Journal of Southeast University (English Edition) Vol. 26, No. 4, pp. 642 - 646 Dec. 2010 ISSN 1003—7985

Multiple homoclinics in a non-periodic Hamiltonian system

Ding Jian[1,2] Feng Guizhen[3] Zhang Fubao[1]

([1] Department of Mathematics, Southeast University, Nanjing 211189, China)
([2] College of Math and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China)
([3] College of Arts and Science, Nanjing Institute of Industry Technology, Nanjing 210046, China)

Abstract: This paper concerns the existence of multiple homoclinic orbits for the second-order Hamiltonian system $\ddot{z}-L(t)z+W_z(t,z)=0$, where $L\in C(\mathbf{R},\mathbf{R}^{N^2})$ is a symmetric matrix-valued function and $W(t,z)\in C^1(\mathbf{R}\times\mathbf{R}^N,\mathbf{R})$ is a nonlinear term. Since there are no periodic assumptions on $L(t)$ and $W(t,z)$ in t, one should overcome difficulties for the lack of compactness of the Sobolev embedding. Moreover, the nonlinearity $W(t,z)$ is asymptotically linear in z at infinity and the system is allowed to be resonant, which is a case that has never been considered before. By virtue of some generalized mountain pass theorem, multiple homoclinic orbits are obtained.

Key words: Hamiltonian system; homoclinic orbits; (C)-condition; asymptotical linearity; generalized mountain pass theorem

1 Introduction and Main Result

This paper concerns the existence of homoclinic orbits for the second-order Hamiltonian system

$$\ddot{z}-L(t)z+W_z(t,z)=0 \tag{1}$$

where $L\in C(\mathbf{R},\mathbf{R}^{N^2})$ is a symmetric matrix-valued function, $W(t,z)\in C^1(\mathbf{R}\times\mathbf{R}^N,\mathbf{R})$.

Recall that a solution $z(t)$ of Eq. (1) is homoclinic to 0 if $z(t)\neq 0$, $z(t)\to 0$ and $\dot{z}(t)\to 0$ as $t\to\pm\infty$.

In recent years, the existence and multiplicity of homoclinic orbits for Hamiltonian systems have been investigated in many papers via the critical point theory[1-6]. In these papers, $W(t,z)$ was generally assumed to satisfy the Ambrosetti-Rabinowitz condition. Precisely, there is a constant $\mu>2$ such that

$$0<\mu W(t,z)\leqslant W_z(t,z)z$$

for all $t\in\mathbf{R}$ and $z\in\mathbf{R}^N\setminus\{0\}$. Later, Refs. [7 - 9] weakened the Ambrosetti-Rabinowitz super quadratic condition. In many papers, researchers were also extensively interested in some other growth conditions, see Ref. [10] for the sub-quadratic case and Refs. [9, 11] for the asymptotically quadratic case. Most of the above papers assumed that $L(t)$ and $W(t,z)$ are either independent of t or periodic in t. It is well known that the major difficulty is to prove the (PS)-condition when one applies the mountain pass theorem. Generally, to prove this condition, one needs some good embedding results. It seems that the problem is a little simple when the system is periodic, which is a compact condition to some degree. Without the periodic condition, the problem becomes rather difficult. It is worth pointing out that recently some authors have treated this problem by analyzing the spectrum $\sigma(A)$ of the linear operator A corresponding to the Hamiltonian system. For example, Ref. [12] and Ref. [13] respectively considered the first-order and the second-order system with variational functionals being super quadratic (or sub-quadratic) at infinity. Under some conditions on L, the authors proved that $\sigma(A)$ consists of eigenvalues which implies some compactly embedding results. Based on these results, they obtained the convergence of the (PS)-sequence and then obtained one homoclinic orbit by using a saddle point theorem.

In this paper, we consider the second-order system (1) without periodic conditions. Unlike Refs. [12] and [13], here the nonlinear term $W(t,z)$ is asymptotically quadratic at infinity. As far as we know, there has not been much work done in treating this case. Furthermore, if $W(t,z)$ is even in z, we obtain multiple homoclinic solutions.

Just as Ref. [13], for L, we make the following assumptions:

L1) For the smallest eigenvalue $l(t)$ of $L(t)$, i.e., $l(t)=\inf_{|\xi|=1}L(t)\xi\xi$, there is a constant $\gamma<1$ such that $l(t)|t|^{\gamma-2}\to\infty$ as $|t|\to\infty$.

L2) For some $a>0$ and $r>0$, one of the following is true:

1) $L\in C^1(\mathbf{R},\mathbf{R}^{N^2})$ and $|L'(t)z|\leqslant a|L(t)z|$ for all $|t|>r$ and all $z\in\mathbf{R}^N$ with $|z|=1$;

2) $L\in C^2(\mathbf{R},\mathbf{R}^{N^2})$ and $((aL(t)-L''(t))z,z)\geqslant 0$ for all $|t|>r$ and all $z\in\mathbf{R}^N$ with $|z|=1$, where $L'(t)=(\mathrm{d}/\mathrm{d}t)L(t)$, $L''(t)=(\mathrm{d}^2/\mathrm{d}t^2)L(t)$.

We denote by A the self-adjoint extension of the operator $-(\mathrm{d}^2/\mathrm{d}t^2)+L(t)$ with the domain $D(A)\subset L^2:=L^2(\mathbf{R},\mathbf{R}^N)$. Let $|A|$ be the absolute value of A and $|A|^{1/2}$ the square root of $|A|$. Set $E=D(|A|^{1/2})$ and define the inner product on E,

$$(x,z)_0=(|A|^{1/2}x,|A|^{1/2}z)_2+(x,z)_2$$

with the corresponding norm

$$\|z\|_0=(z,z)_0^{1/2}$$

where $(\cdot,\cdot)_2$ is the inner product in L^2. Then E is a Hilbert space.

We need the following propositions from Ref. [13].

Proposition 1 Suppose that L satisfies L1), then E is compactly embedded in L^p for all $p\in[1,\infty]$, and for each $p\in[1,\infty]$, there exists a constant $C_p>0$ such that

$$|z|_p\leqslant C_p\|z\|_0$$

Received 2010-03-24.

Biographies: Ding Jian (1978—), male, doctor; Zhang Fubao (corresponding author), male, doctor, professor, zhangfubao@seu.edu.cn.

Citation: Ding Jian, Feng Guizhen, Zhang Fubao. Multiple homoclinics in a non-periodic Hamiltonian system [J]. Journal of Southeast University (English Edition), 2010, 26(4): 642 - 646.

for all $z \in E$, where $|\cdot|_p$ is the norm on $L^p := L^p(\mathbf{R}, \mathbf{R}^N)$.

It is easy to check that $D(A)$ is a Hilbert space under the inner product

$$(x, z)_A = (x, z)_2 + (Ax, Az)_2$$

with the corresponding norm

$$\| z \|_A = (|z|_2^2 + |Az|_2^2)^{1/2}$$

Proposition 2 If L satisfies L1) and L2), then $D(A)$ is continuously embedded in $W^{2,2}$, and, consequently, we obtain

$$|z(t)| \to 0, \quad |\dot{z}(t)| \to 0$$

as $|t| \to \infty$, for all $z \in D(A)$.

By proposition 1, since the self-adjoint operator A in L^2 is bounded from below, it possesses a compact resolvent. Therefore, the spectrum $\sigma(A)$ consists of eigenvalues numbered in $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \to \infty$ (counted in their multiplicities). The corresponding eigenfunctions $(e_i)_{i \in \mathbf{N}}$ form an orthonormal basis in L^2. Let n^- and n^0 be the number of λ_i satisfying $\lambda_i < 0$ and $\lambda_i = 0$, respectively, and $\bar{n} = n^- + n^0$. Set $E^- = \mathrm{span}\{e_1, \ldots, e_{n^-}\}$, $E^0 = \mathrm{span}\{e_{n^-+1}, \ldots, e_{\bar{n}}\} = \ker A$, and $E^+ = \overline{\mathrm{span}\{e_{\bar{n}+1}, \ldots\}}$, then

$$E = E^- \oplus E^0 \oplus E^+$$

We introduce another inner product on E

$$(x, z) = (|A|^{1/2} x, |A|^{1/2} z)_2 + (x^0, z^0)_2$$

The corresponding norm is

$$\| z \| = (z, z)^{1/2} = (\, \| A|^{1/2} z |_2^2 + |z^0|_2^2)^{1/2}$$

where $x = x^- + x^0 + x^+$ and $z = z^- + z^0 + z^+ \in E^- \oplus E^0 \oplus E^+$.

By proposition 1, we can prove the norms $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent (see Ref. [13]). From now on, we will always use the norm $\|\cdot\|$ on E.

Define

$$\Phi(z) := \frac{1}{2} \| z^+ \|^2 - \frac{1}{2} \| z^- \|^2 - \Psi(z)$$

where $\Psi(z) = \int_{\mathbf{R}} W(t, z)\, \mathrm{d}t$. Under our assumptions, it is easy to check that $\Phi \in C^1(E, \mathbf{R})$.

Denote $\widetilde{W}(t, z) := \frac{1}{2} W_z(t, z) z - W(t, z)$. Our assumptions read as follows:

A1) $W(t, z) \geqslant 0$ and $W_z(t, z) = o(|z|)$ as $z \to 0$ uniformly in t;

A2) $W_z(t, z) = V(t) z + R_z(t, z)$ with V being a bounded continuous symmetric $(N \times N)$-matrix valued function and $R_z(t, z) = o(|z|)$ uniformly in t as $|z| \to \infty$;

A3) $b_0 := \inf_{t \in \mathbf{R}} [\inf_{\xi \in \mathbf{R}^N, |\xi| = 1} V(t) \xi \xi] > \inf \sigma(A) \cap (0, \infty)$;

A4) $\sup_{t \in \mathbf{R}, z \neq 0} |R_z(t, z)| / |z| < \infty$;

A5) Either (a) $0 \notin \sigma(A - V)$ or (b) $\widetilde{W}(t, z) \geqslant 0$ for all (t, z) and $\widetilde{W}(t, z) \geqslant r_0$ for some $r_0 > 0$ and all (t, z) with $|z|$ large enough.

We will prove the following theorem.

Theorem 1 If assumptions L1), L2) and A1) to A5) are satisfied, then system (1) possesses at least one homoclinic solution. If in addition $W(t, z)$ is even in z, then (1) has at least l pairs of homoclinic solutions, where l is the number of linearly independent eigenfunctions with corresponding eigenvalues lying in $(0, b_0)$.

Remark 1 A2), A3) and A5) were used by Ding and Jeanjean to study some first-order system in Ref. [14]. A2) shows that W is asymptotically quadratic at infinity; A3) implies that $\sigma(A) \cap (0, b_0) \neq \varnothing$, which ensures the existence of (multiple) critical points of Φ. If 0 belongs to $\sigma(A - V)$, it causes much difficulty in proving the (C)-condition. As a compensation, we use the technical assumption A5). A4) looks more general than (R_5) in Ref. [14], where a more rigorous limit is needed since the eigenvalues of finite multiplicity of A are limited to some bounded interval.

Remark 2 The following function satisfies A2) to A5)

$$W(t, z) = a(t) \frac{z^2}{2} \left(1 - \frac{1}{\ln(e + |z|)}\right)$$

where $a(t)$ is bounded and $\inf a(t) > 0$.

We will use the following generalization of the mountain pass theorem[15] to prove theorem 1.

Theorem 2 Let E be a Banach space with $E = V \oplus X$, where V is finite dimensional. Suppose that $\Phi \in C^1(E, \mathbf{R})$ satisfies the (PS)-condition and

I1) There are constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap X} \geqslant \alpha$;

I2) There is an $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q := (\bar{B}_R \cap V) \oplus \{re \mid 0 < r < R\}$, then $\Phi|_{\partial Q} \leqslant 0$.

Then Φ possesses a critical value $c \geqslant \alpha$ which can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in Q} \Phi(h(u))$$

where

$$\Gamma := \{h \in C(\bar{Q}, E) \mid h = id \text{ on } \partial Q\}$$

To obtain multiple homoclinics, we need the following theorem which is a special case of theorem 9.12 in Ref. [15].

Theorem 3 Let E be a Banach space with $E = V \oplus X$, where V is finite dimensional. $\Phi \in C^1(E, \mathbf{R})$ is even and $\Phi(0) = 0$. In addition, Φ satisfies the (PS)-condition, I1) and

I3) Let $X_0 \subset X$ be a finite dimensional subspace and denote $E_0 = V \oplus X_0$. For each subspace $\widetilde{E}_0$ of E_0, there is an $R(\widetilde{E}_0)$ such that $\Phi \leqslant 0$ on $\widetilde{E}_0 \setminus B_{R(\widetilde{E}_0)}$.

Then Φ possesses at least $\dim X_0$ critical points.

2 Proof of the Main Result

In this section, we use theorem 2 and theorem 3 to prove theorem 1. We consider the functional Φ on $E = V \oplus X$, where $V = E^- \oplus E^0$ and $X = E^+$. Obviously, V is a finite dimensional subspace.

Instead of the (PS)-condition, here we use the (C)-condition. Recall a function Φ satisfies the (C)-condition on E if any sequence $\{z_j\} \subset E$ satisfying $\{\Phi(z_j)\}$ is bounded and

$(1+\|z_j\|)\Phi'(z_j)\to 0$ has a convergent subsequence. Theorem 2 and theorem 3 still hold true under the (C)-condition[16].

We divide the proof of theorem 1 into the following steps:

Step 1 Φ satisfies (C)-condition.

Let $\{z_j\}\subset E$ be a (C)-sequence, that is

$$|\Phi(z_j)|\leqslant C,\quad (1+\|z_j\|)\Phi'(z_j)\to 0 \tag{2}$$

where and, henceforth, C stands for some generic positive constant.

Then there exists a $C_0>0$,

$$C_0\geqslant \Phi(z_j)-\frac{1}{2}\Phi'(z_j)z_j=\int_{\mathbf{R}}\widetilde{W}(t,z_j) \tag{3}$$

Arguing indirectly, we assume that, up to a subsequence, $\|z_j\|\to\infty$. Set $w_j:=z_j/\|z_j\|$. Then $\|w_j\|=1$, $|w_j|\leqslant C_p\|w_j\|=C_p$ for $p\in[1,\infty]$. Choosing a subsequence if necessary, we assume that $w_j\rightharpoonup w$ in E and $w_j\to w$ in L^p for all $p\in[1,\infty]$.

We claim that $w\neq 0$. If not, $w=0$. Choose $n_0\geqslant 1$ and set

$$E_{n_0}=\operatorname{span}\{e^-_{n+1},\ldots,e^-_{n+n_0}\}$$

Denote

$$E^f=E^-\oplus E^0\oplus E_{n_0}$$

Then E^f is finite dimensional. E possesses an orthogonal decomposition

$$E=E^f\oplus E^c,\quad z=z^f+z^c$$

Since $w_j\rightharpoonup 0$, $w_f\rightharpoonup 0$ in E and $w_j^c\to 0$ in L^p for $p\in[1,\infty]$.

By assumptions, there exists a constant $C>0$ such that

$$|W_z(t,z)|\leqslant C|z| \tag{4}$$

and

$$W(t,z)\leqslant C|z|^2 \tag{5}$$

for all $(t,z)\in\mathbf{R}\times\mathbf{R}^N$.

In fact, by A2), there exist $C_1>0$ and $R_1>0$ such that

$$\frac{|R_z(t,z)|}{|z|}\leqslant C_1$$

for all t and $|z|\geqslant R_1$.

By A1) and A2), there is $C_2>0$ and $r>0$ such that

$$\frac{|R_z(t,z)|}{|z|}\leqslant\frac{|W_z(t,z)|}{|z|}+|V(t)|\leqslant C_2$$

for all t and $|z|\leqslant r$.

Combining A4), we obtain

$$\frac{|R_z(t,z)|}{|z|}\leqslant C$$

for some $C>0$. Then (4) and (5) follow easily since $V(t)$ is bounded.

It is easy to check that

$$\frac{\Phi'(z_j)z_j^c}{\|z_j\|^2}=\|w_j^c\|^2-\int_{\mathbf{R}}\frac{W_z(t,z_j)}{|z_j|}w_j^c|w_j|$$

Therefore, by (2) and (4),

$$\|w_j^c\|^2=\int_{\mathbf{R}}\frac{W_z(t,z_j)}{|z_j|}w_j^c|w_j|+o(1)\leqslant C\int_{\mathbf{R}}|w_j^c||w_j|+o(1)\leqslant C|w_j^c|_2^2+o(1)=o(1)$$

Therefore, $1=\|w_j\|^2=\|w_j^f\|^2+\|w_j^c\|^2\to 0$. This contradiction shows that $w\neq 0$.

Since $z_j(t)\to\infty$ if $w(t)\neq 0$, for $\varphi\in C_0^\infty(\mathbf{R},\mathbf{R}^N)$,

$$\int_{\mathbf{R}}\frac{R_z(t,z_j)\varphi}{\|z_j\|}=\int_{\mathbf{R}}\frac{R_z(t,z_j)\varphi}{|z_j|}|w_j|\leqslant \int_{\mathbf{R}}\frac{|R_z(t,z_j)||\varphi|}{|z_j|}|w_j-w|+\int_{w(t)\neq 0}\frac{|R_z(t,z_j)||\varphi|}{|z_j|}|w|\leqslant C|\varphi|_2|w_j-w|_2+o(1)=o(1)$$

Thus,

$$\int_{\mathbf{R}}\frac{W_z(t,z_j)\varphi}{\|z_j\|}=\int_{\mathbf{R}}\frac{V(t)z_j\varphi}{\|z_j\|}+\int_{\mathbf{R}}\frac{R_z(t,z_j)\varphi}{\|z_j\|}\to\int_{\mathbf{R}}V(t)w\varphi$$

for all $\varphi\in C_0^\infty(\mathbf{R},\mathbf{R}^N)$.

Therefore, by (2), we have

$$\ddot{w}-L(t)w+V(t)w=0 \tag{6}$$

The remaining argument on the boundedness of the (C)-sequence is standard[14, 17]. For the readers' convenience, we outline it as follows:

Since $w\neq 0$, (a) of A5) is impossible. Thus we assume that (b) of A5) is satisfied. Denote $Q_j(r):=\{t\in\mathbf{R}\mid z_j(t)<r\}$ and $Q_j^c(r):=\mathbf{R}\backslash Q_j(r)$. For $r\geqslant 0$, define

$$f(r):=\inf\{\widetilde{W}(t,z):t\in\mathbf{R}\text{ and }z\in\mathbf{R}^N\text{ with }|z|\geqslant r\}$$

Then there exists $r_0>0$ such that $f(r_0)>0$. By (3),

$$|Q_j^c(r_0)|\leqslant\frac{C_0}{f(r_0)}$$

Let $\bar{Q}:=\{t:w(t)\neq 0\}$. Since w satisfies (6), the Cauchy uniqueness principle implies $\bar{Q}=\mathbf{R}$. Therefore, there are $\varepsilon>0$ and $I_0\subset\bar{Q}$ such that $|w(t)|\geqslant 2\varepsilon$ for all $t\in I_0$ and

$$\frac{2C_0}{f(r_0)}\leqslant|I_0|<\infty$$

By an Egoroff's theorem, we can find a set $I'_0\subset I_0$ with $|I'_0|>\frac{C_0}{f(r_0)}$ such that $w_j\to w$ uniformly in I'_0. Thus, for almost all j, $|w_j(t)|\geqslant\varepsilon$ and $|z_j(t)|\geqslant r_0$ in I'_0.

Therefore,

$$\frac{C_0}{f(r_0)}<|I'_0|\leqslant|Q_j^c(r_0)|\leqslant\frac{C_0}{f(r_0)}$$

is a contradiction. Thus we conclude that $\{z_j\}$ is bounded.

If necessary, up to a subsequence, we can assume that z_j

$\rightharpoonup z$ in E and $z_j \to z$ in L^p for $p \in [1, +\infty]$. By (2) and (4), it is easy to see that

$$\| z_j^+ - z_k^+ \|^2 = (\Phi'(z_j) - \Phi'(z_k))(z_j^+ - z_k^+) - \int_{\mathbf{R}} (W_z(t, z_j) - W_z(t, z_k), z_j^+ - z_k^+) \leqslant$$
$$o(1) + \int_{\mathbf{R}} | W_z(t, z_j) - W_z(t, z_k) | | z_j^+ - z_k^+ | \leqslant$$
$$o(1) + \int_{\mathbf{R}} C(|z_j| + |z_k|) |z_j^+ - z_k^+| = o(1) \tag{7}$$

Therefore, $\{z_j^+\}$ is a Cauchy sequence in E. Since $\dim(E^- \oplus E^0)$ is finite, $\{z_j\}$ is a Cauchy sequence in E which clearly possesses a convergent subsequence.

Step 2 There are constants $\rho, \alpha > 0$ such that $\Phi |_{\partial B_\rho \cap X} \geqslant \alpha$.

By assumptions, given $p > 2$, for $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$| W_z(t, z) | \leqslant \varepsilon |z| + C_\varepsilon |z|^{p-1}$$

and

$$W(t, z) \leqslant \varepsilon |z|^2 + C_\varepsilon |z|^p$$

Therefore,

$$\Psi(z) \leqslant \varepsilon | z |_2^2 + C_\varepsilon | z |_p^p \leqslant C(\varepsilon \| z \|^2 + C_\varepsilon \| z \|^p)$$

and then the conclusion holds true.

Step 3 There is an $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q := (\bar{B}_R \cap V) \oplus \{re \mid 0 < r < R\}$, then $\Phi |_{\partial Q} \leqslant 0$.

In fact, we can prove that $\Phi(z) \to -\infty$ as $\| z \| \to \infty$, $z \in E_k = V \oplus \operatorname{span}\{e^-_{\bar{n}+1}, \ldots e^-_{\bar{n}+k}\}$ $(1 \leqslant k \leqslant l)$.

If not, assume for some sequence $\{z_j\} \subset E_k$ with $\| z_j \| \to \infty$, there is $b > 0$ such that $\Phi(z_j) \geqslant -b$ for all $j \in \mathbf{N}$. Setting $w_j = z_j / \| z_j \|$, we have $\| w_j \| = 1$. Since $\dim E_k$ is finite, there exist $w \in E$ and a subsequence also denoted by $\{w_j\}$ such that $w_j \rightharpoonup w$. Then $w_j^+ \to w^+$, $w_j^- \to w^-$, $w_j^0 \to w^0$.

We claim that $w^+ \neq 0$. If not, from the following inequation

$$-\frac{b}{\| z_j \|^2} \leqslant \frac{\Phi(z_j)}{\| z_j \|^2} = \frac{1}{2} \| w_j^+ \|^2 - \frac{1}{2} \| w_j^- \|^2 - \int_{\mathbf{R}} \frac{W(t, z_j)}{\| z_j \|^2} \tag{8}$$

we obtain $\| w_j^- \| \to 0$ and then $w_j \to w = w^0$. We also obtain

$$\int_{\mathbf{R}} \frac{W(t, z_j)}{\| z_j \|^2} \to 0 \tag{9}$$

By A2), $W(t, z) = \frac{1}{2} V(t) zz + R(t, z)$ and $R(t, z) / |z|^2 \to 0$ uniformly in t as $|z| \to \infty$. Since $|z_j(t)| \to \infty$ if $w(t) \neq 0$, we have

$$\int_{\mathbf{R}} \frac{R(t, z_j)}{\| z_j \|^2} = \int_{\mathbf{R}} \frac{R(t, z_j)}{|z_j|^2} |w_j|^2 \leqslant \int_{\mathbf{R}} \frac{R(t, z_j)}{|z_j|^2} |w_j - w|^2 + \int_{w(t) \neq 0} \frac{R(t, z_j)}{|z_j|^2} |w|^2 = o(1) \tag{10}$$

By A3),

$$\frac{1}{2} \int_{\mathbf{R}} \frac{V(t) z_j z_j}{\| z_j \|^2} = \frac{1}{2} \int_{\mathbf{R}} \frac{V(t) z_j z_j}{|z_j|^2} |w_j|^2 \geqslant \frac{b_0}{2} |w_j|_2^2 \tag{11}$$

From Eqs. (9) to (11), one has $|w_j|_2 \to 0$. Since $\dim E_k$ is finite, all the norms are equivalent. Then, we obtain $1 = \| w_j \| \leqslant C |w_j|_2 \to 0$; this contradiction implies $w^+ \neq 0$.

Since

$$\| w \|^2 \leqslant \lambda_{\bar{n}+l} |w|_2^2 < b_0 |w|_2^2$$

for all $w \in \operatorname{span}\{e^-_{\bar{n}+1}, \ldots e^-_{\bar{n}+l}\} \setminus \{0\}$, combining (8), (10) and (11), we obtain

$$0 \leqslant \lim_{j\to\infty} \frac{\Phi(z_j)}{\| z_j \|^2} = \lim_{j\to\infty} \left(\frac{1}{2} \| w_j^+ \|^2 - \frac{1}{2} \| w_j^- \|^2 - \frac{1}{2} \int_{\mathbf{R}} \frac{V(t) z_j z_j}{\| z_j \|^2} - \int_{\mathbf{R}} \frac{R(t, z_j)}{\| z_j \|^2} \right) \leqslant$$
$$\frac{1}{2} \| w^+ \|^2 - \frac{1}{2} \| w^- \|^2 - \frac{b_0}{2} |w|_2^2 \leqslant \frac{1}{2} (\lambda^-_{\bar{n}+l} - b_0) |w^+|_2^2 < 0$$

This contradiction implies $\Phi(z) \to -\infty$ as $z \in E_k$, $\| z \| \to \infty$.

Step 4 By step 1 to step 3, using theorem 2, we obtain a critical point z whose critical value is no less than α. Therefore, z is a nontrivial solution of system (1). That is,

$$Az = W_z(t, z)$$

Hence, by (4),

$$|Az|_2^2 = \int_{\mathbf{R}} |W_z(t, z)|^2 \leqslant C |z|_2^2 \leqslant C \| z \|^2 < \infty$$

which implies that $z \in D(A)$. By proposition 2, z is a nontrivial homoclinic orbit.

If $W(t, z)$ is even in z, Φ is also even, $\Phi(0) = 0$ and satisfies all the assumptions of theorem 3. Then we obtain at least l homoclinic orbits of system (1).

Above all, we complete the proof of theorem 1.

References

[1] Omana W, Willem M. Homoclinic orbits for a class of Hamiltonian systems [J]. *Differential Integral Equations,* 1992, **5**(5): 1115 − 1120.

[2] Izydorek M, Janczewska J. Homoclinic solutions for a class of second order Hamiltonian systems [J]. *J Differential Equations,* 2005, **219**(2): 375 − 389.

[3] Rabinowitz P H. Homoclinic orbits for a class of Hamiltonian systems[J]. *Proc Roy Soc Edinburgh Sect A*, 1990, **114**(1/2): 33 − 38.

[4] Korman P, Lazer A C. Homoclinic orbits for a class of symmetric Hamiltonian systems [J]. *Electron J Differential Equations,* 1994, **1994**(1): 1 − 10.

[5] Coti-Zelati V, Ekeland I, Sere E. A variational approach to homoclinic orbits in Hamiltonian systems [J]. *Math Ann,* 1990, **288**(1): 133 − 160.

[6] Fei G H. The existence of homoclinic orbits for Hamiltonian systems with the potentials changing sign[J]. *Chinese Ann*

Math Ser B, 1996, **17**(4): 403 - 410.
[7] Ou Z Q, Tang C L. Existence of homoclinic solution for the second order Hamiltonian systems[J]. *J Math Anal Appl,* 2004, **291**(1): 203 - 213.
[8] Felmer P L, De B e Silva E A. Homoclinic and periodic orbits for Hamiltonian systems[J]. *Ann Sc Norm Super Pisa Cl Sci,* 1998, **26**(2): 285 - 301.
[9] Lü Y, Tang C L. Existence of even homoclinic orbits for second-order Hamiltonian systems[J]. *Nonlinear Anal,* 2007, **67**(7): 2189 - 2198.
[10] Salvatore A. Homoclinic orbits for a special class of nonautonomous Hamiltonian systems[J]. *Nonlinear Anal,* 1997, **30**(8): 4849 - 4857.
[11] Szulkin A, Zou W. Homoclinic orbits for asymptotically linear Hamiltonian systems[J]. *J Functional Anal,* 2001, **187**(1): 25 - 41.
[12] Ding Y H, Li S J. Homoclinic orbits for first order Hamiltonian systems[J]. *J Math Anal Appl,* 1995, **189**(2): 585 - 601.
[13] Ding Y H. Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems[J]. *Nonlinear Anal,* 1995, **25**(11): 1095 - 1113.
[14] Ding Y H, Jeanjean L. Homoclinic orbits for a nonperiodic Hamiltonian system[J]. *J Differential Equations,* 2007, **237**(2): 473 - 490.
[15] Rabinowitz P H. Minimax methods in critical point theory with applications to differential equations[C] // *Expository lectures from the CBMS Regional Conference.* Miami, USA, 1986, **65**: 96 - 100.
[16] Bartolo P, Benci V, Fortunato D. Abstract critical point theorems and applications to some nonlinear problems with “strong” resonance at infinity[J]. *Nonlinear Anal,* 1983, **7**(9): 981 - 1012.
[17] Ding Y H, Szulkin A. Bound states for semilinear Schrodinger equations with sign-changing potential[J]. *Calc Var Partial Differential Equations,* 2007, **29**(3): 397 - 419.

非周期 Hamilton 系统的多同宿轨

丁 建[1,2] 冯桂珍[3] 张福保[1]

([1]东南大学数学系,南京 211189)
([2] 南京信息工程大学数理学院, 南京 210044)
([3] 南京工业职业技术学院人文数理学院, 南京 210046)

摘要:研究了二阶 Hamilton 系统 $\ddot{z}-L(t)z+W_z(t,z)=0$ 多个同宿轨的存在性,其中 $L\in C(\mathbf{R},\mathbf{R}^{N^2})$ 是一对称矩阵值函数,$W(t,z)\in C^1(\mathbf{R}\times\mathbf{R}^N,\mathbf{R})$ 是非线性项. 由于 $L(t)$ 和 $W(t,z)$ 关于 t 没有周期性假设,需要克服 Sobolev 嵌入缺乏紧性的困难. 而且,这里非线性项 $W(t,z)$ 关于 z 在无穷远处是渐进线性的且系统允许出现共振,这一情形之前未被考虑过. 借助于广义的山路定理,得到了多个同宿轨.

关键词: Hamilton 系统;同宿轨;(C)-条件;渐近线性;广义山路定理

中图分类号:O175