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Stability analysis for affine fuzzy system based on fuzzy Lyapunov functions

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Abstract: An analysis method based on the fuzzy Lyapunov functions is presented to analyze the stability of the continuous affine fuzzy systems. First, a method is introduced to deal with the consequent part of the fuzzy local model. Thus, the stability analysis method of the homogeneous fuzzy system can be used for reference. Stability conditions are derived in terms of linear matrix inequalities based on the fuzzy Lyapunov functions and the modified common Lyapunov functions, respectively. The results demonstrate that the stability result based on the fuzzy Lyapunov functions is less conservative than that based on the modified common Lyapunov functions via numerical examples. Compared with the method which does not expand the consequent part, the proposed method is simpler but its feasible region is reduced. Finally, in order to expand the application of the fuzzy Lyapunov functions, the piecewise fuzzy Lyapunov function is proposed, which can be used to analyze the stability for triangular or trapezoidal membership functions and obtain the stability conditions. A numerical example validates the effectiveness of the proposed approach.

Key words: affine fuzzy system; stability analysis; linear matrix inequalities; fuzzy Lyapunov function

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The successful application of many advanced control algorithms relies on the accurate object model. However, a simple model which can reflect the overall dynamic characteristics of the nonlinear system is difficult to obtain by the conventional linear modeling methods. The Takagi-Sugeno (T-S) fuzzy model is an effective representation for complex nonlinear systems, and fuzzy logical control has been proved to be a successful control approach for certain nonlinear systems[1-3].

The continuous T-S local linear dynamic models are defined by rules in the following IF-THEN form:

$$R_i\text{: IF } z_1(t) \text{ is } M_{i1} \text{ and } z_2(t) \text{ is } M_{i2},\ \dots,\ z_q(t) \text{ is } M_{iq} \quad \text{THEN } \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{a}_i \tag{1}$$

where $R_i(i=1, 2, \dots, l)$ denotes the i-th fuzzy rule; $z(t) = \{z_1(t), z_2(t), \dots, z_q(t)\}$ is the premise variable vector and $M_{i1}, M_{i2}, \dots, M_{iq}$ are fuzzy variables; $\boldsymbol{x}^{\mathrm{T}}(t) = \{\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \dots, \boldsymbol{x}_n(t)\}$ is the state vector; $\boldsymbol{A}_i$ is the matrix with appropriate dimensions in the local system; and $\boldsymbol{a}_i$ is the affine term.

In general, the T-S fuzzy system can be divided into two categories: the homogeneous fuzzy system and the affine fuzzy system[4]. It is difficult to analyze the stability and design the controller for the nonlinear system when the affine term is considered. Therefore, most of the previous works have just focused on the homogeneous fuzzy system. The T-S fuzzy model can be built by data identification or linearization of the original nonlinear system at some operating points, and the affine term exists in normal conditions. If the affine term is ignored, the approximation capability of the T-S fuzzy model will be weakened.

It is not until recently that some works have paid attention to the stability of the affine fuzzy system. In Refs. [5 – 6], the stability and synthesis of the affine system are presented for continuous and discrete cases, respectively. It is noted that the approaches based on the common quadratic Lyapunov functions lead to conservative conclusions. In Ref. [7], an analytical method is presented for the discrete-time T-S fuzzy dynamic systems based on the piecewise Lyapunov function. The stability result is less conservative than that based on the common Lyapunov functions. In this paper, the stability analysis of the affine fuzzy system is presented for the continuous case based on fuzzy Lyapunov functions.

1 Processing Method of Consequent Part

If the affine fuzzy local model is described as Eq. (1), the global model of the system can be expressed as

$$\dot{\boldsymbol{x}}(t) = \frac{\sum_{i=1}^{l} W_i(z(t))\{\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{a}_i\}}{\sum_{i=1}^{l} W_i(z(t))} = \sum_{i=1}^{l} h_i(z(t))\{\boldsymbol{A}_i x(t) + \boldsymbol{a}_i\} \tag{2}$$

where $W_i(z(t)) = \prod_{j=1}^{n} M_j^i(z_j(t))$; $h_i(z(t)) = \dfrac{W_i(z(t))}{\sum_{i=1}^{l} W_i(z(t))}$, $h_i(z(t)) \geqslant 0$, $\sum_{i=1}^{n} h_i(z(t)) = 1$.

For the affine fuzzy local model, its equilibrium point is not at the original point. Thus, it is more difficult to deal with the stability analysis and the controller design[4]. To facilitate the stability analysis, the following assumption is introduced.

Assumption 1 Let L_0 be the set of indices for the fuzzy rules that contain the origin $\boldsymbol{x}=0$, $L_0 \in \{h_i(0) \neq 0\}$ for $i \in L_0$, and the affine term $\boldsymbol{a}_i = 0$. L_1 is the set of indices for the

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fuzzy rules that exclude the origin.

Remark 1 Assumption 1 assures that the origin $\boldsymbol{x} = \boldsymbol{0}$ is the equilibrium point of the global affine fuzzy system.

Now, a processing method of the consequent part is introduced [8]. Define that

$$\widetilde{\boldsymbol{A}}_i = \begin{bmatrix} \boldsymbol{A}_i & a_i \\ 0 & 0 \end{bmatrix},\ \tilde{\boldsymbol{x}} = [\boldsymbol{x} \quad 1]^{\mathrm{T}}$$

Then, using these notations, the affine fuzzy local and global model can be expressed respectively as

$$\left.\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_i \boldsymbol{x}(t) & i \in L_0 \\ \dot{\tilde{\boldsymbol{x}}}(t) &= \widetilde{\boldsymbol{A}}_i \tilde{\boldsymbol{x}}(t) & i \in L_1 \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} \dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{l} h_i(z(t)) \boldsymbol{A}_i \boldsymbol{x}(t) & i \in L_0 \\ \dot{\tilde{\boldsymbol{x}}}(t) &= \sum_{i=1}^{l} h_i(z(t)) \widetilde{\boldsymbol{A}}_i \tilde{\boldsymbol{x}}(t) & i \in L_1 \end{aligned}\right\} \tag{4}$$

Remark 2 In the above affine fuzzy local model, the expression of $\dot{\boldsymbol{x}}(t)$ is divided into two forms $i \in L_0$ and $i \in L_1$ according to the differences in the fuzzy rules. In fact, when $i \in L_1$, the expression of $\dot{\tilde{\boldsymbol{x}}}(t)$ can take the place of that when $i \in L_0$, and the corresponding relationships hold that $\boldsymbol{a}_i = \boldsymbol{0}$, $\widetilde{\boldsymbol{A}}_i = [\boldsymbol{A}_i \quad 0;\ 0 \quad 0]$ and $\tilde{\boldsymbol{x}} = [\boldsymbol{x} \quad 0]^{\mathrm{T}}$.

Remark 3 By Eq. (4), we can see that the expression of the affine fuzzy model is similar to that of the homogeneous fuzzy model. The stability analysis and the controller design methods of the homogeneous fuzzy system can be applied to the affine fuzzy system. In this paper, the stability analysis of the affine fuzzy system is based on the fuzzy Lyapunov functions. The basic idea is somewhat similar to that of Refs. [9 − 10], but the main contents are completely new.

First, consider a function with the following form:

$$V(\boldsymbol{x}(t)) = \begin{cases} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{P}\, x & i \in L_0 \\ \tilde{\boldsymbol{x}}^{\mathrm{T}} \widetilde{\boldsymbol{P}}\, \tilde{\boldsymbol{x}} & i \in L_1 \end{cases}$$

where matrices $\boldsymbol{P}$ and $\widetilde{\boldsymbol{P}}$ are positive definite. It is an extension of globally quadratic Lyapunov functions, and it is proved to be a Lyapunov function candidate based on the standard Lyapunov theory[11].

Theorem 1 The continuous affine fuzzy system of Eq. (2) is globally exponentially stable if there exist positive definite matrices $\boldsymbol{P}$ and $\widetilde{\boldsymbol{P}}$ such that the following linear matrix inequalities (LMIs) are satisfied for $i = 1, 2, \ldots, l$,

$$\left.\begin{aligned} \boldsymbol{A}_i^{\mathrm{T}} \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_i &< 0 & i \in L_0 \\ \widetilde{\boldsymbol{A}}_i^{T} \widetilde{\boldsymbol{P}} + \widetilde{\boldsymbol{P}} \widetilde{\boldsymbol{A}}_i &< 0 & i \in L_1 \end{aligned}\right\} \tag{5}$$

Remark 4 The result can be readily obtained referring to the stability analysis of homogeneous fuzzy systems[12], so the proving process is not listed due to lack of space.

2 Stability Analysis Based on Fuzzy Lyapunov Functions

In order to compare the above stability conclusions, a stability analysis method for the affine fuzzy system based on fuzzy Lyapunov functions is presented. The fuzzy Lyapunov function candidate is defined as

$$V(\boldsymbol{x}(t)) = \begin{cases} \sum_{i=1}^{l} h_i(z(t)) \boldsymbol{x}^{\mathrm{T}} \boldsymbol{P}_i \boldsymbol{x} & i \in L_0 \\ \sum_{i=1}^{l} h_i\,(z(t)) \tilde{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{P}_i \tilde{\boldsymbol{x}} & i \in L_1 \end{cases} \tag{6}$$

Notice that the function shares the same membership functions with the affine fuzzy model of a dynamic system.

Theorem 2 Assuming that $\dot{h}_\rho \leqslant \varphi_\rho$, the continuous affine fuzzy system of Eq. (2) is globally exponentially stable if there exists a set of positive definite matrices $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_l$ which satisfy

$$\left.\begin{aligned} &\sum_{\rho=1}^{l} \varphi_\rho \boldsymbol{P}_\rho + \frac{1}{2}(\boldsymbol{A}_j^{\mathrm{T}} \boldsymbol{P}_i + \boldsymbol{P}_i \boldsymbol{A}_j + \boldsymbol{A}_i^{\mathrm{T}} \boldsymbol{P}_j + \boldsymbol{P}_j \boldsymbol{A}_i) < 0 & \rho, i, j \in L_0 \\ &\sum_{\rho=1}^{l} \varphi_\rho \boldsymbol{P}_\rho + \frac{1}{2}(\widetilde{\boldsymbol{A}}_j^{\mathrm{T}} \boldsymbol{P}_i + \boldsymbol{P}_i \widetilde{\boldsymbol{A}}_j + \widetilde{\boldsymbol{A}}_i^{\mathrm{T}} \boldsymbol{P}_j + \boldsymbol{P}_j \widetilde{\boldsymbol{A}}_i) < 0 & \rho, i, j \in L_1 \end{aligned}\right\} \tag{7}$$

where $\rho = 1, 2, \ldots, l$; $1 \leqslant i \leqslant j \leqslant 1$.

Proof Considering the fuzzy Lyapunov function candidate (6), the time derivative of $\boldsymbol{V}(x)$ is given as

$$\dot{\boldsymbol{V}}(x) = \dot{\boldsymbol{V}}_1(\boldsymbol{x})\,\big|_{i \in L_0} + \dot{\boldsymbol{V}}_2(\boldsymbol{x})\,\big|_{i \in L_1}$$

When $i \in L_1$,

$$\begin{aligned} \dot{\boldsymbol{V}}_2(x) &= \sum_{\rho=1}^{l} \dot{h}_\rho\, \tilde{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{P}_\rho \tilde{\boldsymbol{x}} + \sum_{i=1}^{l} h_i [\dot{\tilde{\boldsymbol{x}}}^{\mathrm{T}} \boldsymbol{P}_i \tilde{\boldsymbol{x}} + \tilde{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{P}_i \dot{\tilde{\boldsymbol{x}}}] = \\ &\sum_{\rho=1}^{l} \dot{h}_\rho\, \tilde{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{P}_\rho \tilde{\boldsymbol{x}} + \sum_{i=1}^{l} h_i \Big[\sum_{j=1}^{l} h_j \tilde{\boldsymbol{x}}^{\mathrm{T}} \widetilde{\boldsymbol{A}}_j^{\mathrm{T}} \boldsymbol{P}_i \tilde{\boldsymbol{x}} + \tilde{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{P}_i \sum_{j=1}^{l} h_j \widetilde{\boldsymbol{A}}_j \tilde{\boldsymbol{x}} \Big] = \\ &\sum_{\rho=1}^{l} \dot{h}_\rho\, \tilde{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{P}_\rho \tilde{\boldsymbol{x}} + \sum_{i=1}^{l} \sum_{j=1}^{l} h_i h_j\, \tilde{\boldsymbol{x}}^{\mathrm{T}} [\widetilde{\boldsymbol{A}}_j^{\mathrm{T}} \boldsymbol{P}_i + \boldsymbol{P}_i \widetilde{\boldsymbol{A}}_j] \tilde{\boldsymbol{x}} \end{aligned} \tag{8}$$

In the universe of the ρ-th membership function, if there exists

$$\dot{h}_\rho(z(t)) \leqslant \varphi_\rho$$

then Eq. (8) can be rewritten as

$$\begin{aligned} \dot{\boldsymbol{V}}_2(\boldsymbol{x}) \leqslant\ & \sum_{\rho=1}^{l} \varphi_\rho\, \tilde{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{P}_\rho \tilde{\boldsymbol{x}} + \frac{1}{2} \sum_{i=1}^{l} \sum_{j \geqslant i} h_i h_j\, \tilde{\boldsymbol{x}}^{\mathrm{T}} [\widetilde{\boldsymbol{A}}_j^{\mathrm{T}} \boldsymbol{P}_i + \\ & \boldsymbol{P}_i \widetilde{\boldsymbol{A}}_j + \widetilde{\boldsymbol{A}}_i^{\mathrm{T}} \boldsymbol{P}_j + \boldsymbol{P}_j \widetilde{\boldsymbol{A}}_i] \tilde{\boldsymbol{x}} = \\ & \sum_{i=1}^{l} \sum_{j \geqslant i} h_i h_j\, \tilde{\boldsymbol{x}}^{\mathrm{T}} \Big[\sum_{\rho=1}^{l} \varphi_\rho \boldsymbol{P}_\rho + \frac{1}{2}(\widetilde{\boldsymbol{A}}_j^{\mathrm{T}} \boldsymbol{P}_i + \\ & \boldsymbol{P}_i \widetilde{\boldsymbol{A}}_j + \widetilde{\boldsymbol{A}}_i^{\mathrm{T}} \boldsymbol{P}_j + P_j \widetilde{\boldsymbol{A}}_i) \Big] \tilde{\boldsymbol{x}} \end{aligned}$$

Similar conclusion can be obtained when $i \in L_0$,

$$\begin{aligned} \dot{\boldsymbol{V}}_1(\boldsymbol{x}) \leqslant\ & \sum_{i=1}^{l} \sum_{j \geqslant i} h_i h_j \boldsymbol{x}^{\mathrm{T}} \Big[\sum_{\rho=1}^{l} \varphi_\rho \boldsymbol{P}_\rho + \frac{1}{2}(\boldsymbol{A}_j^{\mathrm{T}} \boldsymbol{P}_i + \boldsymbol{P}_i \boldsymbol{A}_j + \\ & \boldsymbol{A}_i^{\mathrm{T}} \boldsymbol{P}_j + \boldsymbol{P}_j \boldsymbol{A}_i) \Big] \boldsymbol{x} \end{aligned}$$

If the premises of Theorem 2 hold, $\dot{\boldsymbol{V}}(\boldsymbol{x})$ is strictly negative for all state-space, and the continuous affine fuzzy system is globally exponentially stable. Thus, the proof is completed.

Remark 5 In the proving process of Theorem 2, $\dot{\boldsymbol{V}}(\boldsymbol{x})$ contains the time derivative of premise membership functions, and φ_ρ can be obtained according to the membership functions[13].

Remark 6 The stability analysis of the discrete fuzzy system based on fuzzy Lyapunov functions does not include the derivation of $\boldsymbol{V}(\boldsymbol{x})$, so the sufficient stability conditions will not contain the time derivative of premise membership functions.

Example 1 The stability of an affine fuzzy system is checked by Theorems 1 and 2. Consider the fuzzy system (2) with the following subsystems:

$$R_i\text{: IF } \boldsymbol{x}_1(t) \text{ is } M_i \text{ THEN } \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{a}_i \qquad i = 1, 2, 3$$

where

$$\boldsymbol{x}^{\mathrm{T}}(t) = [\boldsymbol{x}_1(t), \boldsymbol{x}_2(t)],\ \boldsymbol{a}_1 = \boldsymbol{a}_3 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\boldsymbol{A}_1 = \boldsymbol{A}_3 = \begin{bmatrix} -8 & 1 \\ 5 & -1 \end{bmatrix},\ \boldsymbol{A}_2 = \begin{bmatrix} -1 & -2 \\ 2 & -8 \end{bmatrix}$$

The membership functions are shown in Fig. 1.

$$\boldsymbol{M}_2(x_1) = (1 + \cos(x_1))/2 \qquad |x_1| \leqslant \pi$$

$$\boldsymbol{M}_1(x_1) = M_3(x_1) = \begin{cases} (1 - \cos(x_1))/2 & 0 < |x_1| \leqslant \pi \\ 1 & |x_1| > \pi \end{cases}$$

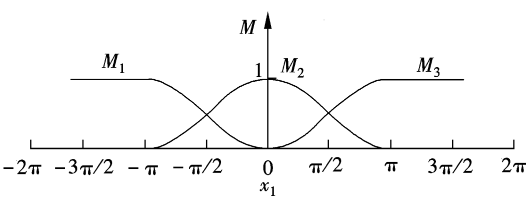

Fig. 1 Membership functions

The stability conditions for the homogeneous fuzzy system cannot be directly applied to this example. There exist no positive definite matrices that satisfy Theorem 1 using the LMI optimization technique. However, the simulation indicates that the system is stable. Fig. 2 shows the state response of the affine system for an initial value of $\boldsymbol{x} = (-5, 5)^{\mathrm{T}}$.

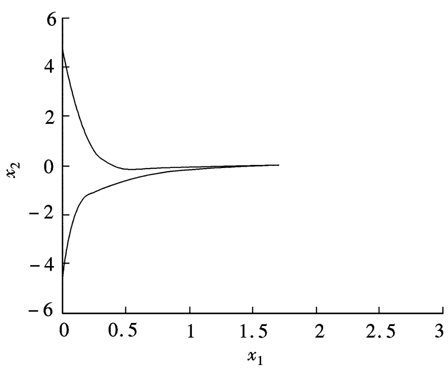

Fig. 2 System response from an initial condition

Using Theorem 2 and choosing $\varphi_1 = 0$, $\varphi_2 = \varphi_3 = 0.5$, a set of positive definite matrices can be solved as follows:

$$\boldsymbol{p}_1 = 10^6 \times \begin{bmatrix} 0.0691 & 0.0102 & 0.0708 \\ 0.0102 & 0.0241 & 0.1157 \\ 0.0708 & 0.1157 & 1.4398 \end{bmatrix}$$

$$\boldsymbol{p}_2 = 10^5 \times \begin{bmatrix} 5.5514 & -0.8091 \\ -0.8091 & 1.2334 \end{bmatrix}$$

$$\boldsymbol{p}_3 = 10^4 \times \begin{bmatrix} 4.8417 & -0.2793 & 0.3104 \\ -0.2793 & 0.4683 & 2.0923 \\ 0.3104 & 2.0923 & 9.8675 \end{bmatrix}$$

It is verified that the affine fuzzy system is globally exponentially stable. Fig. 3 shows the behavior of the system for two initial conditions of $\boldsymbol{x} = (-5, 5)^{\mathrm{T}}$ and $\boldsymbol{x} = (5, -5)^{\mathrm{T}}$.

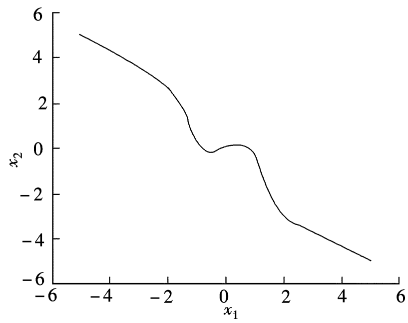

Fig. 3 Trajectories from two initial conditions

In this example, the results reveal that Theorem 2 is less conservative than Theorem 1. But it is noted that this conclusion is not always valid for any affine fuzzy system.

In addition, comparing Theorem 2 with the stability conclusions in Refs. [5 − 6], it can be obtained that the sufficient stability conditions will reduce the stability margin through the processing method of the consequent part. One possible reason is that the dimension of matrices $\boldsymbol{A}_i$ is expanded with zeros when the consequent part is processed. It becomes more difficult to search positive definite matrices $\boldsymbol{P}_i$ by the LMI optimization technique. Another reason is that the term $\sum_{\rho=1}^{l} \varphi_\rho \boldsymbol{P}_\rho$ may be the main conservative factor when considering fuzzy Lyapunov functions. In Ref. [14], a new fuzzy Lyapunov function is proposed for the homogeneous fuzzy system to eliminate the time derivative of the membership functions, and it is believed that the result is less conservative.

Remark 7 Notice that the membership functions should be continuous and derivable in the universe for the continuous fuzzy system when the stability analysis is based on fuzzy Lyapunov functions. The method in Theorem 2 is not suitable for some common membership functions, e. g. triangular or trapezoidal membership functions.

3 Stability Analysis Based on Piecewise Fuzzy Lyapunov Functions

In the piecewise Lyapunov functions approach, the global state space is divided into several fuzzy regions by taking advantage of the structure information of membership functions in the premise rules[15−16]. This treatment can be used as the fuzzy Lyapunov functions for reference. Considering the triangular or trapezoidal membership functions, some assumptions are given.

Assumption 2 In any case, at most two fuzzy rules are activated by the input vector. The membership function at any point is

$$\sum_{i=1}^{n} M_i(z(t)) = 1$$

where $M(z(t))$ is the membership function of the premise variable, and n is the number of rules at one point.

Then, the whole premise space is divided into several regions in the axes by the universe of fuzzy variables. Each region between two points is called a fuzzy subregion S_k. An example about the segmentation of state space is illustrated in Fig. 4.

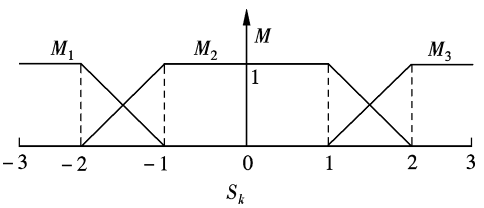

Fig. 4 Membership functions and fuzzy subregions

Remark 8 It is worth noting that the segmentation method of state space in this paper is different from that of the piecewise Lyapunov functions approach.

In each fuzzy subregion, the membership functions are continuous and derivable. The fuzzy Lyapunov functions can be used to analyze stability of the system easily at this time, and the stability result is presented in the following.

Theorem 3 Assuming that $\dot{h}_{\rho k} \leqslant \varphi_{\rho k}$, the continuous open-loop affine fuzzy system of Eq. (2) is globally exponentially stable in each fuzzy subregion S_k ($k = 1, 2, \ldots, m$; m is the number of the fuzzy subregions) if there exists a set of positive definite matrices $\boldsymbol{P}_{1k}, \ldots, \boldsymbol{P}_{nk}$ which satisfy

$$\left.\begin{aligned} &\sum_{\rho=1}^{n} \varphi_{\rho k}\boldsymbol{P}_{\rho k} + \frac{1}{2}(\boldsymbol{A}_{jk}^{\mathrm{T}}\boldsymbol{P}_{ik} + \boldsymbol{P}_{ik}\boldsymbol{A}_{jk} + \boldsymbol{A}_{ik}^{\mathrm{T}}\boldsymbol{P}_{jk} + \boldsymbol{P}_{jk}\boldsymbol{A}_{ik}) < 0 \\ &\qquad \rho k, ik, jk \in \boldsymbol{L}_0 \\ &\sum_{\rho=1}^{n} \varphi_{\rho k}\boldsymbol{P}_{\rho k} + \frac{1}{2}(\tilde{\boldsymbol{A}}_{jk}^{\mathrm{T}}\boldsymbol{P}_{ik} + \boldsymbol{P}_{ik}\tilde{\boldsymbol{A}}_{jk} + \tilde{\boldsymbol{A}}_{ik}^{\mathrm{T}}\boldsymbol{P}_{jk} + \boldsymbol{P}_{jk}\tilde{\boldsymbol{A}}_{ik}) < 0 \\ &\qquad \rho k, ik, jk \in L_1 \end{aligned}\right\} \tag{9}$$

where $\rho = 1, 2; 1 \leqslant i \leqslant j \leqslant 2$.

Proof In the k-th fuzzy subregion S_k, the local affine fuzzy model is

$$\left.\begin{aligned} \dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{n} h_{ik}(z(t))\boldsymbol{A}_{ik}x(t) \qquad ik \in L_0 \\ \dot{\tilde{\boldsymbol{x}}}(t) &= \sum_{i=1}^{n} h_{ik}(z(t))\tilde{\boldsymbol{A}}_{ik}\tilde{\boldsymbol{x}}(t) \qquad ik \in L_1 \end{aligned}\right\} \tag{10}$$

According to Eq. (6), the fuzzy Lyapunov function candidate of this fuzzy subregion is defined as

$$V_k(\boldsymbol{x}(t)) = \begin{cases} \sum_{i=1}^{n} h_{ik}(z(t))\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}_{ik}\boldsymbol{x} & ik \in L_0 \\ \sum_{i=1}^{n} h_{ik}(z(t))\tilde{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{P}_{ik}\tilde{\boldsymbol{x}} & ik \in L_1 \end{cases} \tag{11}$$

Similarly, as the proving process of Theorem 2 is concerned, assuming that $\dot{h}_{\rho k}(z(t)) \leqslant \varphi_{\rho k}$, the time derivative of $\boldsymbol{V}_k(x)$ is easily obtained,

$$\begin{aligned} \dot{\boldsymbol{V}}_k(\boldsymbol{x}) &= \dot{\boldsymbol{V}}_{1k}(\boldsymbol{x})\big|_{ik \in L_0} + \dot{\boldsymbol{V}}_{2k}(\boldsymbol{x})\big|_{ik \in L_1} \leqslant \\ &\sum_{i=1}^{n}\sum_{j \geqslant i} h_{ik}h_{jk}\boldsymbol{x}^{\mathrm{T}}\Big[\sum_{\rho=1}^{n}\varphi_{\rho k}\boldsymbol{P}_{\rho k} + \frac{1}{2}(\boldsymbol{A}_{jk}^{\mathrm{T}}\boldsymbol{P}_{ik} + \\ &\boldsymbol{P}_{ik}\boldsymbol{A}_{jk} + \boldsymbol{A}_{ik}^{\mathrm{T}}\boldsymbol{P}_{jk} + \boldsymbol{P}_{jk}\boldsymbol{A}_{ik})\Big]\boldsymbol{x}\big|_{ik \in L_0} + \\ &\sum_{i=1}^{n}\sum_{j \geqslant i} h_{ik}h_{jk}\tilde{\boldsymbol{x}}^{\mathrm{T}}\Big[\sum_{\rho=1}^{n}\varphi_{\rho k}\boldsymbol{P}_{\rho k} + \frac{1}{2}(\tilde{\boldsymbol{A}}_{jk}^{\mathrm{T}}\boldsymbol{P}_{ik} + \\ &\boldsymbol{P}_{ik}\tilde{\boldsymbol{A}}_{jk} + \tilde{\boldsymbol{A}}_{ik}^{\mathrm{T}}\boldsymbol{P}_{jk} + \boldsymbol{P}_{jk}\tilde{\boldsymbol{A}}_{ik})\Big]\tilde{\boldsymbol{x}}\big|_{ik \in L_1} \end{aligned}$$

Therefore, when inequalities (9) hold, $\boldsymbol{V}_k(\boldsymbol{x})$ is a fuzzy Lyapunov function of the fuzzy subregion.

Especially, when $n = 1$, the time derivative of the membership function $\dot{h}_{1k} = 0$, and the corresponding sufficient condition for stability is

$$\boldsymbol{A}_{1k}^{\mathrm{T}}\boldsymbol{P}_{1k} + \boldsymbol{P}_{1k}\boldsymbol{A}_{1k} < 0 \quad \text{or} \quad \tilde{\boldsymbol{A}}_{1k}^{\mathrm{T}}\boldsymbol{P}_{1k} + \boldsymbol{P}_{1k}\tilde{\boldsymbol{A}}_{1k} < 0$$

Define the characteristic function $\lambda_k(x)$ of the fuzzy subregion S_k:

$$\lambda_k(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in S_k \\ 0 & \text{otherwise} \end{cases}$$

For the whole affine fuzzy system, the global fuzzy Lyapunov functions can be constructed as

$$\boldsymbol{V}(\boldsymbol{x}) = \sum_{k=1}^{m} \lambda_k(\boldsymbol{x})\boldsymbol{V}_k(\boldsymbol{x})$$

If Theorem 3 holds, $\dot{\boldsymbol{V}}(\boldsymbol{x})$ is strictly negative for all $\boldsymbol{x} \neq 0$, and the affine fuzzy system is globally exponentially asymptotically stable in the equilibrium state.

Example 2 Consider an affine fuzzy system similar to example 1, and the system matrices are given as

$$\boldsymbol{A}_1 = \boldsymbol{A}_3 = \begin{bmatrix} -1 & 3 \\ -3 & -4 \end{bmatrix}, \ \boldsymbol{A}_2 = \begin{bmatrix} -10 & 2 \\ 3 & -2 \end{bmatrix}$$
$$\boldsymbol{a}_1 = \boldsymbol{a}_3 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The membership functions are shown in Fig. 5. As mentioned above, Theorem 2 cannot be applied to this example. Using Theorem 3 and choosing $\varphi_{11} = \varphi_{13} = \varphi_{15} = 0, \varphi_{12} = \varphi_{22} = \varphi_{14} = \varphi_{24} = 1$, a set of positive definite matrices $\boldsymbol{P}_i$ can be computed by LMI optimization technology.

$$\boldsymbol{p}_{11} = 10^3 \times \begin{bmatrix} 0.4098 & 0.0859 & 0.1114 \\ 0.0859 & 0.2364 & -0.0943 \\ 0.1114 & -0.0943 & 1.0258 \end{bmatrix}$$

$$\boldsymbol{p}_{15} = 10^3 \times \begin{bmatrix} 0.3407 & 0.0919 & 0.0815 \\ 0.0919 & 0.2135 & -0.0796 \\ 0.0815 & -0.0796 & 1.0224 \end{bmatrix}$$

$$\boldsymbol{p}_{12} = \begin{bmatrix} 630.6877 & 168.8295 & 151.6640 \\ 168.8295 & 347.5441 & -122.2047 \\ 151.6640 & -122.2047 & 124.1348 \end{bmatrix}$$

$$\boldsymbol{p}_{24} = \begin{bmatrix} 542.3160 & 153.9656 & 125.6785 \\ 153.9656 & 332.7974 & -119.9810 \\ 125.6785 & -119.9810 & 112.9431 \end{bmatrix}$$

$$\boldsymbol{p}_{13} = \begin{bmatrix} 126.0433 & 129.0283 \\ 129.0283 & 484.6664 \end{bmatrix}$$

$$\boldsymbol{p}_{14} = \begin{bmatrix} 151.6448 & 190.9366 \\ 190.9366 & 664.3442 \end{bmatrix}$$

$$\boldsymbol{p}_{22} = \begin{bmatrix} 151.6448 & 190.9366 \\ 190.9366 & 664.3442 \end{bmatrix}$$

Thus, it can be verified that the system is exponentially stable. The above method expands the application of the fuzzy Lyapunov functions in analyzing the stability of the affine fuzzy system.

4 Conclusion

A method which deals with the consequent part of the affine fuzzy systems is introduced for stability analysis. The sufficient stability conditions based on the fuzzy Lyapunov functions and the modified common Lyapunov functions are proposed. The results demonstrate that the stability result based on fuzzy Lyapunov functions is less conservative. Meanwhile, the sufficient stability conditions derived from the consequent part processing method will reduce the stability region compared with the method which does not expand the consequent part. Furthermore, in order to expand the application of fuzzy Lyapunov functions, the piecewise fuzzy Lyapunov function is proposed.

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基于模糊 Lyapunov 函数的仿射模糊模型稳定性分析

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摘要:研究了基于模糊 Lyapunov 函数分析连续仿射模糊系统稳定性的方法. 首先, 对模糊系统局部模型的后件部分进行扩展处理, 以便于借鉴齐次模糊模型的稳定性分析方法. 然后,分别得到基于改进公共 Lyapunov 函数与模糊 Lyapunov 函数的系统稳定条件,该条件可表示为一组线性矩阵不等式. 通过算例对所得稳定条件进行对比,结果表明: 基于模糊 Lyapunov 函数得到的稳定条件与基于改进公共 Lyapunov 函数的相比具有较小保守性; 对后件部分进行扩展处理后,尽管稳定性证明方法较简便, 但与不进行后件处理得到的稳定条件相比, 可行解范围有所减小. 最后, 为了增大模糊 Lyapunov 函数的应用范围, 提出了对模糊空间进行划分的方法, 该方法可对隶属度函数为三角形或梯形的模糊系统进行稳定性分析,得到了基于分段模糊 Lyapunov 函数的系统稳定条件, 并通过算例验证了所提方法的有效性.

关键词:仿射模糊模型;稳定性分析;线形矩阵不等式;模糊 Lyapunov 函数

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