Journal of Southeast University (English Edition) Vol. 27, No. 2, pp. 188 – 191 June 2011 ISSN 1003—7985

A new method for integration of a Birkhoffian system

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Abstract: The idea of the gradient method for integrating the dynamical equations of a nonconservative system presented by Vujanović is transplanted to a Birkhoffian system, which is a new method for the integration of Birkhoff's equations. First, the differential equations of motion of the Birkhoffian system are written out. Secondly, $2n$ Birkhoff's variables are divided into two parts, and assume that a part of the variables is the functions of the remaining part of the variables and time. Thereby, the basic quasi-linear partial differential equations are established. If a complete solution of the basic partial differential equations is sought out, the solution of the problem is given by a set of algebraic equations. Since one can choose n arbitrary Birkhoff's variables as the functions of n remains of variables and time in a specific problem, the method has flexibility. The major difficulty of this method lies in finding a complete solution of the basic partial differential equation. Once one finds the complete solution, the motion of the systems can be obtained without doing further integration. Finally, two examples are given to illustrate the application of the results.

Key words: Birkhoffian system; integration method; basic partial differential equation

doi: 10.3969/j.issn.1003 – 7985.2011.02.015

It is well known that there is a complete set of highly effective integration methods for holonomic conservative systems. When a non-potential force or a nonholonomic constraint exits, many classical methods run into serious difficulties. Even if they can still be used, they should be subjected to extremely harsh restrictions[1], for example, the Hamilton-Jacobi method on the promotion of nonholonomic mechanical systems[2]. On the assumption that the generalized momentum in the sense of Hamiltonian mechanics can be expressed as a function of generalized coordinates and time, Vujanović[3] presented a gradient method in nonconservative mechanics. This method provides an important tool for the integration of the dynamical equations of holonomic nonconservative systems. Mei[4] extended the gradient method to the nonholonomic system. In recent years, although the studies of integration methods for constrained mechanical systems have yielded a number of important results[5–22], the application documents are still rare on the gradient method. In this paper, we further transplant the idea of the gradient method to a Birkhoffian system. We divide $2n$ Birkhoff's variables into two parts, and assume that one of the variables is the function of the remaining n variables and time. Thus the gradient method is successfully extended to the Birkhoffian system and a new method for integrating Birkhoff's dynamical equations is introduced.

Received 2011-01-05.

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Foundation item: The National Natural Science Foundation of China (No. 10972151).

Citation: Zhang Yi. A new method for integration of a Birkhoffian system [J]. Journal of Southeast University (English Edition), 2011, 27(2): 188 – 191. [doi: 10.3969/j.issn.1003 – 7985.2011.02.015]

1 Differential Equation of Motion of a Birkhoffian System

The differential equation of motion of a Birkhoffian system is[23]

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=0 \qquad \mu,\nu=1,2,\ldots,2n \tag{1}$$

where $B=B(t,a)$ is called Birkhoffian, $R_\mu=R_\mu(t,a)$ is Birkhoff's function and

$$\Omega_{\mu\nu}=\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu} \tag{2}$$

is called Birkhoff's tensor. Suppose that the system (1) is nonsingular, i.e.,

$$\det(\Omega_{\mu\nu})\neq 0 \tag{3}$$

then all of $\dot{a}^\mu$ can be solved by Eq. (1), and we obtain

$$\dot{a}^\mu=\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}\right) \qquad \mu=1,2,\ldots,2n \tag{4}$$

where $\Omega^{\mu\nu}\Omega_{\nu\tau}=\delta_{\mu\tau}$.

2 New Method for Integrating Birkhoff's Equations

Assume that $2n$ Birkhoff's variable a^μ can be divided into two parts, a^j ($j=1,2,\ldots,n$) and a^σ ($\sigma=n+1,\ldots,2n$). Without loss of generality, suppose that the variable a^j can be expressed as a function of the variable a^σ and time t; that is

$$a^j=\phi^j(a^\sigma,t) \qquad j=1,2,\ldots,n;\ \sigma=n+1,\ldots,2n \tag{5}$$

Hence, we have

$$\dot{a}^j=\frac{\partial\phi^j}{\partial a^\sigma}\dot{a}^\sigma+\frac{\partial\phi^j}{\partial t} \tag{6}$$

Substituting Eq. (4) into Eq. (6), we obtain the basic system of quasi-linear partial differential equations,

$$\frac{\partial\phi^j}{\partial t}+\frac{\partial\phi^j}{\partial a^\sigma}\Omega^{\sigma\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}\right)-\Omega^{j\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}\right)=0$$
$$j=1,2,\ldots,n;\ \sigma=n+1,\ldots,2n;\ \nu=1,2,\ldots,2n \tag{7}$$

By utilizing Eq. (5), a complete solution of Eq. (7) can be expressed as

$$a^j=\phi^j(a^\sigma,t,C_A)$$

$$j = 1, 2, \ldots, n; \sigma = n + 1, \ldots, 2n; A = 1, 2, \ldots, 2n \quad (8)$$

where C_A are the constants of integration. When Eq. (8) is substituted into Eq. (7), Eq. (7) will be reduced to an identity. By applying the initial condition

$$a^{\mu}(0) = a_0^{\mu} \qquad \mu = 1, 2, \ldots, 2n \quad (9)$$

and substituting Eq. (9) into Eq. (8), we can express n constants of integration, for example, $C_{n+1}, \ldots, C_{2n}$, in terms of the rest of constant $C_k (k = 1, 2, \ldots, n)$ and initial condition a_0^{μ}; therefore, we have

$$a^j = \phi^j(a^{\sigma}, t, a_0^{\mu}, C_k)$$
$$j, k = 1, 2, \ldots, n; \ \sigma = n + 1, \ldots, 2n; \ \mu = 1, 2, \ldots, 2n \quad (10)$$

Thus we can obtain the following theorem.

Theorem 1 If the equation

$$\frac{\partial \phi^j}{\partial C_k} = 0 \qquad j \text{ fixed}; \ k = 1, 2, \ldots, n \quad (11)$$

is a linear algebraic equation for $a^{n+1}, \ldots, a^{2n}$, and the following formula is set up within the domain of t, a^{σ}, i. e.,

$$\det\left(\frac{\partial^2 \phi^j}{\partial C_k \partial a^{\sigma}}\right) \neq 0$$
$$j \text{ fixed}; \ k = 1, 2, \ldots, n; \ \sigma = n + 1, \ldots, 2n \quad (12)$$

Then Eq. (10) in combination with Eq. (11) gives a general solution of the Birkhoffian system (1) in the initial condition (9).

Proof Differentiating Eq. (11) with respect to t, we obtain

$$\frac{\partial^2 \phi^j}{\partial C_k \partial t} + \frac{\partial^2 \phi^j}{\partial C_k \partial a^{\sigma}} \dot{a}^{\sigma} = 0 \quad (13)$$

Taking partial derivation of the basic partial differential equation (7) with respect to C_k, we have

$$\frac{\partial^2 \phi^j}{\partial t \partial C_k} + \frac{\partial^2 \phi^j}{\partial a^{\sigma} \partial C_k} \Omega^{\sigma\nu}\left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right) + \frac{\partial \phi^j}{\partial a^{\sigma}} \frac{\partial}{\partial a^l}\left[\Omega^{\sigma\nu}\left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right)\right]\frac{\partial \phi^l}{\partial C_k} - \frac{\partial}{\partial a^l}\left[\Omega^{j\nu}\left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right)\right]\frac{\partial \phi^l}{\partial C_k} = 0$$
$$j \text{ fixed}; \ k, l = 1, 2, \ldots n; \ \sigma = n + 1, \ldots, 2n; \ \nu = 1, 2, \ldots, 2n \quad (14)$$

By using Eq. (11), Eq. (14) becomes

$$\frac{\partial^2 \phi^j}{\partial t \partial C_k} + \frac{\partial^2 \phi^j}{\partial a^{\sigma} \partial C_k} \Omega^{\sigma\nu}\left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right) = 0 \quad (15)$$

Comparing Eqs. (13) and (15), we obtain

$$\dot{a}^{\sigma} = \Omega^{\sigma\nu}\left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right)$$
$$\sigma = n + 1, \ldots, 2n; \ \nu = 1, 2, \ldots, 2n \quad (16)$$

Eq. (16) gives the last n equations of Eq. (4).

Differentiating Eq. (10) with respect to t, we have

$$\frac{\partial \phi^j}{\partial t} + \frac{\partial \phi^j}{\partial a^{\sigma}} \dot{a}^{\sigma} - \dot{a}^j = 0 \quad (17)$$

Substituting Eq. (16) into Eq. (17) and utilizing the basic partial differential equation (7), we have

$$\dot{a}^j = \Omega^{j\nu}\left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right) \qquad j = 1, 2, \ldots, n; \nu = 1, 2, \ldots, 2n \quad (18)$$

Eq. (18) is the first n equations of Eq. (4).

3 Examples

Example 1 Birkhoffian and Birkhoff's functions of a four-dimensional Birkhoffian system are respectively[23-24]

$$B = \frac{1}{2}\left(a^3 - \frac{1}{b}\arctan bt\right)^2 + \frac{1}{2}\left[a^4 - \frac{1}{2b}\ln(1 + b^2t^2)\right]^2$$
$$R_1 = a^3, \ R_2 = a^4, \ R_3 = R_4 = 0 \quad (19)$$

where b is a constant. Let us try to find the motion of the system with the method of this paper.

Birkhoff's equation (1) gives

$$\left.\begin{aligned} &\dot{a}^3 = 0, \ \dot{a}^4 = 0, \ \dot{a}^1 - a^3 + \frac{1}{b}\arctan bt = 0 \\ &\dot{a}^2 - a^4 + \frac{1}{2b}\ln(1 + b^2t^2) = 0 \end{aligned}\right\} \quad (20)$$

Let

$$a^1 = \phi^1(a^3, a^4, t), \ a^2 = \phi^2(a^3, a^4, t) \quad (21)$$

Then the basic partial differential equation (7) gives

$$\frac{\partial \phi^1}{\partial t} - a^3 + \frac{1}{b}\arctan bt = 0, \ \frac{\partial \phi^2}{\partial t} - a^4 + \frac{1}{2b}\ln(1 + b^2t^2) = 0 \quad (22)$$

The complete solution of the quasi-linear partial differential equation (22) is easily seen as

$$\left.\begin{aligned} a^1 = \phi^1 = C_3 - \frac{t}{b}\arctan bt + {}& \\ \frac{1}{2b^2}\ln(1 + b^2t^2) + (t + C_1)a^3 + C_2 a^4 & \\ a^2 = \phi^2 = C_4 - \frac{1}{b^2}\arctan bt + \frac{t}{b} - {}& \\ \frac{t}{2b}\ln(1 + b^2t^2) + C_2 a^3 + (t + C_1)a^4 & \end{aligned}\right\} \quad (23)$$

where C_1, C_2, C_3, C_4 are the constants of integration. Using the inertial conditions of the system, we obtain

$$\left.\begin{aligned} C_3 = a_0^1 - C_1 a_0^3 - C_2 a_0^4 \\ C_4 = a_0^2 - C_2 a_0^3 - C_1 a_0^4 \end{aligned}\right\} \quad (24)$$

Substituting Eq. (24) into Eq. (23), we obtain

$$\left.\begin{aligned} a^1 = \phi^1 = a_0^1 - C_1 a_0^3 - C_2 a_0^4 - \frac{t}{b}\arctan bt + {}& \\ \frac{1}{2b^2}\ln(1 + b^2t^2) + (t + C_1)a^3 + C_2 a^4 & \\ a^2 = \phi^2 = a_0^2 - C_2 a_0^3 - C_1 a_0^4 - \frac{1}{b^2}\arctan bt + {}& \\ \frac{t}{b} - \frac{t}{2b}\ln(1 + b^2t^2) + C_2 a^3 + (t + C_1)a^4 & \end{aligned}\right\} \quad (25)$$

From

$$\frac{\partial \phi_1}{\partial C_1}=0,\ \frac{\partial \phi_1}{\partial C_2}=0 \quad \text{or} \quad \frac{\partial \phi_2}{\partial C_1}=0,\ \frac{\partial \phi_2}{\partial C_2}=0$$

we have

$$-a_0^3+a^3=0,\ -a_0^4+a^4=0 \tag{26}$$

Therefore, we have

$$a^3=a_0^3,\ a^4=a_0^4 \tag{27}$$

Substituting Eq. (27) into Eq. (25), we obtain

$$\left.\begin{aligned} a^1&=a_0^1-\frac{t}{b}\arctan bt+\frac{1}{2b^2}\ln(1+b^2t^2)+ta_0^3\\ a^2&=a_0^2-\frac{1}{b^2}\arctan bt+\frac{t}{b}-\frac{t}{2b}\ln(1+b^2t^2)+ta_0^4\end{aligned}\right\} \tag{28}$$

Eqs. (27) and (28) are the general solutions of the problem under consideration, which are in correspondence with the results given by Mei using the field method[23].

Example 2 Birkhoffian and Birkhoff's functions of a Birkhoffian system are

$$B=a^2a^3+\frac{1}{2}(a^4)^2,\ R_1=a^2,\ R_2=0,\ R_3=a^4,\ R_4=0 \tag{29}$$

Let us try to find the motion of the system with the above method.

Birkhoff's equation (1) gives

$$-\dot a^2=0,\ \dot a^1-a^3=0,\ -\dot a^4-a^2=0,\ \dot a^3-a^4=0 \tag{30}$$

Let

$$a^1=\phi^1(a^2,a^4,t),\ a^3=\phi^3(a^2,a^4,t) \tag{31}$$

Then the basic partial differential equation (7) gives

$$\frac{\partial\phi^1}{\partial t}-\frac{\partial\phi^1}{\partial a^4}a^2-\phi^3=0,\ \frac{\partial\phi^3}{\partial t}-\frac{\partial\phi^3}{\partial a^4}a^2-a^4=0 \tag{32}$$

Suppose that the solution of Eq. (32) is of the form

$$\left.\begin{aligned}\phi^1&=f_1(t)+f_2(t)a^2+f_3(t)a^4\\ \phi^3&=g_1(t)+g_2(t)a^2+g_3(t)a^4\end{aligned}\right\} \tag{33}$$

Substituting Eq. (33) into Eq. (32) and equating to zero terms which contain a^2, a^4 and free terms, we obtain

$$\left.\begin{aligned}&\dot f_1-g_1=0,\ \ \dot f_2-f_3-g_2=0,\ \ \dot f_3-g_3=0\\ &\dot g_1=0,\ \ \dot g_2-g_3=0,\ \ \dot g_3-1=0\end{aligned}\right\} \tag{34}$$

Integrating Eq. (34), we obtain

$$\left.\begin{aligned} f_1&=C_1+C_4t\\ f_2&=C_2+(C_5+C_3)t+C_6t^2+\frac{1}{3}t^3\\ f_3&=C_3+C_6t+\frac{1}{2}t^2\\ g_1&=C_4\\ g_2&=C_5+C_6t+\frac{1}{2}t^2\\ g_3&=C_6+t\end{aligned}\right\} \tag{35}$$

where $C_i (i=1, 2, \ldots, 6)$ are the constants of integration. The initial condition is

$$a^\mu(0)=a_0^\mu \qquad \mu=1,2,\ldots,4 \tag{36}$$

Substituting Eq. (35) into Eq. (33) and making use of the initial condition (36) to cancel C_1, C_4, we obtain

$$\left.\begin{aligned} a^1=\phi^1&=a_0^1-C_2a_0^2-C_3a_0^4+(a_0^3-C_5a_0^2-C_6a_0^4)\ t+\\ &\left[C_2+(C_5+C_3)t+C_6t^2+\frac{1}{3}t^3\right]a^2+\left(C_3+C_6t+\frac{1}{2}t^2\right)a^4\\ a^3=\phi^3&=a_0^3-C_5a_0^2-C_6a_0^4+\left(C_5+C_6t+\frac{1}{2}t^2\right)a^2+(C_6+t)a^4\end{aligned}\right\} \tag{37}$$

It is easy to verify that $\partial\phi^j/\partial C_\mu=0$ ($j=1,2;\ \mu=2,3,5,6$) leads to

$$-a_0^2+a^2=0,\ -a_0^4+ta^2+a^4=0 \tag{38}$$

Eq. (38) yields

$$a^2=a_0^2,\ a^4=a_0^4-ta_0^2 \tag{39}$$

Substituting Eq. (39) into Eq. (37), we obtain

$$\left.\begin{aligned} a^1&=a_0^1-\frac{1}{6}t^3a_0^2+ta_0^3+\frac{1}{2}t^2a_0^4\\ a^3&=a_0^3-\frac{1}{2}t^2a_0^2+ta_0^4\end{aligned}\right\} \tag{40}$$

Eqs. (39) and (40) give the solution of the problem.

4 Conclusion

Vujanović[3] applied a gradient method to integrate the dynamical equations of the holonomic nonconservative system. This paper transplants the idea of the gradient method to a Birkhoffian system, and presents a new method for integrating the Birkhoffian system. The advantages of this method are that the basic system of partial differential equations is quasi-linear, the method has flexibility, and we can choose n arbitrary Birkhoff's variables a^j as the functions of the remaining variables a^σ and t in a specific problem. The major difficulty of this method lies in finding a complete solution of the basic partial differential equation. Once one finds the complete solution, one can obtain the motion of the systems without doing further integration.

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Birkhoff 系统积分的新方法

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摘要:将 Vujanović提出的用于积分完整非保守系统动力学方程的梯度法思想移植到 Birkhoff 系统,给出了 Birkhoff 系统积分的一种新方法. 首先,列写出 Birkhoff 系统的运动微分方程;其次,将 $2n$ 个 Birkhoff 变量分成 2 部分,并假设其中一部分变量是其余变量及时间的函数,由此建立拟线性的基本偏微分方程组;如果求出此基本偏微分方程的完全解,则问题的解由一组代数方程给出. 该方法具有灵活性,对于具体问题可选 Birkhoff 变量中任意 n 个变量作为余下 n 个变量和时间的函数,其主要困难在于求基本偏微分方程的完全解. 一旦求出完全解,就可以不用进一步积分而求得系统的运动. 最后,举例说明结果的应用.

关键词: Birkhoff 系统; 积分方法; 基本偏微分方程
中图分类号:O316