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Drazin invertibility for matrices over an arbitrary ring

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Abstract: In order to study the Drazin invertibility of a matrix with the generalized factorization over an arbitrary ring, the necessary and sufficient conditions for the existence of the Drazin inverse of a matrix are given by the properties of the generalized factorization. Let $\boldsymbol{T}=\boldsymbol{PAQ}$ be a square matrix with the generalized factorization, then $\boldsymbol{T}$ has Drazin index k if and only if k is the smallest natural number such that $\boldsymbol{A}_k$ is regular and $\boldsymbol{U}_k(\boldsymbol{V}_k)$ is invertible if and only if k is the smallest natural number such that $\boldsymbol{A}_k$ is regular and $\widetilde{\boldsymbol{U}}_k(\widetilde{\boldsymbol{V}}_k)$ is invertible if and only if k is the smallest natural number such that $\boldsymbol{A}_k$ is regular and $\hat{\boldsymbol{U}}_k(\hat{\boldsymbol{V}}_k)$ is invertible. The formulae to compute the Drazin inverse are also obtained. These results generalize recent results obtained for the Drazin inverse of a matrix with a universal factorization.

Key words: ring; generalized factorization; Drazin inverse; group inverse

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Throughout this paper and unless otherwise specified, R denotes an arbitrary ring with identity 1, $M_{m\times n}(R)$ and $M_n(R)$ the set of all $m\times n$ matrices and the ring of all $n\times n$ matrices over R, respectively. Given an $m\times n$ matrix $\boldsymbol{A}$ over a ring R, $\boldsymbol{A}$ is called (von Neumann) regular if there exists an $n\times m$ matrix $\boldsymbol{A}^-$ such that $\boldsymbol{AA}^-\boldsymbol{A}=\boldsymbol{A}$. $\boldsymbol{A}^-$ is called a von Neumann regular inverse of $\boldsymbol{A}$ and the set of all the von Neumann regular inverses of $\boldsymbol{A}$ will be denoted by $\boldsymbol{A}\{1\}$. An $n\times n$ matrix $\boldsymbol{T}$ over the ring R is said to have Drazin index k if k is the smallest natural number such that there exists a (unique) solution $\boldsymbol{T}^{D_k}$ of the system of equations: 1) $\boldsymbol{T}^k=\boldsymbol{T}^{k+1}\boldsymbol{Z}$; 2) $\boldsymbol{ZTZ}=\boldsymbol{Z}$; 3) $\boldsymbol{TZ}=\boldsymbol{ZT}$. $\boldsymbol{T}^{D_k}$ is called a Drazin inverse of index k of $\boldsymbol{T}$. If $k=1$, then $\boldsymbol{T}^{D_1}$ is denoted by $\boldsymbol{T}^{\#}$ and is called the group inverse of $\boldsymbol{T}$.

The Drazin inverse and the group inverse of a square matrix are studied in Refs. [1 - 2]. Chen[3] discussed the Drazin invertibility and the group invertibility of a matrix with GDH-factorization. In Ref. [4], the sufficient and necessary conditions are given for a matrix with a universal factorization to have a Drazin inverse. Motivated by the previous studies, we study the Drazin invertibility of a matrix with generalized factorization in this paper. $\boldsymbol{T}=\boldsymbol{PAQ}$ is called a GDH-factorization if $\boldsymbol{P}$ is right high and $\boldsymbol{Q}$ is left high. $\boldsymbol{P}$ is right high if $\boldsymbol{P}x=0$ implies $x=0$ and $\boldsymbol{Q}$ is left high if $x\boldsymbol{Q}=0$ implies $x=0$. $\boldsymbol{T}=\boldsymbol{PAQ}$ is called a universal factorization if there exist matrices $\boldsymbol{P}'$ and $\boldsymbol{Q}'$ such that $\boldsymbol{P}'\boldsymbol{PA}=\boldsymbol{A}=\boldsymbol{AQQ}'$. $\boldsymbol{T}=\boldsymbol{PAQ}$ is called a generalized factorization if the following conditions are satisfied $\boldsymbol{P}_1\boldsymbol{A}=\boldsymbol{P}_2\boldsymbol{A}$ whenever $\boldsymbol{P}_1\boldsymbol{AQ}=\boldsymbol{P}_2\boldsymbol{AQ}$ and $\boldsymbol{AQ}_1=\boldsymbol{AQ}_2$ whenever $\boldsymbol{PAQ}_1=\boldsymbol{PAQ}_2$. Clearly GDH-factorization and universal-factorization are both generalized factorizations, but the converse is not true[5].

The main results, theorem 1, theorem 2 and theorem 3, give the necessary and sufficient conditions for a matrix with the generalized factorization to have a Drazin inverse and the formula for obtaining the Drazin inverse if the conditions are satisfied. Hence, the theorems in Refs. [6 - 7] can be deduced.

Theorem 1 Let $\boldsymbol{A}\in M_{m\times n}(R)$ be a square matrix and $\boldsymbol{T}=\boldsymbol{PAQ}$ be a generalized factorization. Let $\boldsymbol{A}_1=\boldsymbol{A}$, $\boldsymbol{A}_i=\boldsymbol{AQ}(\boldsymbol{PAQ})^{i-2}\boldsymbol{PA}$, $i>1$. The following statements are equivalent:

1) $\boldsymbol{T}$ has Drazin index k;

2) k is the smallest natural number such that $\boldsymbol{A}_k$ is regular and $\boldsymbol{U}_k=\boldsymbol{A}_k^-\boldsymbol{A}_k\boldsymbol{QTPA}_k+\boldsymbol{I}_n-\boldsymbol{A}_k^-\boldsymbol{A}_k$ is invertible;

3) k is the smallest natural number such that $\boldsymbol{A}_k$ is regular and $\boldsymbol{V}_k=\boldsymbol{A}_k\boldsymbol{QTPA}_k\boldsymbol{A}_k^-+\boldsymbol{I}_n-\boldsymbol{A}_k\boldsymbol{A}_k^-$ is invertible.

In this case, we obtain

$$\begin{aligned}\boldsymbol{T}^{D_k}=\boldsymbol{PA}_k\boldsymbol{U}_k^{-1}\boldsymbol{Q}=\boldsymbol{PV}_k^{-1}\boldsymbol{A}_k\boldsymbol{Q}=\boldsymbol{T}^{k+1}\boldsymbol{PA}_k\boldsymbol{U}_k^{-2}\boldsymbol{Q}=\\ \boldsymbol{PV}_k^{-2}\boldsymbol{A}_k\boldsymbol{QT}^{k+1}=\boldsymbol{T}^{k+1}\boldsymbol{PV}_k^{-1}\boldsymbol{A}_k\boldsymbol{U}_k^{-1}\boldsymbol{Q}=\\ \boldsymbol{PV}_k^{-1}\boldsymbol{A}_k\boldsymbol{U}_k^{-1}\boldsymbol{QT}^{k+1}\end{aligned}$$

Proof 1)$\Leftrightarrow$2) Suppose that $\boldsymbol{T}$ has Drazin index k, let $\boldsymbol{Z}=\boldsymbol{T}^{D_k}$, then $\boldsymbol{T}^k=\boldsymbol{T}^k\boldsymbol{Z}^k\boldsymbol{T}^k$ and $\boldsymbol{PA}_k\boldsymbol{Q}=\boldsymbol{PA}_k\boldsymbol{QZ}^k\boldsymbol{PA}_k\boldsymbol{Q}$. Since $\boldsymbol{T}=\boldsymbol{PAQ}$ is a generalized factorization, $\boldsymbol{PA}_k\boldsymbol{Q}$ is also a generalized factorization. So, we obtain $\boldsymbol{A}_k=\boldsymbol{A}_k\boldsymbol{QZ}^k\boldsymbol{PA}_k$, i.e., $\boldsymbol{A}_k$ is regular. Let $\boldsymbol{A}_k^-\in\boldsymbol{A}_k\{1\}$, using the same methods as shown in Ref. [1], we obtain

$$(\boldsymbol{A}_k^-\boldsymbol{A}_k\boldsymbol{QTPA}_k)(\boldsymbol{A}_k^-\boldsymbol{A}_k\boldsymbol{QZ}^{k+1}\boldsymbol{PA}_k)^{k+1}=\boldsymbol{A}_k^-\boldsymbol{A}_k$$

and

$$(\boldsymbol{A}_k^-\boldsymbol{A}_k\boldsymbol{QZ}^{k+1}\boldsymbol{PA}_k)^{k+1}(\boldsymbol{A}_k^-\boldsymbol{A}_k\boldsymbol{QTPA}_k)=\boldsymbol{A}_k^-\boldsymbol{A}_k$$

So $\boldsymbol{A}_k^-\boldsymbol{A}_k\boldsymbol{QTPA}_k$ is invertible in $\boldsymbol{A}_k^-\boldsymbol{A}_kM_n(R)\boldsymbol{A}_k^-\boldsymbol{A}_k$. By theorem 1 in Ref. [8], we obtain that $\boldsymbol{U}_k=\boldsymbol{A}_k^-\boldsymbol{A}_k\boldsymbol{QTPA}_k+\boldsymbol{I}_n-\boldsymbol{A}_k^-\boldsymbol{A}_k$ is invertible in $M_n(R)$. Conversely, suppose that $\boldsymbol{U}_k$ is invertible, and let $\boldsymbol{Z}=\boldsymbol{PA}_k\boldsymbol{U}_k^{-1}\boldsymbol{Q}$. First, we obtain

$$\begin{aligned}\boldsymbol{A}_k\boldsymbol{U}_k=\boldsymbol{A}_k\boldsymbol{QTPA}_k\boldsymbol{A}_k^-\boldsymbol{A}_k\boldsymbol{U}_k=\boldsymbol{A}_k^-\boldsymbol{A}_k\boldsymbol{QTPA}_k=\\ \boldsymbol{U}_k\boldsymbol{A}_k^-\boldsymbol{A}_k\boldsymbol{U}_k^{-1}\boldsymbol{A}_k^-\boldsymbol{A}_k=\boldsymbol{A}_k^-\boldsymbol{A}_k\boldsymbol{U}_k^{-1}\end{aligned}$$

and

$$\boldsymbol{T}^{k+1}\boldsymbol{Z}=\boldsymbol{P}(\boldsymbol{A}_k\boldsymbol{QTPA}_k)\boldsymbol{U}_k^{-1}\boldsymbol{Q}=\boldsymbol{PA}_k\boldsymbol{Q}=\boldsymbol{T}^k$$

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Next, we obtain

$$\begin{aligned} ZTZ &= PA_k(A_k^- A_k U_k^{-1}) QTPA_k U_k^{-1} Q = \\ & PA_k U_k^{-1}(A_k^- A_k QTPA_k) U_k^{-1} Q = Z \end{aligned}$$

Note that

$$\begin{aligned} U_k^2 = &(A_k^- A_k QPA_k + I_n - A_k^- A_k)\cdot \\ &(A_k^- A_k QT^2 PA_k + I_n - A_k^- A_k) \end{aligned}$$

and

$$\begin{aligned} U_k^2 = &(A_k^- A_k QT^2 PA_k + I_n - A_k^- A_k)\cdot \\ &(A_k^- A_k QPA_k + I_n - A_k^- A_k) \end{aligned}$$

Let

$$\widetilde{U}_k = A_k^- A_k QPA_k + I_n - A_k^- A_k$$

and

$$S = A_k^- A_k QT^2 PA_k + I_n - A_k^- A_k$$

we obtain

$$U_k^2 = \widetilde{U}_k \cdot S = S \cdot \widetilde{U}_k$$

Since U_k is invertible, so is $\widetilde{U}_k$ and $\widetilde{U}_k^{-1} = U_k^{-2} S = S U_k^{-2}$. Immediately, we obtain $TZ = PA_k \widetilde{U}_k^{-1} Q = ZT$. Hence, $T^{D_k} = PA_k U_k^{-1} Q$.

2)⇔3) Let $A_k^- \in A_k\{1\}$, and suppose that U_k is invertible. Then by theorem 1 in Ref. [8], $A_k^- A_k QTPA_k$ is invertible in $A_k^- A_k M_n(R) A_k^- A_k$, so there exists $Z \in M_n(R)$ such that $A_k^- A_k QTPA_k Z A_k^- A_k = A_k^- A_k = A_k^- A_k Z A_k^- A_k QTPA_k$. Multiplying by A_k on the left side and A_k^- on the right side, respectively, we know that $A_k Z A_k^- A_k A_k^-$ is an inverse of $A_k QTPA_k A_k^-$ in $A_k A_k^- M_m(R) A_k A_k^-$. So by theorem 1 in Ref. [8], V_k is invertible. The converse is analogous. Since

$$A_k U_k = A_k QTPA_k = V_k A_k$$

hence

$$T^{D_k} = PA_k U_k^{-1} Q = PV_k^{-1} A_k Q$$

And

$$A_k U_k^{-1} = A_k QTPA_k U_k^{-2}$$

so

$$T^{D_k} = T^{k+1} PA_k U_k^{-2} Q$$

Similarly, we have

$$T^{D_k} = PV_k^{-2} A_k QT^{k+1}$$

We also obtain

$$T^{D_k} = T^{k+1} PA_k U_k^{-2} Q = PV_k^{-2} A_k QT^{k+1}$$

Theorem 2 Let $A \in M_{m\times n}(R)$ be a square matrix and $T = PAQ$ be a generalized factorization. Let $A_1 = A$, $A_i = AQ(PAQ)^{i-2}PA$, $i > 1$. The following statements are equivalent:

1) T has Drazin index k;

2) k is the smallest natural number such that A_k is regular and $\widetilde{U}_k = A_k^- A_k QPA_k + I_n - A_k^- A_k$ is invertible;

3) k is the smallest natural number such that A_k is regular and $\widetilde{V}_k = A_k QPA_k A_k^- + I_n - A_k A_k^-$ is invertible.

In this case, we obtain

$$\begin{aligned} T^{D_k} = & T^{k-1} PA_k \widetilde{U}_k^{-2} Q = T^{k-1} P\widetilde{V}_k^{-2} A_k Q = \\ & T^{k-1} P\widetilde{V}_k^{-1} A_k \widetilde{U}_k^{-1} Q \end{aligned}$$

Proof By theorem 1, it is sufficient to prove that $\widetilde{U}_k$ is invertible iff U_k is invertible. Note that $U_k^2 = \widetilde{U}_k \cdot S = S \cdot \widetilde{U}_k$ where $S = A_k^- A_k QT^2 PA_k + I_n - A_k^- A_k$. So if U_k is invertible, then $\widetilde{U}_k$ is invertible. By induction, we obtain

$$\widetilde{U}_k^p = A_k^- A_k QT^{(p-1)k} PA_k + I_n - A_k^- A_k$$

and

$$S^q = A_k^- A_k QT^{2q+(q-1)k} PA_k + I_n - A_k^- A_k$$

Let $p = k+2$, $q = k$, then we have $\widetilde{U}_k^p = S^q$. Suppose that $\widetilde{U}_k$ is invertible, so is S, thus, U_k is invertible. Now we prove $T^{D_k} = T^{k-1} PA_k \widetilde{U}_k^{-2} Q$. First, $\widetilde{U}_k S^{-1} = S^{-1} \widetilde{U}_k$, by theorem 1, $T^{D_k} = T^{k+1} PA_k S^{-1} \widetilde{U}_k^{-1} Q$. Since $PA_k S = T^{k+2} PA_k = T^2 PA_k \widetilde{U}_k$, then we obtain

$$PA_k \widetilde{U}_k^{-1} = T^2 PA_k S^{-1}$$

So

$$PA_k \widetilde{U}_k^{-2} Q = T^2 PA_k S^{-1} \widetilde{U}_k^{-1} Q$$

Thus

$$T^{D_k} = T^{k-1}(T^2 PA_k S^{-1} \widetilde{U}_k^{-1} Q) = T^{k-1} PA_k \widetilde{U}_k^{-2} Q$$

Since

$$A_k \widetilde{U}_k = A_k QPA_k = \widetilde{V}_k A_k, \; A_k \widetilde{U}_k^{-1} = \widetilde{V}_k^{-1} A_k$$

we obtain

$$T^{D_k} = T^{k-1} P\widetilde{V}_k^{-2} A_k Q = T^{k-1} P\widetilde{V}_k^{-1} A_k \widetilde{U}_k^{-1} Q$$

Corollary 1 Let $T \in M_n(R)$, the following statements are equivalent:

1) T has Drazin index k;

2) k is the smallest natural number such that T^k is regular and $\widetilde{U}_k = (T^k)^- T^{2k} + I_n - (T^k)^- T^k$ is invertible;

3) k is the smallest natural number such that T^k is regular and $\widetilde{V}_k = T^{2k}(T^k)^- + I_n - T^k(T^k)^-$ is invertible.

In this case, we obtain

$$T^{D_k} = T^{2k-1} \widetilde{U}_k^{-2} = T^{k-1} \widetilde{V}_k^{-2} T^k = T^{k-1} \widetilde{V}_k^{-1} T^k \widetilde{U}_k^{-1}$$

Theorem 3 Let $A \in M_{m\times n}(R)$ be a square matrix and $T = PAQ$ be a generalized factorization. Let $A_1 = A$, $A_i = AQ(PAQ)^{i-2}PA$, $i > 1$. The following statements are equivalent:

1) T has Drazin index k;

2) k is the smallest natural number such that A_k is regular and $\hat{U}_k = A_k^- A_k QPA + I_n - A_k^- A_k$ is invertible;

3) k is the smallest natural number such that A_k is regular

and $\hat{V}_k = AQPA_kA_k^- + I_n - A_kA_k^-$ is invertible.

In this case, we obtain

$$T^{D_k} = PA_k\hat{U}_k^{-(k+1)}Q = P\tilde{V}_k^{-(k+1)}A_kQ$$

Proof Note that $A_k\hat{U}_k = A_kQPA = AQPA_k = \hat{V}_kA_k$, on the other hand, $(\hat{U}_k)^2 = A_k^-AQPA_kQPA + I_n - A_k^-A_k = A_k^-A_kQTPA + I_n - A_k^-A_k$. By induction, we obtain

$$(\hat{U}_k)^t = A_k^-A_kQT^{t-1}PA + I_n - A_k^-A_k$$

so

$$(\hat{U}_k)^k = A_k^-A_kQT^{k-1}PA + I_n - A_k^-A_k = \tilde{U}_k$$

Similarly $(\hat{V}_k)^k = \tilde{V}_k$. So by theorem 2, (1) ⇔ (2) ⇔ (3). Since $A_k\hat{U}_k = A_kQPA$, then

$$PA_k = PA_kQPA\hat{U}_k^{-1} = TPA_k\hat{U}_k^{-1}$$

By theorem 2,

$$T^{D_k} = T^{k-1}PA_k\hat{U}_k^{-2k}Q = T^{k-2}(TPA_k\hat{U}_k^{-1})\hat{U}_k^{-2k+1}Q = \dots = PA_k\hat{U}_k^{-(k+1)}Q$$

Since $A_k\hat{U}_k^{-1} = \hat{V}_k^{-1}A_k$, we obtain $T^{D_k} = P\tilde{V}_k^{-(k+1)}A_kQ$.

Let $A \in M_{m\times n}(R)$ be a square matrix and $T = PAQ$ be a universal factorization, then by theorem 2 we can obtain the theorem in Ref. [7]. It is clear that for any $T \in M_n(R)$, $T = I_nTI_n$ is a generalized factorization. Now by theorem 1 and theorem 3, we immediately obtain theorems 1, 2 and 3 in Ref. [6]. And we also obtain the characterization on the group inverse of a matrix with generalized factorization.

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环上矩阵的 Drazin 可逆性

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摘要:为了研究任意环上具有广义分解的矩阵的 Drazin 可逆性,利用广义分解的一些性质,给出了任意环上具有广义分解的矩阵的 Drazin 可逆的充分必要条件:设 $T = PAQ$ 为具有广义分解的矩阵,则 T 的 Drazin 指标为 k 当且仅当 k 为使得 A_k 正则且 $U_k(V_k)$ 可逆的最小自然数当且仅当 k 为使得 A_k 正则且 $\tilde{U}_k(\tilde{V}_k)$ 可逆的最小自然数当且仅当 k 为使得 A_k 正则且 $\hat{U}_k(\hat{V}_k)$ 可逆的最小自然数. 同时给出了几种计算 Drazin 逆的公式,推广了任意环上具有泛分解的矩阵 Drazin 逆的结果.

关键词:环;广义分解;Drazin 逆; 群逆

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