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Domain-based noise removal method using fourth-order partial differential equation

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Abstract: Due to the fact that the fourth-order partial differential equation (PDE) for noise removal can provide a good trade-off between noise removal and edge preservation and avoid blocky effects often caused by the second-order PDE, a domain-based fourth-order PDE method for noise removal is proposed. First, the proposed method segments the image domain into two domains, a speckle domain and a non-speckle domain, based on the statistical properties of isolated speckles in the Laplacian domain. Then, depending on the domain type, different conductance coefficients in the proposed fourth-order PDE are adopted. Moreover, the frequency approach is used to determine the optimum iteration stopping time. Compared with the existing fourth-order PDEs, the proposed fourth-order PDE can remove isolated speckles and keeps the edges from being blurred. Experimental results show the effectiveness of the proposed method.

Key words: fourth-order partial differential equation; conductance coefficient; speckle domain; image denoising
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The use of the partial differential equations (PDEs) for image denoising has become a major research topic in the past few years. The PDEs include anisotropic diffusion equations[1–4], total variation models[5] and curve evolution equations[6]. One of the most popular and successful methodologies for image denoising is the use of the anisotropic diffusion equations which was first introduced by Perona and Malik[1]. Let u denote the image intensity function, t the time, and $c(\cdot)$ the conductance coefficient, and the following second-order nonlinear diffusion model (the PM second-order PDE) is considered.

$$\frac{\partial u}{\partial t} - \nabla \cdot (c(|\nabla u|)\nabla u) = 0 \tag{1}$$

where $\nabla \cdot$ and ∇ denote the divergence and the gradient, respectively. It is designed with an explicit goal of achieving a good trade-off between noise removal and edge preservation. Although the PM second-order PDE and its variances are better in image denoising, these methods tend to cause blocky effects in processed images. In Refs. [7 – 8], it is noted that the PM second-order PDE is a second-order model. This feature guarantees its ability to reconstruct images with discontinuities, but it is responsible for the block effect.

An effective solution to this problem was introduced by You and Kaveh[8] in which a fourth-order PDE is used for noise removal. The YK fourth-order PDE replaces the gradient operator in the PM second-order PDE with a Laplacian operator. Due to the fact that the Laplacian of an image at a pixel is zero only if the image is planar in its neighborhood, the YK fourth-order PDE attempts to remove noise and preserve edges by approximating an observed image with a piecewise planar image. Therefore, the denoised image appears less blocky and more natural than that processed by the PM second-order PDE. However, the YK fourth-order PDE tends to leave the processed images with isolated black and white speckles which may be characterized as pixels whose intensity values are either much larger or smaller than those of their neighboring pixels. Although You and Kaveh proposed using median filtering to process the denoised image by the YK fourth-order PDE, the median filter can degrade the image to some degree. The main reason is that the ordering process destroys spatial neighborhood information and some structures.

In this paper, a novel fourth-order PDE is presented. The proposed fourth-order PDE preserves the advantage of the fourth-order PDE and avoids leaving isolated black and white speckles. Moreover, the new method preserves fine details, sharp corners, curved structures and thin lines. The numerical experiments show that the proposed method can outperform the other methods in terms of the quality of the denoised image.

1 Fourth-Order PDEs

In recent years, a number of authors have presented the analogous fourth-order PDE for image denoising[7–11]. The theoretical analysis in Refs.[8, 11] shows that fourth-order equations have advantages over the second-order equations in some aspects. First, the fourth-order linear diffusion damps oscillations much faster than the second-order diffusion. Secondly, the second-order PDE evolves toward a piecewise constant approximation in smooth regions. Unlike the second-order PDE, the fourth-order PDE evolves toward a piecewise smooth image if the image support is infinite. It is well known that piecewise smooth images look more natural than the piecewise constant images[8]. Therefore, the block effects are reduced and the image appears more natural.

In Ref.[8], You and Kaveh proposed the following fourth-order PDE,

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$$\frac{\partial u}{\partial t} = -\nabla^2(J_1(\nabla^2 u)) = -\nabla^2(c_1(|\nabla^2 u|)\nabla^2 u) \quad (2)$$

where ∇^2 denotes the Laplacian operator; $J(\cdot)$ is the flux function; and the conductance coefficient $c_1(\cdot)$ in the YK fourth-order PDE is a function of the absolute value of the Laplacian of the image intensity; that is

$$c_1(|\nabla^2 u|) = \frac{1}{1 + (|\nabla^2 u| / k)^2} \quad (3)$$

where $k > 0$ is the Laplacian threshold. A small k will cause slow diffusion within homogeneous regions, while a large k leads to low contrast edges or textures to be smoothed out. The plots of the conductance coefficient $c_1(\cdot)$ and flux function $J_1(\cdot)$ for different values of the Laplacian threshold k are shown in Fig. 1. The YK fourth-order PDE replaces the gradient operator in the PM second-order PDE with a Laplacian operator. Due to the fact that the Laplacian of an image at a pixel is zero only if the image is planar in its neighborhood, the YK fourth-order PDE attempts to remove noise and preserve edges by approximating an observed image with a piecewise planar image. Therefore, the denoised image appears less blocky and more natural than that processed by the PM second-order PDE.

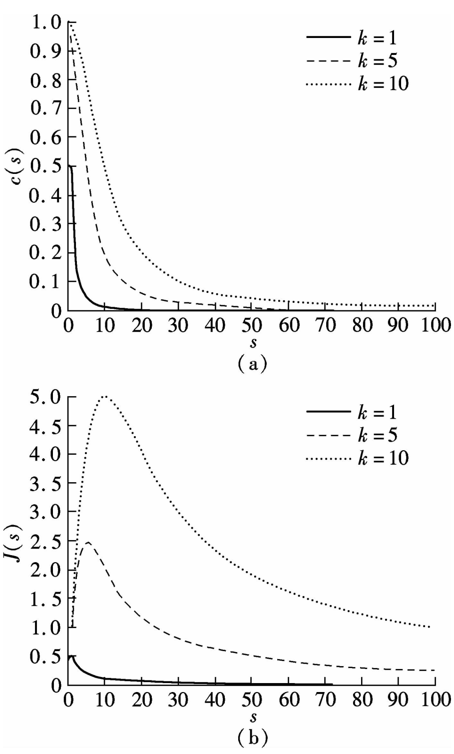


Fig. 1 Conductance coefficient c_1 and flux function J_1 for different Laplacian threshold values. (a) Conductance coefficient c_1; (b) Flux function J_1

However, the YK fourth-order PDE tends to leave the processed images with isolated black and white speckles which may be characterized as pixels whose intensity values are either much larger or smaller than those of their neighboring pixels. For instance, if the pixel located on the verge in the neighborhood is almost black, and if it is next to a white edge in the image, then the center pixel appears darker than its neighboring pixels because the intensity value of the center pixel is weakened by the verge pixel in the neighborhood. Then the value of the Laplacian at the pixel will be very large, and as a result, the value of flux at the pixel will be close to zero at the pixel. Consequently, the value of the pixel will remain when using Eq. (2).

In order to remove the isolated speckles, You and Kaveh proposed using median filtering to process the images by the YK model. A recently developed method known as the hybrid model of the fourth-order PDE[12] tries to deal with this problem by attaching a relaxed median filter (RMF)[13] at the end of the YK fourth-order (denoted as "the YK model + RMF"). These two methods can degrade the image to some degree. The main reason is that the ordering process destroys some structures and spatial neighborhood information. Lu et al.[14] proposed a new fourth-order PDE based on the changes of the Laplacian (denoted as "LC fourth-order PDE"), which has the following formula,

$$\frac{\partial u}{\partial t} = -\nabla \cdot (c_1(|\nabla \Delta u|)\nabla \Delta u) \quad (4)$$

where $\Delta = \nabla^2$ is the Laplacian operator. In fact, as much as the order of the derivative of the image is higher, the sensitivity to the noise is higher. Therefore, the image processed by the LC fourth-order PDE cannot effectively remove noise when the image noise level is higher. As mentioned earlier, the YK fourth-order PDE leaves the processed images with isolated black and white speckles, while the improved versions of the YK fourth-order PDE can degrade the image to some degree when removing the isolated speckles. In the following section, we will address this problem.

2 Proposed Fourth-Order PDE

The ability of edge preservation in the fourth-order PDE-based denoising method strongly depends on the conductance coefficient. The desirable conductance coefficient should be diffused more in smooth areas and less around intensity transitions, so that small variations in image intensity such as noise and unwanted texture are smoothed and edges are preserved.

Let $\Omega \in \mathbf{R}^2$ be a rectangular image domain and $\Omega_T = \Omega \times [0, T]$. We propose a new formula for the conductance coefficient in the form of

$$c_2(x, y, t) = \begin{cases} \dfrac{1}{\sqrt{1 + (|\nabla^2 u(x, y, t)| / k)^2}} & (x, y, t) \in \Omega'_T = \Omega' \times [0, T] \\ \dfrac{1}{1 + (|\nabla^2 u(x, y, t)| / k)^2} & (x, y, t) \in \Omega_T - \Omega'_T \end{cases} \quad (5)$$

where $T > 0$ is the stopping time; Ω'_T is the isolated speckle domain which will be determined later.

As mentioned in the previous section, isolated speckles have significantly smaller or larger intensity values than their neighboring pixels. Correspondingly, isolated speckles have significantly smaller or larger Laplacian values than their neighboring pixels in the Laplacian domain. In order to exploit this property, we present the following algorithm to determine domain Ω'_T. Let U denote the Laplacian map of image u, and the mean and variance of the neighboring points around point (x, y) in the Laplacian domain U are

$$m = \frac{U_{x+1,y} + U_{x-1,y} + U_{x,y-1} + U_{x,y+1}}{4} \tag{6}$$

and

$$\sigma^2 = \frac{U^2_{x+1,y} + U^2_{x-1,y} + U^2_{x,y-1} + U^2_{x,y+1}}{4} - m^2 \tag{7}$$

For domain segmentation using m and σ^2, we propose the following scheme:

$$(x, y) \in \begin{cases} \Omega'_T & |U_{x,y} - m| > a\sigma \\ \Omega_T - \Omega'_T & \text{otherwise} \end{cases} \tag{8}$$

where a is a constant that may be adjusted for a specific application.

The first term in Eq. (5) implies the diffusion of the isolated speckles. In this domain, our proposed fourth-order PDE has the following form:

$$\frac{\partial u}{\partial t} = -\nabla^2 \left(\frac{1}{\sqrt{1 + (|\nabla^2 u| / k)^2}} \nabla^2 u \right) \tag{9}$$

In the domain $\Omega_T - \Omega'_T$, our proposed fourth-order PDE has the following form:

$$\frac{\partial u}{\partial t} = -\nabla^2 \left(\frac{1}{1 + (|\nabla^2 u| / k)^2} \nabla^2 u \right) \tag{10}$$

which is similar to Eq. (2).

The implementation of the fourth-order PDE process is iterative. Thus, implementation of an iterative algorithm greatly depends on the termination time T, which causes what we often refer to as the termination problem. It can be stopped manually by setting T to be a fixed number. However, in real applications, different images require different termination times. Several criteria for estimating the optimal stopping time are preferred[15-17]. In this paper, the frequency approach proposed in Ref. [17] is used to determine the optimum iteration stopping time when compared with other methods.

Given an image, $u(x, y, 0)$ denotes the original intensity of pixel (x, y), and we use the explicit Euler scheme with a forward difference method for the time derivative, the central difference scheme with a 3×3 kernel for the spatial derivatives, and the 8-nearest neighborhood discretization of the Laplacian operator for computing the Laplacian of the image. We summarize the proposed fourth-order PDE method as follows:

Step 1 Initialization.

Input a given image u. $u(x, y, 0)$ denotes the original intensity of pixel (x, y).

Set parameters a, k, and T.

Step 2 Iterate until $t = T$.

Determine the isolated speckles domain Ω'_T.

For each pixel (x, y), if $(x, y) \in \Omega'_T$, then Eq. (9) is chosen for denoising; otherwise, choose Eq. (10).

A block diagram of the proposed fourth-order PDE algorithm is shown in Fig. 2.

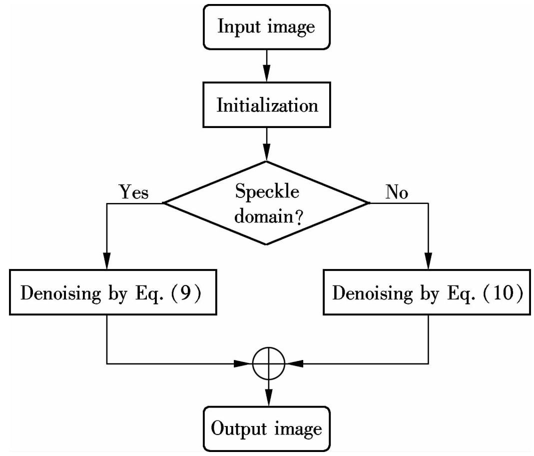


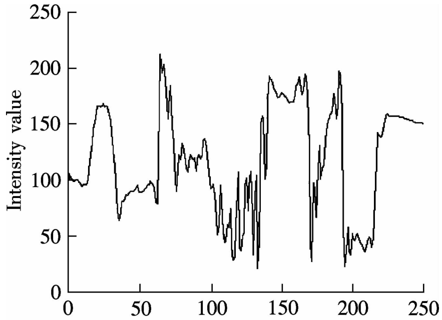
Fig. 2 Block diagram of proposed fourth-order PDE method

3 Experiments

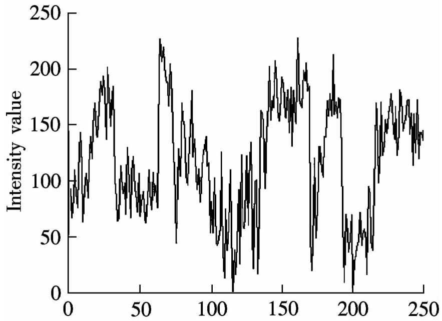
In this section, we present numerical results obtained by applying the proposed fourth-order PDE to image denosing. We test the proposed method on the "Lena" image with a size of 256×256 and the "license plate" image with a size of 240×306.

Fig. 3 shows a comparative analysis of the proposed method with the YK fourth-order PDE method. Figs. 3(a) and (b) show the original "Lena" image and the corresponding noisy image. The results for the YK fourth-order PDE and the proposed fourth-order PDE are displayed in Figs. 3(c) and (d). In order to appraise the nonlinear behavior of the two methods, four plots of a horizontal line from the image are depicted in Fig. 4. Figs. 4(a) and (b) show the row number 100 of the original image and the noisy image, respectively. The corresponding rows of processed images using the YK fourth-order PDE and the proposed fourth-order PDE method are presented in Figs. 4(c) and (d). From Figs. 3 and 4, it can be seen that the YK fourth-order PDE leaves the processed image with isolated speckles, while the

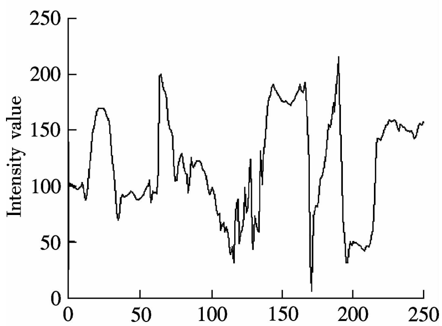
Fig. 3 Comparison of proposed method and YK fourth-order PDE for "Lena" image. (a) Original image; (b) Noised image; (c) YK fourth-order PDE with $k = 0.5$; (d) The proposed fourth-order PDE with $k = 0.5$ and $a = 4$



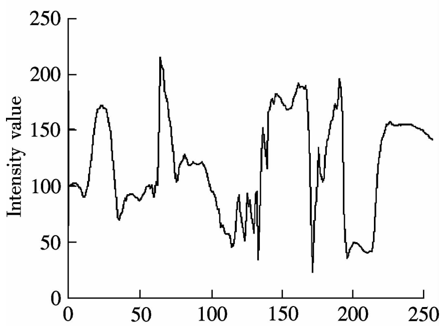
(a)



(b)



(c)



(d)

Fig. 4 Plot of gray level values obtained along one line of Fig. 3. (a) Original image; (b) Noised image; (c) YK fourth-order PDE; (d) The proposed fourth-order PDE

proposed fourth-order PDE exhibits a better result. To quantify the achieved performance improvements, we adopt improvement in signal-to-noise ratio (ISNR), which is defined as

$$\mathrm{ISNR} = 10\log 10\left(\frac{\sum_{i,j}[u(i,j) - u_0(i,j)]^2}{\sum_{i,j}[u(i,j) - u_{\mathrm{new}}(i,j)]^2}\right) \tag{11}$$

where $u_0(\cdot)$ is the initial image (noised image) and $u_{\mathrm{new}}(\cdot)$ is the denoised image. The greater the value of ISNR, the better the restored image. The ISNR values of the YK fourth-order PDE and the proposed fourth-order PDE are 6.388 1 and 7.203 5 dB, respectively. The performance of the proposed fourth-order PDE is obviously better.

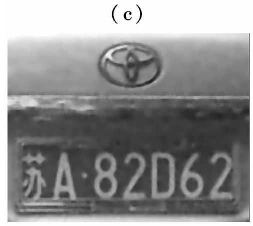
To verify the effectiveness of the proposed fourth-order PDE method for image denoising, it is evaluated by comparison with the PM second-order PDE[1], the YK fourth-order PDE[8], the YK fourth-order PDE + RMF[12], and the LC fourth-order PDE[14]. Fig. 5(a) shows the original "license plate" image and we generate a noisy image, as shown in Fig. 5(b). The results yielded by the PM second-order PDE with $k=5$ and the YK fourth-order PDE with $k=0.5$ are shown in Figs. 5(c) and (d), respectively. We observe that the PM second-order PDE can cause the processed image to be blocky, but the YK fourth-order PDE can avoid this blocky effect. However, both the PM second-order PDE and the YK fourth-order PDE tend to leave the processed image with isolated speckles. Fig. 5(e) is the denoised image by using "YK fourth-order PDE + RMF". It can remove the isolated speckles but it can degrade the image to some degree. Fig. 5(f) is the denoised image by using the LC fourth-order PDE with $k=0.5$. Fig. 5(g) is the denoised image using the proposed fourth-order PDE with $a=4$, while other parameters are the same as those obtained in Figs. 5 (d) and (c). The PSNR corresponding to these

(a)

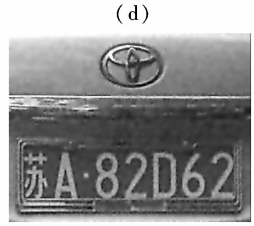
(b)

(c)

(d)



(e)



(f)

(g)

Fig. 5 Comparison of different methods for image denoising. (a) Original image; (b) Noised image; (c) PM second-order PDE[1]; (d) YK fourth-order PDE[8]; (e) YK fourth-order PDE + RMF[12]; (f) LC fourth-order PDE[14]; (g) The proposed fourth-order PDE results are listed in Tab. 1. From Fig. 5 and Tab. 1, we can see that the proposed method shows the best performance.

Tab. 1 ISNR values for "license plate" image

Image	ISNR/dB
PM second-order PDE	5.926 6
YK fourth-order PDE	6.072 3
YK fourth-order PDE + RMF	5.951 7
LC fourth-order PDE	6.304 6
Proposed fourth-order PDE	6.993 6

4 Conclusion

A denoising method based on the fourth-order PDE is proposed. The proposed method segments the image domain into two domains, a speckle domain and a non-speckle domain, based on the statistical properties of isolated speckles in the Laplacian domain. According to the domain type, a different conductance coefficient is used in the fourth-order PDE. The proposed method inherits the advantages of the fourth-order PDE, which can avoid the blocky effects widely seen in images processed by the second-order PDE. Compared with the existing fourth-order PDEs, the proposed fourth-order PDE can remove isolated speckles and keep the edges from being blurred. Experimental results verify the effectiveness of the proposed method.

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基于区域的四阶偏微分方程去噪方法

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摘要:由于四阶偏微分方程的图像去噪方法在有效降噪的同时能很好地保持特征并避免二阶偏微分方程处理图像常出现的块状效应,提出了一种基于区域的四阶偏微分方程的图像去噪方法. 首先,该方法根据斑点噪声在拉普拉斯域中所具有的统计特性,将图像区域分为斑点区域和非斑点区域;然后,针对不同的区域采用不同的传导系数. 此外,采用频域法来确定迭代停止时间. 与已有的四阶偏微分方程图像去噪方法相比,所提出的方法能够更好地去除斑点和保持边缘. 实验结果验证了所提出方法的有效性.

关键词:四阶偏微分方程; 传导系数;斑点域; 图像去噪

中图分类号:TP391