$

$$\boldsymbol{\omega}_{x^1} = \begin{bmatrix} 0 & 0 & 0\\ 0 & -\tilde{\kappa}_3\big|_{x^1} & \tilde{\kappa}_2\big|_{x^1}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ \boldsymbol{I}^* = \begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix},\ \boldsymbol{I}^{\#} = \begin{Bmatrix} 0\\ 1\\ 0\\ 0\end{Bmatrix}$$

where ${}^{x^1}\tilde{\boldsymbol{\Lambda}}_1$ is the first line of ${}^{x^1}\tilde{\boldsymbol{\Lambda}}$.

Furthermore, the sectional independent displacement derivative matrix $\boldsymbol{d}_{\mathrm{S}}$ can be written as

$$\boldsymbol{d}_{\mathrm{S}} = [\boldsymbol{u}^{\mathrm{T}}\quad \boldsymbol{u}^{\mathrm{T}}_{,1}\quad \boldsymbol{u}^{\mathrm{T}}_{,11}\quad \theta^1\quad \theta^1_{,1}]^{\mathrm{T}} = \boldsymbol{A}_1\boldsymbol{u}_{\mathrm{S}} = \boldsymbol{A}_1\boldsymbol{N}_{\mathrm{S}}\boldsymbol{\Lambda}_{\mathrm{T}}\boldsymbol{T}_{\omega}\boldsymbol{u}_{\mathrm{e}} = \boldsymbol{B}\boldsymbol{u}_{\mathrm{e}} \tag{7}$$

$$\boldsymbol{A}_1 = \begin{bmatrix} \boldsymbol{I} & ()_{,1} & ()_{,11} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & 1 & ()_{,1}\end{bmatrix}^{\mathrm{T}}$$

1.2 $\widetilde{K}_i^j$ and ${}^{x^1}\widetilde{\Lambda}_i^j$ expressions of isoparametric curve

The analyses described above can be applied to all kinds of curved beams. It is important to carefully consider how to calculate $\widetilde{K}_i^j$ and ${}^{x^1}\widetilde{\Lambda}_i^j$. These two quantities are determined by curves. In the following, we will discuss the $\widetilde{K}_i^j{}^{x^1}\widetilde{\Lambda}_i^j$ calculation of isoparametric curves. The original ${}^{x^1}\widetilde{\Lambda}_i^j$ expression cannot be used after deformation in the case with the updated Lagrangian formulation.

In order to solve this problem, a strategy similar to the isoparametric element method is used. With the aid of cubic Hermitian polynomials, the original curve expression is substituted approximately with the polynomial constituted by nodal coordinates $X_i^j(j=1,2)$ on C_f and its derivative with respect to x^1

$$X^i \doteq H_j^* Y_j^i \qquad j=1,2,3,4$$

$$Y_1^i = X_1^i,\ Y_2^i = X_{1,1}^i,\ Y_3^i = X_2^i,\ Y_4^i = X_{2,1}^i$$

$$H_1^* = 1-3\zeta\zeta+2\zeta\zeta\zeta,\ H_2^* = (\zeta-2\zeta\zeta+\zeta\zeta\zeta)L$$

$$H_3^* = 3\zeta\zeta-2\zeta\zeta\zeta,\ H_4^* = (\zeta\zeta\zeta-\zeta\zeta)L$$

where X^i is the coordinate of the original curve point on C_f. The Hermitian polynomials parameter ζ whose value ranges from 0 to 1 should be noted as $\zeta \neq x^1/L$. This is because

$$\mathrm{d}X^i = H_{j,\zeta}^* Y_j^i\,\mathrm{d}\zeta, \quad \mathrm{d}x^1 = \sqrt{\mathrm{d}X^i\mathrm{d}X^i} = \mathrm{d}\zeta\sqrt{H_{j,\zeta}^* Y_j^i H_{k,\zeta}^* Y_k^i}$$

And generally, $\sqrt{H_{j,\zeta}^* Y_j^i H_{k,\zeta}^* Y_k^i} = L$ does not hold true at all the points.

So calculating the derivatives is less convenient.

$$H_{j,1}^* = H_{j,\zeta}^*\zeta_{,1}, \quad H_{j,11}^* = H_{j,\zeta\zeta}^*\zeta_{,1}\zeta_{,1} + H_{j,\zeta}^*\zeta_{,11}$$

$$H_{j,111}^* = H_{j,\zeta\zeta\zeta}^*\zeta_{,1}\zeta_{,1}\zeta_{,1} + 3H_{j,\zeta\zeta}^*\zeta_{,1}\zeta_{,11} + H_{j,\zeta}^*\zeta_{,111}$$

$$\zeta_{,1} = \frac{1}{\sqrt{s}},\ \zeta_{,11} = -\frac{s_{,\zeta}}{2ss},\ \zeta_{,111} = \frac{s_{,\zeta}}{sss\sqrt{s}} - \frac{s_{,\zeta\zeta}}{2ss\sqrt{s}}$$

$$s = H_{j,\zeta}^* Y_j^i H_{k,\zeta}^* Y_k^i, \quad s_{,\zeta} = 2H_{j,\zeta\zeta}^* Y_j^i H_{k,\zeta}^* Y_k^i$$

$$s_{,\zeta\zeta} = 2(H_{j,\zeta\zeta\zeta}^* Y_j^i H_{k,\zeta}^* Y_k^i + H_{j,\zeta\zeta}^* Y_j^i H_{k,\zeta\zeta}^* Y_k^i)$$

Furthermore, considering the Lagrange interpolation of the initial angle of torsion, the expressions of the curvature, the torsion and ${}^{x^1}\widetilde{\Lambda}_i^j$ are given directly below:

$$\widetilde{\kappa} = \sqrt{H_{j,11}^* Y_j^i H_{k,11}^* Y_k^i}$$

$$\widetilde{\upsilon} = -\frac{1}{\widetilde{\kappa}\widetilde{\kappa}}\begin{vmatrix} H_{j,1}^* Y_j^1 & H_{j,1}^* Y_j^2 & H_{j,1}^* Y_j^3 \\ H_{j,11}^* Y_j^1 & H_{j,11}^* Y_j^2 & H_{j,11}^* Y_j^3 \\ H_{j,111}^* Y_j^1 & H_{j,111}^* Y_j^2 & H_{j,111}^* Y_j^3 \end{vmatrix}$$

$$\widetilde{\kappa}_1 = \widetilde{\upsilon} + L_{j,1}^*\alpha_j,\ \widetilde{\kappa}_2 = \widetilde{\kappa}\sin(L_j^*\alpha_j),\ \widetilde{\kappa}_3 = \widetilde{\kappa}\cos(L_j^*\alpha_j)$$

$${}^{x^1}\widetilde{\Lambda}_1^i = H_{j,1}^* Y_j^i$$

$${}^{x^1}\widetilde{\Lambda}_2^i = [\cos(L_l^*\alpha_l)H_{j,11}^* Y_j^i + \sin(L_l^*\alpha_l)H_{m,1}^* H_{n,11}^* Y_m^j Y_n^k \varepsilon_{jki}]/\widetilde{\kappa}$$

$${}^{x^1}\widetilde{\Lambda}_3^i = [-\sin(L_l^*\alpha_l)H_{j,11}^* Y_j^i + \cos(L_l^*\alpha_l)H_{m,1}^* H_{n,11}^* Y_m^j Y_n^k \varepsilon_{jki}]/\widetilde{\kappa}$$

where ε_{ijk} is the Eddington permutation tensor.

2 Large Deformation Finite Element Incremental Formulations

The arbitrary spatial curved beam in the total form is

$$\int_L \delta\boldsymbol{d}_S^{\mathrm{T}}(\boldsymbol{Z}_S^{\mathrm{T}} + \boldsymbol{Z}_Q^{\mathrm{T}})\boldsymbol{F}_S\mathrm{d}x^1 = \int_L \delta\boldsymbol{u}_S^{\mathrm{T}}\boldsymbol{Z}_q^{\mathrm{T}}\boldsymbol{q}_S\mathrm{d}x^1 + \delta\boldsymbol{u}_e^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{F}_e \tag{8}$$

where $\boldsymbol{Z}_S$, $\boldsymbol{Z}_q$ and $\boldsymbol{Z}$ are the coefficient matrices that do not contain the unknown quantity. The specific expressions of these matrices are

$$\boldsymbol{Z}_S = \begin{bmatrix} -\widetilde{\boldsymbol{\kappa}} + \boldsymbol{C}_2^{\mathrm{T}}\boldsymbol{C}_1 & \boldsymbol{I} + \boldsymbol{C}_2^{\mathrm{T}}\boldsymbol{C}_2 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ -\widetilde{\boldsymbol{\kappa}}\boldsymbol{C}_1 & \boldsymbol{C}_1 - \widetilde{\boldsymbol{\kappa}}\boldsymbol{C}_2 & \boldsymbol{C}_2 & -\widetilde{\boldsymbol{\kappa}}\boldsymbol{C}_3 & \boldsymbol{C}_3 \end{bmatrix}$$

$$\boldsymbol{Z}_q = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{C}_1 & \boldsymbol{C}_3 \end{bmatrix},\ \boldsymbol{Z} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix},\ \boldsymbol{C}_3 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\boldsymbol{C}_1 = \begin{bmatrix} 0 & 0 & 0 \\ \widetilde{\kappa}_2 & -\widetilde{\kappa}_1 & 0 \\ \widetilde{\kappa}_3 & 0 & -\widetilde{\kappa}_1 \end{bmatrix},\ \boldsymbol{C}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

where $\boldsymbol{Z}_Q$ is the coefficient matrix related to the section displacement,

$$\boldsymbol{Z}_Q^{\mathrm{T}} = [\boldsymbol{Z}_Q^1\boldsymbol{d}_S \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{Z}_Q^2\boldsymbol{d}_S \quad \boldsymbol{Z}_Q^3\boldsymbol{d}_S]$$

$\boldsymbol{Z}_Q^i(i=1,\ 2,\ 3)$ are irrelevant with regards to the unknown quantity and they must satisfy

$$\boldsymbol{Z}_Q^1 + x^3\boldsymbol{Z}_Q^2 - x^2\boldsymbol{Z}_Q^3 = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{C}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & 0 \\ \boldsymbol{C}_1 & \boldsymbol{C}_2 & \boldsymbol{0} & \boldsymbol{C}_3 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}_1 & \boldsymbol{C}_2 & \boldsymbol{0} & \boldsymbol{C}_3 \end{bmatrix}$$

$$\boldsymbol{Q} = \begin{bmatrix} -\widetilde{\boldsymbol{\kappa}}\widetilde{\boldsymbol{\kappa}} & g^*\widetilde{\boldsymbol{\kappa}} & \boldsymbol{0} & -\widetilde{\boldsymbol{\kappa}}\widetilde{\boldsymbol{\kappa}}\boldsymbol{Q}_1 & \boldsymbol{0} \\ & g^*\boldsymbol{I} & \boldsymbol{0} & -2\widetilde{\boldsymbol{\kappa}}\boldsymbol{Q}_1 & \boldsymbol{0} \\ & & \boldsymbol{0} & -\boldsymbol{Q}_1 & \boldsymbol{0} \\ & & & \boldsymbol{0} & \boldsymbol{0} \\ & & & & \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{Q}_1 = \begin{bmatrix} 0 & x^3 & -x^2 \\ -x^3 & 0 & 0 \\ x^2 & 0 & 0 \end{bmatrix},\ g^* = 1 + x^2\widetilde{\kappa}_3 - x^3\widetilde{\kappa}_2$$

where $\boldsymbol{q}_S$ is the load matrix, and $\boldsymbol{F}_e$ is the element nodal force matrix. $\boldsymbol{F}_S$ is the section internal force matrix and it can be expressed as

$$\boldsymbol{F}_S = [\widetilde{F}_1 \quad \widetilde{F}_2 \quad \widetilde{F}_3 \quad \widetilde{M}_1 \quad \widetilde{M}_2 \quad \widetilde{M}_3]^{\mathrm{T}}$$

The relationship between $\boldsymbol{F}_S$ and the section displacement, also known as the constitutive relationship, is

$$\boldsymbol{F}_S = \boldsymbol{D}_S\boldsymbol{Z}_S\boldsymbol{d}_S + \frac{1}{2}\boldsymbol{D}_Q\boldsymbol{Z}_Q(\boldsymbol{d}_S)\boldsymbol{d}_S \tag{9}$$

$\boldsymbol{D}_{\mathrm{S}}$ and $\boldsymbol{D}_{\mathrm{Q}}$ are determined by the material property and the section property.

$$\boldsymbol{D}_{\mathrm{S}} = \begin{bmatrix} E_0 A & 0 & 0 & 0 & E_0 S^2 & -E_0 S^3 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & GI^p & 0 & 0 \\ & & & & E_0 I^2 & -E_0 I^{23} \\ & & & & & E_0 I^3 \end{bmatrix}$$

The specific elements of $\boldsymbol{D}_{\mathrm{Q}}$ and $\boldsymbol{D}_{\mathrm{S}}$ are consistent, except that $GI^p = 0$.

Eq. (8) is in the total form, and it needs to be rewritten into an incremental form for solving nonlinear problems. There are two kinds of incremental formulations: the total Lagrangian formulation and the updated Lagrangian formulation.

2.1 TL formulation

For the TL incremented formulation, when the element node has experienced an incremental displacement $\Delta\boldsymbol{u}_{\mathrm{e}}$ relative to the previous time, the independent displacement derivative matrix $\boldsymbol{d}_{\mathrm{S}}$ of x^1 section, the section internal force $\boldsymbol{F}_{\mathrm{S}}$ and the nodal force $\boldsymbol{F}_{\mathrm{e}}$ will have an increment,

$$\boldsymbol{d}_{\mathrm{S}} = \boldsymbol{d}_{\mathrm{S0}} + \Delta\boldsymbol{d}_{\mathrm{S}},\ \boldsymbol{F}_{\mathrm{S}} = \boldsymbol{F}_{\mathrm{S0}} + \Delta\boldsymbol{F}_{\mathrm{S}},\ \boldsymbol{F}_{\mathrm{e}} = \boldsymbol{F}_{\mathrm{e0}} + \Delta\boldsymbol{F}_{\mathrm{e}}$$

where $\boldsymbol{d}_{\mathrm{S0}}$, $\boldsymbol{F}_{\mathrm{S0}}$ and $\boldsymbol{F}_{\mathrm{e0}}$ are the original physical quantities at the previous time, which are called initial displacement, initial internal force and initial nodal force, respectively. And the load also has an increment $\boldsymbol{q}_{\mathrm{S}} = \boldsymbol{q}_{\mathrm{S0}} + \Delta\boldsymbol{q}_{\mathrm{S}}$.

Following the rules (6) and (7), the incremental displacement matrix $\Delta\boldsymbol{u}_{\mathrm{S}}$ of x^1 section and the incremental independent displacement derivative matrix $\Delta\boldsymbol{d}_{\mathrm{S}}$ also obtain the increment $\Delta\boldsymbol{u}_{\mathrm{S}} = \boldsymbol{N}\boldsymbol{\Lambda}_{\mathrm{S}}\boldsymbol{\Lambda}_{\mathrm{T}}\boldsymbol{T}_{\omega}\Delta\boldsymbol{u}_{\mathrm{e}}$, $\Delta\boldsymbol{d}_{\mathrm{S}} = \boldsymbol{B}\Delta\boldsymbol{u}_{\mathrm{e}}$. By substituting all these formulae into the virtual work equation (8), and following the variation algorithm, the virtual work equation of the TL incremental form can be obtained:

$$\int_L (\delta\Delta\boldsymbol{d}_{\mathrm{S}}^{\mathrm{T}}\boldsymbol{Z}_{\mathrm{S}}^{\mathrm{T}} + \delta\Delta\boldsymbol{d}_{\mathrm{S}}^{\mathrm{T}}\boldsymbol{Z}_{\mathrm{Q0}}^{\mathrm{T}} + \delta\Delta\boldsymbol{d}_{\mathrm{S}}^{\mathrm{T}}\Delta\boldsymbol{Z}_{\mathrm{Q}}^{\mathrm{T}})(\boldsymbol{F}_{\mathrm{S0}} + \Delta\boldsymbol{F}_{\mathrm{S}})\,\mathrm{d}x^1 = \int_L \delta\Delta\boldsymbol{u}_{\mathrm{S}}^{\mathrm{T}}\boldsymbol{Z}_{\mathrm{q}}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{S0}} + \Delta\boldsymbol{q}_{\mathrm{S}})\,\mathrm{d}x^1 + \delta\Delta\boldsymbol{u}_{\mathrm{e}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{F}_{\mathrm{e0}} + \Delta\boldsymbol{F}_{\mathrm{e}}) \tag{10}$$

$$\boldsymbol{Z}_{\mathrm{Q}} = \boldsymbol{Z}_{\mathrm{Q0}} + \Delta\boldsymbol{Z}_{\mathrm{Q}}$$

$$\boldsymbol{Z}_{\mathrm{Q0}}^{\mathrm{T}} = [\boldsymbol{Z}_{\mathrm{Q}}^1\boldsymbol{d}_{\mathrm{S0}}\quad \boldsymbol{0}\quad \boldsymbol{0}\quad \boldsymbol{0}\quad \boldsymbol{Z}_{\mathrm{Q}}^2\boldsymbol{d}_{\mathrm{S0}}\quad \boldsymbol{Z}_{\mathrm{Q}}^3\boldsymbol{d}_{\mathrm{S0}}]$$

$$\Delta\boldsymbol{Z}_{\mathrm{Q}}^{\mathrm{T}} = [\boldsymbol{Z}_{\mathrm{Q}}^1\Delta\boldsymbol{d}_{\mathrm{S}}\quad \boldsymbol{0}\quad \boldsymbol{0}\quad \boldsymbol{0}\quad \boldsymbol{Z}_{\mathrm{Q}}^2\Delta\boldsymbol{d}_{\mathrm{S}}\quad \boldsymbol{Z}_{\mathrm{Q}}^3\Delta\boldsymbol{d}_{\mathrm{S}}]$$

Similar to the constitutive relationship (9), the increments can also be written as

$$\Delta\boldsymbol{F}_{\mathrm{S}} = \boldsymbol{D}_{\mathrm{S}}\boldsymbol{Z}_{\mathrm{S}}\Delta\boldsymbol{d}_{\mathrm{S}} + \frac{1}{2}\boldsymbol{D}_{\mathrm{Q}}\boldsymbol{Z}_{\mathrm{Q}}(\boldsymbol{d}_{\mathrm{S}})\Delta\boldsymbol{d}_{\mathrm{S}}$$

We substitute it into Eq. (10), and rewrite Eq. (10) into the following form to make the physical meaning more explicit.

$$\boldsymbol{K}_{\mathrm{L}}\Delta\boldsymbol{u}_{\mathrm{e}} + \boldsymbol{K}_{\sigma}\Delta\boldsymbol{u}_{\mathrm{e}} + (\boldsymbol{K}_{\mathrm{N1}} + \boldsymbol{K}_{\mathrm{N2}} + \boldsymbol{K}_{\mathrm{N3}})\Delta\boldsymbol{u}_{\mathrm{e}} = \bar{\boldsymbol{P}} - \boldsymbol{P} \tag{11}$$

The physical meaning of each item is as follows:

Linear element stiffness matrix

$$\boldsymbol{K}_{\mathrm{L}} = \int_L \boldsymbol{B}^{\mathrm{T}}\boldsymbol{Z}_{\mathrm{S}}^{\mathrm{T}}\boldsymbol{D}_{\mathrm{S}}\boldsymbol{Z}_{\mathrm{S}}\boldsymbol{B}\,\mathrm{d}x^1$$

Initial internal force stiffness matrix

$$\boldsymbol{K}_{\sigma} = \int_L \boldsymbol{B}^{\mathrm{T}}(\tilde{F}_{10}\boldsymbol{Z}_{\mathrm{Q}}^1 + \widetilde{M}_{20}\boldsymbol{Z}_{\mathrm{Q}}^2 + \widetilde{M}_{30}\boldsymbol{Z}_{\mathrm{Q}}^3)\boldsymbol{B}\,\mathrm{d}x^1$$

Initial displacement stiffness matrices

$$\boldsymbol{K}_{\mathrm{N1}} = \frac{1}{2}\int_L \boldsymbol{B}^{\mathrm{T}}\boldsymbol{Z}_{\mathrm{S}}^{\mathrm{T}}\boldsymbol{D}_{\mathrm{Q}}\boldsymbol{Z}_{\mathrm{Q0}}\boldsymbol{B}\,\mathrm{d}x^1$$

$$\boldsymbol{K}_{\mathrm{N2}} = \int_L \boldsymbol{B}^{\mathrm{T}}\boldsymbol{Z}_{\mathrm{Q0}}^{\mathrm{T}}\boldsymbol{D}_{\mathrm{S}}\boldsymbol{Z}_{\mathrm{S}}\boldsymbol{B}\,\mathrm{d}x^1$$

$$\boldsymbol{K}_{\mathrm{N3}} = \frac{1}{2}\int_L \boldsymbol{B}^{\mathrm{T}}\boldsymbol{Z}_{\mathrm{Q0}}^{\mathrm{T}}\boldsymbol{D}_{\mathrm{Q}}\boldsymbol{Z}_{\mathrm{Q0}}\boldsymbol{B}\,\mathrm{d}x^1$$

Equivalent nodal force of load at current time

$$\bar{\boldsymbol{P}} = \boldsymbol{T}_{\omega}^{\mathrm{T}}\boldsymbol{\Lambda}_{\mathrm{T}}^{\mathrm{T}}\int_L [\boldsymbol{\Lambda}_{\mathrm{S}}^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{Z}_{\mathrm{q1}}^{\mathrm{T}} + (\boldsymbol{A}_2\boldsymbol{N}_{\mathrm{S}})^{\mathrm{T}}\boldsymbol{Z}_{\mathrm{q2}}^{\mathrm{T}}](\boldsymbol{q}_{\mathrm{S0}} + \Delta\boldsymbol{q}_{\mathrm{S}})\,\mathrm{d}x^1 + \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{F}_{\mathrm{e0}} + \Delta\boldsymbol{F}_{\mathrm{e}})$$

The effect of the section internal force at the previous time on the nodal force

$$\boldsymbol{P} = \int_L \boldsymbol{B}^{\mathrm{T}}(\boldsymbol{Z}_{\mathrm{S}}^{\mathrm{T}} + \boldsymbol{Z}_{\mathrm{Q0}}^{\mathrm{T}})\boldsymbol{F}_{\mathrm{S0}}\,\mathrm{d}x^1$$

2.2 UL formulation

Similar to the TL form, when an incremental displacement $\Delta\boldsymbol{u}_{\mathrm{e}}$ at the element nodal point occurs relative to the previous time, there will also be an increment at the independent displacement derivative matrix $\boldsymbol{d}_{\mathrm{S}}$ of x^1 section, the section internal force $\boldsymbol{F}_{\mathrm{S}}$ and the nodal force $\boldsymbol{F}_{\mathrm{e}}$ are

$$\boldsymbol{d}_{\mathrm{S}} = \Delta\boldsymbol{d}_{\mathrm{S}},\ \boldsymbol{F}_{\mathrm{S}} = \boldsymbol{F}_{\mathrm{S0}} + \Delta\boldsymbol{F}_{\mathrm{S}},\ \boldsymbol{F}_{\mathrm{e}} = \boldsymbol{F}_{\mathrm{e0}} + \Delta\boldsymbol{F}_{\mathrm{e}}$$

Compared with the TL formulation, the independent displacement derivative matrix $\boldsymbol{d}_{\mathrm{S}}$ does not have an initial value.

Other procedures are the same as with the TL formulation; therefore, the virtual work equation of the UL form is

$$\int_L \delta\Delta\boldsymbol{d}_{\mathrm{S}}^{\mathrm{T}}(\boldsymbol{Z}_{\mathrm{S}}^{\mathrm{T}} + \Delta\boldsymbol{Z}_{\mathrm{Q}}^{\mathrm{T}})(\boldsymbol{F}_{\mathrm{S0}} + \Delta\boldsymbol{F}_{\mathrm{S}})\,\mathrm{d}x^1 = \int_L \delta\Delta\boldsymbol{u}_{\mathrm{S}}^{\mathrm{T}}\boldsymbol{Z}_{\mathrm{q}}^{T}(\boldsymbol{q}_{\mathrm{S}} + \Delta\boldsymbol{q}_{\mathrm{S}})\,\mathrm{d}x^1 + \delta\Delta\boldsymbol{u}_{\mathrm{e}}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{F}_{\mathrm{e}} + \Delta\boldsymbol{F}_{\mathrm{e}}) \tag{12}$$

Without the initial displacement, the constitutive relationship of the incremental form is simple,

$$\Delta\boldsymbol{F}_{\mathrm{S}} = \boldsymbol{D}_{\mathrm{L}}\Delta\boldsymbol{d}_{\mathrm{S}}$$

The UL formulation is

$$\boldsymbol{K}_{\mathrm{L}}\Delta\boldsymbol{u}_{\mathrm{e}} + \boldsymbol{K}_{\sigma}\Delta\boldsymbol{u}_{\mathrm{e}} = \bar{\boldsymbol{P}} - \boldsymbol{P} \tag{13}$$

Except that $\boldsymbol{P} = \int_L \boldsymbol{B}^{\mathrm{T}}\boldsymbol{Z}_{\mathrm{S}}^{\mathrm{T}}\boldsymbol{F}_{\mathrm{S0}}\,\mathrm{d}x^1$ is different from that in the TL formulation, other integrals and their physical meanings are the same as those in the TL formulation.

In Eqs. (11) and (13), each nodal point has seven degrees of freedom. They are three point displacements, three rotational angles, and a derivative of u^1, respectively. The seventh degree of freedom will bring problems to the subsequent process. The best solution is to carry out a static condensation during element analysis, which is well discussed in Ref. [8].

3 Examples

Comparative computational results of some simple structures are given below.

3.1 Planar circular arc

The planar circular arc is the first to be verified. The sizes of two planar circular arc structures are shown in Fig. 10 and the material parameter is $E = 20$ GN/m^2. Their supporting and loading are also the same. Both sides of the circular arc are fixed, and the bottom support uplifts by 1 m in $\boldsymbol{e}_3$ direction.

In order to verify the correctness and accuracy, an exact result needs to be identified as the benchmark. As it will take too much time and effort to obtain a perfectly precise benchmark, commercial software called MIDAS/Civil2006 is used to calculate the circular arc that is divided into 500 straight beam elements. The result is taken as the relatively accurate benchmark.

The bending moment M_2 at the bottom support is selected as the observed quantity. The computational results of different dividing quantities of curved and straight beam elements are plotted in Fig. 2. The results with the straight beam elements are calculated with MIDAS/Civil2006. We can see that, in order to obtain the generally accurate result with a maximum error of 5%, the circular arc needs to be divided into only about 10 straight beam elements, which is acceptable. However, to obtain a more accurate result, with an error less than 1%, there need to be 20 or even 30 straight beam elements. In contrast, the error is less than 1% with 2 to 4 curved beam elements. Obviously, the computational accuracy with the curved beam element is much higher than that with the straight beam element.

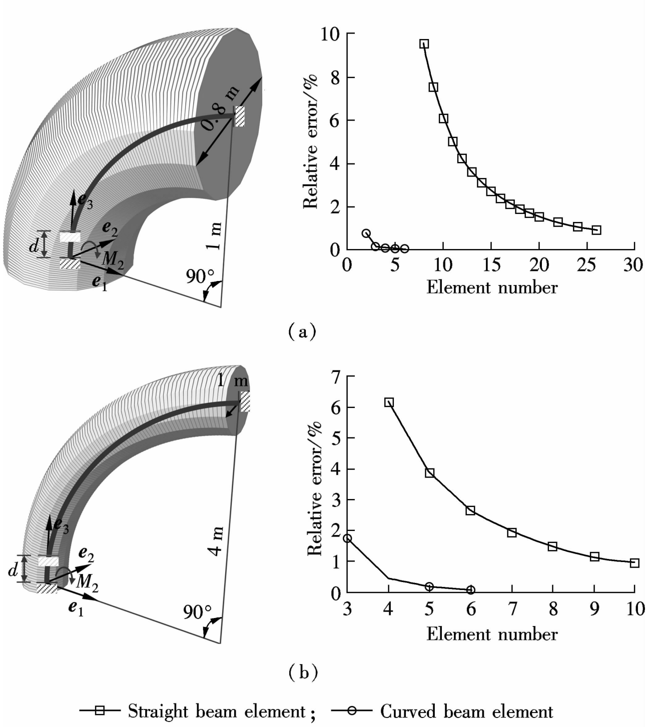

Fig. 2 Comparison of computational errors for planar arc structures. (a) Planar arc with small slenderness ratio; (b) Planar arc with large slenderness ratio

3.2 Spatial spring

The previous example is planar, and the following example is about spatial structure. A spatial spring and its dimensions are shown in Fig. 3. The axis of the spring is the Archimedean spiral. The material parameters are $E = 20$ GN/m^2, and $G = 7.6923$ GN/m^2. The supporting, loading and the relatively accurate benchmark are the same as in the previous example.

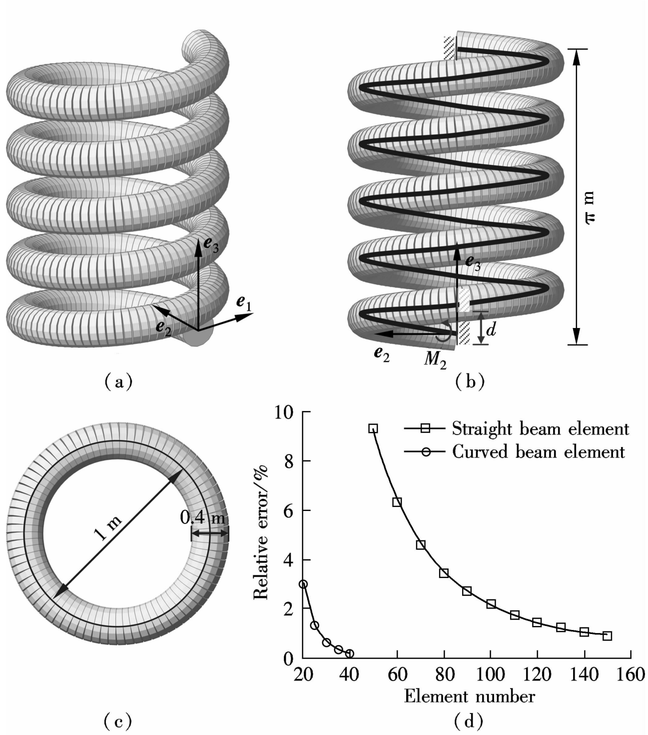

Fig. 3 Comparison of computational errors for spring structures. (a) Spatial spring structure; (b) Facade projection of spatial spring; (c) Planar projection of spatial spring; (d) Computational errors

The bending moment M_2 at the bottom support is selected again. The plotting processes of Fig. 3 and Fig. 2 are similar. Fig. 3 indicates that a computational result with an error less than 3% can be achieved with about 20 curved beam elements. About 90 straight beam elements are required for an equally accurate result. For an error of less than 1%, over 150 straight beam elements are required, which is hardly bearable, due to the fact that the same level of accuracy can be achieved with fewer than 30 curved beam elements.

3.3 Williams frame

The Williams frame (see Fig. 4) is a classical geometrical nonlinear example. Yang et al. [9–10] analyzed this example. Yang et al. [9] divided each bar into 10 straight beam elements of the UL formulation, and the result matches Williams' analytical result[11] well. However, Teh et al. [10] simulated each bar with one straight beam element of the UL formulation, and the result is unsatisfactory. The straight beam element cannot reflect such a situation that deformed bar turns curve. In this paper, the load-displacement curves of simulating each bar with one and two curved beam elements of the UL formulation match Williams' analytical result well (see Fig. 4).

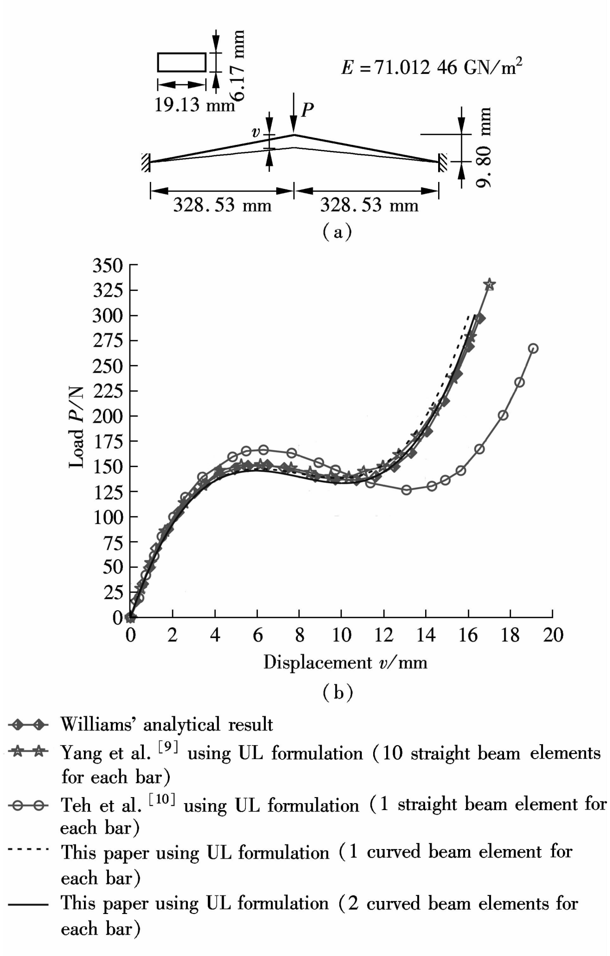

Fig. 4 Load-displacement curves of Williams frame.
(a) Williams frame; (b) Load-displacement curves

4 Conclusions

1) Establishing the spatial curved beam finite element formulation with displacement vector interpolation is feasible, which is improved from component interpolation for the straight beam.

2) The TL and UL incremental formulations for spatial curved beams established in this paper are correct.

3) Whether in the linear analysis, or in the geometrical nonlinear analysis, the accuracy of the curved beam element is obviously higher than that of the straight beam element. In regard to the examples in this paper, to achieve equally acceptable accuracy, the ratio of the required number of curved beam elements to that of straight beam elements is 1∶5.

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大变形空间曲梁有限元分析

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摘要:为了进行空间曲梁大变形有限元分析,将直梁单元位移分量插值的思想改进为位移矢量插值用以建立曲梁单元的位移场,分别建立了适用于任意曲线形式的全拉格朗日和修正拉格朗日增量格式空间曲梁有限元列式. 通过采用等参数曲线代替实际曲线的策略,使得修正拉格朗日增量格式曲梁单元可以应用于更广泛的场合. 算例对比结果表明,曲梁单元的建立过程正确,曲梁单元的精度要明显高于直梁单元. 一般情况下,仅用直梁单元数量 1/5 的曲梁单元就可以达到相同的计算精度.

关键词:空间曲梁;全拉格朗日增量格式;修正拉格朗日增量格式;几何非线性;等参数曲线

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