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On braided Lie algebras

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Abstract: Let $(\mathcal{C}, C)$ be a braided monoidal category. The relationship between the braided Lie algebra and the left Jacobi braided Lie algebra in the category $(\mathcal{C}, C)$ is investigated. First, a braided C^2-commutative algebra in the category $(\mathcal{C}, C)$ is defined and three equations on the braiding in the category $(\mathcal{C}, C)$ are proved. Secondly, it is verified that $(A, [\,,\,])$ is a left (strict) Jacobi braided Lie algebra if and only if $(A, [\,,\,])$ is a braided Lie algebra, where A is an associative algebra in the category $(\mathcal{C}, C)$. Finally, as an application, the structures of braided Lie algebras are given in the category of Yetter-Drinfel'd modules and the category of Hopf bimodules.

Key words: Hopf algebra; braided monoidal category; braided Lie algebra

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Braided monoidal categories are situated at the intersection of quantum group theory and low-dimensional topology[1]. So the study of special objects with additional structures of braided monoidal categories, like (Lie) algebras, is very important. After Majid[2] introduced braided Lie algebras from a geometrical point of view, the study of braided Lie algebras has been an interesting topic[2-10]. From the algebraic point of view, Zhang et al.[11] introduced braided m-Lie algebras and left (strict) Jacobi braided Lie algebras, which are different from the braided Lie algebras defined by Majid[2], and gave the necessary and sufficient conditions for the braided m-Lie algebras to be left (strict) Jacobi braided Lie algebras. One naturally asks if there exists some relationship between the left (strict) Jacobi braided Lie algebras according to Ref. [11] and the braided Lie algebras studied by Bahturin et al. This paper will give a positive answer to the above question.

1 Braided Monoidal Categories

In this section, we recall some notions and two special examples of braided monomial categories from Hopf algebras. For the basic definitions of the category theory, see Refs. [1,12]. In what follows, C denotes a monoidal category with tensor product $\otimes$ and base object I. For every object A in the category C, there are two functors:

$$A\otimes - : \mathcal{C}\rightarrow\mathcal{C},\quad -\otimes A: \mathcal{C}\rightarrow\mathcal{C}$$

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The monoidal category $\mathcal{C}$ is called a braided monoidal category for every object A in $\mathcal{C}$, we have natural isomorphisms

$$C_{A,-}: A\otimes - \rightarrow -\otimes A,\quad C_{-,A}: -\otimes A\rightarrow A\otimes -$$

which satisfy the following:

$$C_{-,V}(W) = C_{W,-}(V)$$

$$C_{V\otimes W,-} = (C_{V,-}\otimes \mathrm{id})(\mathrm{id}\otimes C_{W,-})$$

$$C_{-,V\otimes W} = (\mathrm{id}\otimes C_{-,W})(C_{-,V}\otimes \mathrm{id})$$

In any braided monoidal category $(\mathcal{C}, C)$, the braiding C is a solution of the Yang-Baxter equation, i.e., there is

$$(C_{V,V}\otimes \mathrm{id})(\mathrm{id}\otimes C_{V,V})(C_{V,V}\otimes \mathrm{id}) = (\mathrm{id}\otimes C_{V,V})(C_{V,V}\otimes \mathrm{id})(\mathrm{id}\otimes C_{V,V})$$

Note that if C is a braiding, then so is the inverse C^{-1}. A braided monoidal category $(\mathcal{C}, C)$ is called symmetric if $C_{W,V}C_{V,W} = \mathrm{id}$, for any V, W in $\mathcal{C}$. In particular, $C_{V,V}$ is called symmetric on V if $C^2_{V,V} = \mathrm{id}$, for any V in $\mathcal{C}$.

For the Hopf algebra theory, see Refs. [1,13]. Let H be a Hopf algebra. We shall use Sweedler's notations. For any right H-comodule (M, r), $r(m) = m_0\otimes m_{(1)}$, $m\in M$. Similarly, for any left H-comodule (V, r), $r(v) = v_{(-1)}\otimes v_0$, $v\in V$. The following are two examples of braided monoidal categories from Hopf algebras.

Let H be a Hopf algebra. A right-right Yetter-Drinfel'd module V is a right H-module and a right H-comodule, whose action we denote by $R: M\otimes H\rightarrow M$, and the coaction we denote by r: $M\rightarrow M\otimes H$, such that

$$m_0h_1\otimes m_{(1)}h_2 = (mh_2)_0\otimes h_1(mh_2)_{(1)}$$

for any h in H and any m in M.

Example 1 Let H be a Hopf algebra with bijective antipode S and denote the category of right-right Yetter-Drinfel'd modules by $\mathrm{YD}(H)$. By Ref. [1], the category $\mathrm{YD}(H)$ is a braided monoidal category with a braiding C, where the braiding is given by $C_{M,N}(m\otimes n) = n_0\otimes mn_{(1)}$ for any M, N in $\mathrm{YD}(H)$.

Let H be a Hopf algebra. A Hopf bimodule M over H is a bimodule M, whose actions we denote by $L: H\otimes M\rightarrow M$ and $R: M\otimes H\rightarrow M$, and a bicomodule whose coactions we denote by r: $M\rightarrow H\otimes M$ and l: $M\rightarrow M\otimes H$, such that r and l are H-bimodule maps.

Example 2 Let H be a Hopf algebra with bijective antipode S and $M(H)$ denote the category of Hopf bimodules. By Ref. [14], the category $M(H)$ is a braided monoidal category with a braiding C, where $C_{M,N}(m\otimes n) =$

$m_{(-2)}n_0S(n_{(2)})\otimes_H S(m_{(-1)})m_0n_{(2)}$ for any M, N in $M(H)$.

2 Braided Lie Algebras

Let $\mathcal{C}$ be a monoidal category $\mathcal{C}$. An algebra in the category $\mathcal{C}$ is a triple $A=(A, m, \mu)$, where

1) A is an object in a monoidal category $\mathcal{C}$;

2) $m: A\otimes A\to A$ and $\mu: I\to A$ are morphisms in $\mathcal{C}$ such that

$$m(\mathrm{id}\otimes\mu) = \mathrm{id} = m(\mu\otimes\mathrm{id})$$

$$m(\mathrm{id}\otimes m) = m(m\otimes\mathrm{id})$$

In the sequel, $(\mathcal{C}, C)$ always denotes a braided monoidal category with a braiding C and the inverse of the braiding C is denoted by C^{-1}, which is also a braiding in the category $\mathcal{C}$.

Definition 1[11] A left (strict) Jacobi braided Lie algebra L in a braided monoidal category $(\mathcal{C}, C)$ is an object of the category $(\mathcal{C}, C)$ together with a bracket operation $[,]: L\otimes L\to L$ which is a morphism in the category $(\mathcal{C}, C)$ satisfying

1) $[,]=-[,]C_{L,L}$;

2) $[,](\mathrm{id}\otimes[,])+[,](\mathrm{id}\otimes[,])(\mathrm{id}\otimes C_{L,L}^{-1})(C_{L,L}\otimes\mathrm{id}) = -[,](\mathrm{id}\otimes[,])(C_{L,L}^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes C_{L,L})$.

Let (A, m, μ) be an algebra in a braided monoidal category $(\mathcal{C}, C)$. Define $[,]: A\otimes A\to A$, where $[,]=m-mC_{A,A}$. It is easy to see that $[,]$ is a morphism in the category $(\mathcal{C}, C)$.

Lemma 1[11] Let (A, m, μ) be an algebra in a braided monoidal category $(\mathcal{C}, C)$. Then the following conditions on $(A, [,])$ are equivalent:

1) $(A, [,])$ is a left (strict) Jacobi braided Lie algebra;

2) $mC_{A,A}^{-1}=mC_{A,A}$;

3) $[,]C_{A,A}=[,]C_{A,A}^{-1}$.

Now recall the concept of a braided Lie algebra in a braided monoidal category $(\mathcal{C}, C)$ [2,12].

Definition 2 A braided Lie algebra L in a braided monoidal category $(\mathcal{C}, C)$ is an object of the category $(\mathcal{C}, C)$ together with a bracket operation $[,]: L\otimes L\to L$ which is a morphism in the category $(\mathcal{C}, C)$ satisfying

1) $[,]=-[,]C_{L,L}$;

2) $[,](\mathrm{id}\otimes[,])+[,](\mathrm{id}\otimes[,])C_{L\otimes L,L}+[,](\mathrm{id}\otimes[,])C_{L,L\otimes L}=0$.

Let (A, m, μ) be an algebra in a braided monoidal category $(\mathcal{C}, C)$.

We mainly discuss the relationships between a braided Lie algebra $(A, [,])$ and a left (strict) Jacobi braided Lie algebra $(A, [,])$. First, we introduce another definition.

Definition 3 An algebra (A, m, μ) in a braided monoidal category $(\mathcal{C}, C)$ is called braided C^2-commutative if $m=mC_{A,A}^2$. It is easy to see that an algebra A is braided C^2-commutative if A is braided-commutative according to Refs. [2, 10], i. e, $m=mC_{A,A}$.

Lemma 2 Let $(\mathcal{C}, C)$ be a braided monoidal category and (A, m, μ) be a braided C^2-commutative algebra in the category $(\mathcal{C}, C)$. Then the following two equations hold:

$$mC_{A,A}(\mathrm{id}\otimes m)(C_{A,A}^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes C_{A,A}) = mC_{A,A}(\mathrm{id}\otimes m)C_{A\otimes A,A} \tag{1}$$

$$mC_{A,A}(\mathrm{id}\otimes mC_{A,A})(C_{A,A}^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes C_{A,A}) = mC_{A,A}(\mathrm{id}\otimes mC_{A,A})C_{A\otimes A,A} \tag{2}$$

Proof We can easily obtain

$$mC_{A,A}(\mathrm{id}\otimes m)C_{A\otimes A,A} = m(m\otimes\mathrm{id}) = m(\mathrm{id}\otimes m)$$

and so

$$m(\mathrm{id}\otimes m)(C_{A,A}^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes C_{A,A}^{-1}) = m(\mathrm{id}\otimes m)C_{A\otimes A,A}$$

By the definition of a braiding, we can obtain

$$mC_{A,A}(\mathrm{id}\otimes m)(C_{A,A}^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes C_{A,A}^{-1}) = mC_{A,A}(\mathrm{id}\otimes m)C_{A\otimes A,A}$$

Similarly, Eq. (2) can also be verified.

Lemma 3 Let $(\mathcal{C}, C)$ be a braided monoidal category and (A, m, μ) be a braided C^2-commutative algebra in the category $(\mathcal{C}, C)$. Then the following equation holds:

$$[,](\mathrm{id}\otimes[,])+[,](\mathrm{id}\otimes[,])C_{A,A\otimes A}+[,](\mathrm{id}\otimes[,])C_{A\otimes A,A} = [,](\mathrm{id}\otimes[,])+(\mathrm{id}+(\mathrm{id}\otimes C_{A,A}^{-1})(C_{A,A}\otimes\mathrm{id})+(C_{A,A}^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes C_{A,A}))$$

Proof First, since A is a braided C^2-commutative algebra, we can obtain

$$[,](\mathrm{id}\otimes[,])\ (\mathrm{id}\otimes C_{A,A}^{-1})(C_{A,A}\otimes\mathrm{id}) = [,](\mathrm{id}\otimes[,])C_{A,A\otimes A}$$

Secondly, by Lemma 2, we have

$$[,](\mathrm{id}\otimes[,])(C_{A,A}^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes C_{A,A}) = [,](\mathrm{id}\otimes[,])C_{A\otimes A,A}$$

Thirdly, the proof is completed by the following computation:

$$\begin{aligned}&[,](\mathrm{id}\otimes[,])+[,](\mathrm{id}\otimes[,])C_{A\otimes A,A}+[,](\mathrm{id}\otimes[,])C_{A,A\otimes A} = \\ &[,](\mathrm{id}\otimes[,])+[,](\mathrm{id}\otimes[,])(\mathrm{id}\otimes C_{A,A}^{-1})(C_{A,A}\otimes\mathrm{id})+ \\ &[,](\mathrm{id}\otimes[,])(C_{A,A}^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes C_{A,A}) = \\ &[,](\mathrm{id}\otimes[,])+(\mathrm{id}+(\mathrm{id}\otimes C_{A,A}^{-1})(C_{A,A}\otimes\mathrm{id})+ \\ &(C_{A,A}^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes C_{A,A}))\end{aligned}$$

Theorem 1 Let (A, m, μ) be an algebra in a braided monoidal category $(\mathcal{C}, C)$. Then the following are equivalent:

1) $(A, [,])$ is a left (strict) Jacobi braided Lie algebra in the category $(\mathcal{C}, C)$;

2) $(A, [,])$ is a braided Lie algebra in the category $(\mathcal{C}, C)$.

Proof It follows from lemmas 1 and 3.

Example 3 Let H be a Hopf algebra with bijective antipode. Then the category of Yetter-Drinfel'd modules $\mathrm{YD}(H)$ is a braided monoidal category with a braiding C as shown in Example 1. Let (A, m, μ) be a braided C^2-commutative algebra in the category $\mathrm{YD}(H)$. By Theorem 1, $(A, [,])$ is a braided Lie algebra in the category $\mathrm{YD}(H)$, where $[a, b] = b_0(ab_{(1)})$ for any a, b in A.

Example 4 Let H be a Hopf algebra with bijective antipode. Then the category of Hopf modules in the category $M(H)$ is a braided monoidal category with a braiding C as shown in Example 2. Let (A, m, μ) be a braided C^2-commutative algebra in the category $M(H)$. By Theorem 1, $(A, [,])$ is a braided Lie algebra in the category $M(H)$, where $[a, b] = (a_{(-2)} b_0 S(b_{(2)}))(S(a_{(-1)}) a_0 b_{(2)})$ for any a, b in A.

Corollary 1 Let $(\mathcal{C}, C)$ be a braided monoidal category and (A, m, μ) an algebra in the category $(\mathcal{C}, C)$. Assume that the braiding C is symmetric on A. Then $(A, [,])$ is a braided Lie algebra in the category $(\mathcal{C}, C)$.

Remark 1 By Corollary 1, all braided Lie algebras studied in Refs. [3 - 5, 8 - 10] are special cases of Theorem 1, respectively.

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关于辫子李代数

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摘要:设$(\mathcal{C}, C)$为辫子张量范畴,研究辫子张量范畴中辫子李代数和左 Jacobi 辫子李代数之间的关系. 首先,引入了一个新的定义即辫子张量范畴中的辫子平方交换的代数并得到 3 个关于辫子的等式. 其次, 证明了对于辫子张量范畴中的结合代数 A, $(A,[,])$ 是辫子李代数当且仅当$(A,[,])$ 是左 Jacobi 辫子李代数. 最后,作为上述结果的应用,给出了 Yetter-Drinfel'd 模范畴和 Hopf 双模范畴中辫子李代数的具体结构.

关键词: Hopf 代数;辫子张量范畴;辫子李代数

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