Journal of Southeast University (English Edition) Vol. 27, No. 3, pp. 343 – 346 Sept. 2011 ISSN 1003—7985

Twisted smash product for Hopf quasigroups

Fang Xiaoli Wang Shuanhong

(Department of Mathematics, Southeast University, Nanjing 211189, China)

Abstract: In order to study algebraic structures of parallelizable sphere s^7, the notions of quasimodules and biquasimodule algebras over Hopf quasigroups, which are not required to be associative, are introduced. The lack of associativity of quasimodules is compensated for by conditions involving the antipode. The twisted smash product for Hopf quasigroups is constructed using biquasimodule algebras, which is a generalization of the twisted smash for Hopf algebras. The twisted smash product and tensor coproduct is turned into a Hopf quasigroup if and only if the following conditions $(h_1 \rightharpoonup a)\otimes h_2 = (h_2 \rightharpoonup a)\otimes h_1$, $(a \leftharpoonup S(h_1))\otimes h_2 = (a \leftharpoonup S(h_2))\otimes h_1$ hold. The obtained results generalize and improve the corresponding results of the twisted smash product for Hopf algebras.

Key words: Hopf quasigroup; quasimodule; twisted smash product

doi: 10.3969/j. issn. 1003 – 7985. 2011. 03. 023

It is a well-known fact that the only parallelizable spheres are s^1, s^3, s^7. The first two are groups and it is known that s^7 is something weaker (a Moufang loop or a Moufang quasigroup). Recently, Klim and Majid[1] introduced the notions of Hopf quasigroups and Hopf coquasigroups in order to capture the quasigroup features of the (algebraic) 7-sphere. They observed that the axioms of Hopf (co) quasigroup unified quasigroups[2-3] and the Malcev algebras[4] just as Hopf algebras historically unified groups and Lie algebras, and they also observed that some basic properties analogous to theorems[5] of Hopf algebras can be generalized to Hopf (co) quasigroups. These are generalizations of Hopf algebras that are not required to be (co) associative. The lack of (co) associativity is compensated for by conditions involving the antipode. In the note, we first introduce the notion of the twisted smash product for Hopf quasigroups. Then we give a necessary and sufficient condition making the twisted smash product and tensor coproduct into a Hopf quasigroup, which includes some important results in several references.

Now, we first recall the definition of a Hopf quasigroup[1].

A Hopf quasigroup is a possibly-nonassociative but unital algebra H with product $\mu: H\otimes H \mapsto H$ and unit $1: k \mapsto H$ equipped with algebra homomorphisms $\Delta: H \mapsto H\otimes H$, $\varepsilon: H \mapsto k$ forming a coassociative coalgebra and a map $S: H\rightarrow H$ such that the following conditions hold.

$$\mu(\mathrm{id}\otimes\mu)(S\otimes\mathrm{id}\otimes\mathrm{id})(\Delta\otimes\mathrm{id}) = \varepsilon\otimes\mathrm{id} = \mu(\mathrm{id}\otimes\mu)(\mathrm{id}\otimes S\otimes\mathrm{id})(\Delta\otimes\mathrm{id}) \tag{1}$$

$$\mu(\mu\otimes\mathrm{id})(\mathrm{id}\otimes S\otimes\mathrm{id})(\mathrm{id}\otimes\Delta) = \mathrm{id}\otimes\varepsilon = \mu(\mu\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes S)(\mathrm{id}\otimes\Delta) \tag{2}$$

We use the Sweedler notation[6] for coproduct: for all $h \in H$, $\Delta(h) = h_1\otimes h_2$ (summation implicit). Thus, in terms of the Sweedler notation, the Hopf quasigroup conditions (1) and (2) can be expressed by

$$S(h_1)(h_2 g) = h_1(S(h_2)g) = (g(h_1))S(h_2) = (gS(h_1))h_2 = g\varepsilon(h)$$

As for the standard Hopf algebra, map S is called an antipode. It is proved in Ref. [1] that the antipode is antimultiplicative and anticomultiplicative and it is immediately shown that, for all

$$h \in H,\ S(h_1)h_2 = h_1 S(h_2) = \varepsilon(h)1$$

Let H be a Hopf quasigroup. A vector space V is a left H-quasimodule [7] if there is a linear map $\alpha: H\otimes V \mapsto V$ written as $\alpha(h\otimes v) = h\rightharpoonup v$ such that

$$h_1 \rightharpoonup (S(h_2) \rightharpoonup v) = \varepsilon(h)v = S(h_1) \rightharpoonup (h_2 \rightharpoonup v),\ 1 \rightharpoonup v = v \tag{3}$$

for all $h, g \in H, v \in V$.

Similarly, we can define a right H-quasimodule, that is, there is a linear map $\beta: V\otimes H \mapsto V$ written as $\beta(v\otimes h) = v\leftharpoonup h$ such that

$$(v \leftharpoonup h_1) \leftharpoonup S(h_2) = (v \leftharpoonup S(h_1)) \leftharpoonup h_2 = \varepsilon(h)v,\ v \leftharpoonup 1 = v \tag{4}$$

for all $h, g \in H, v \in V$. If a vector space V is a left H-quasimodule and a right H-quasimodule, and the following condition holds,

$$(h \rightharpoonup v) \leftharpoonup g = h \rightharpoonup (v \leftharpoonup g) \tag{5}$$

where $h, g \in H, v \in V$, then we call it an H-biquasimodule.

An algebra A (not necessarily associative) is a left H-quasimodule algebra if A is a left H-quasimodule and the following conditions hold,

$$h \rightharpoonup (ab) = (h_1 \rightharpoonup a)(h_2 \rightharpoonup b),\ h \rightharpoonup 1 = \varepsilon(h)1 \tag{6}$$

for all $h \in H, a, b \in A$.

Similarly, we give the notion of a right H-quasimodule algebra, that is A is a right H-quasimodule and the following conditions hold,

Received 2010-09-10.

Biographies: Fang Xiaoli (1979—), male, doctor; Wang Shuanhong (corresponding author), male, doctor, professor, shuanhwang@seu.edu.cn.

Foundation items: The National Natural Science Foundation of China (No. 10971188), the Natural Science Foundation of Zhejiang Province (No. Y6110323), Jiangsu Planned Projects for Postdoctoral Research Funds (No. 0902081C), Zhejiang Provincial Education Department Project (No. Y200907995), Qiantang Talents Project of Science Technology Department of Zhejiang Province (No. 2011R10051).

Citation: Fang Xiaoli, Wang Shuanhong. Twisted smash product for Hopf quasigroups[J]. Journal of Southeast University (English Edition), 2011, 27 (3): 343 – 346. [doi: 10.3969/j. issn. 1003 – 7985. 2011. 03. 023]

$$(ab) \leftarrow h = (a \leftarrow h_1)(b \leftarrow h_2), 1 \leftarrow h = \varepsilon(h)1 \quad (7)$$

for all $h \in H, a, b \in A$.

Let H be a Hopf quasigroup with antipode S, and A is an algebra (not necessarily associative). A is called an H-biquasimodule algebra if the following conditions hold:

1) A is an H-biquasimodule with the left H-quasimodule structure map "$\rightarrow$" and with the right H-quasimodule structure map "$\leftarrow$";

2) A is not only left H-quasimodule algebra with the left module action "$\rightarrow$" but also right H-quasimodule algebra with the right quasimodule action "$\leftarrow$".

A coalgebra C is a left H-quasimodule coalgebra if C is a left H-quasimodule and

$$\Delta(h \rightarrow c) = h_1 \rightarrow c_1 \otimes h_2 \rightarrow c_2, \ \varepsilon(h \rightarrow c) = \varepsilon(h)\varepsilon(c) \quad (8)$$

for all $h \in H, c \in C$. A left H-quasimodule Hopf quasigroup is a Hopf quasigroup which is a left H-quasimodule algebra and left H-quasimodule coalgebra. By a similar method, we can define a right H-quasimodule coalgebra and a right H-quasimodule Hopf quasigroup.

Theorem 1 Let H be a Hopf quasigroup. Let A be a left H-biquasimodule Hopf quasigroup and a right H-quasimodule Hopf quasigroup such that, for all $h \in H$ and $a \in A$,

$$g \rightarrow (S(h) \rightarrow a) = (gS(h)) \rightarrow a \quad (9)$$

$$(a \leftarrow S(h)) \leftarrow g = a \leftarrow (S(h)g) \quad (10)$$

Then the twisted product $A\Theta H$ built on $A \otimes H$ with the tensor coproduct and unit and

$$(a\Theta h)(b\Theta h) = a(h_1 \rightarrow b \leftarrow S(h_3)) \otimes h_2 g \quad (11)$$

$$S(a \otimes h) = (1 \otimes S_H(h))\Theta(S_A(a) \otimes 1) \quad (12)$$

is a Hopf quasigroup if and only if the following conditions hold,

$$(h_1 \rightarrow a) \otimes h_2 = (h_2 \rightarrow a) \otimes h_1 \quad (13)$$

$$(a \leftarrow S(h_1)) \otimes h_2 = (a \leftarrow S(h_2)) \otimes h_1 \quad (14)$$

Proof(sufficiency) To see that Δ is an algebra homomorphism, for all $a, b \in A, h, l \in H$, we compute

$$\begin{aligned}
&\Delta((a \otimes h)\Theta(b \otimes l)) = \Delta(a(h_1 \rightarrow b \rightarrow S(h_3)) \otimes h_2 l) = \\
&a(h_1 \rightarrow b \leftarrow S(h_3)))_1 \otimes (h_2 l)_1 \otimes (h_1 \rightarrow b \leftarrow S(h_3))_2 \otimes (h_2 l)_2) = \\
&a_1(h_1 \rightarrow b \leftarrow S(h_3))_1 \otimes h_{21} l_1 \otimes a_2(h_1 \rightarrow b \leftarrow S(h_3)_2) \otimes h_{22} l_2 = \\
&a_1(h_{11} \rightarrow b_1 \leftarrow S(h_3)_1) \otimes h_{21} l_1 \otimes a_2(h_{12} \rightarrow b_2 \leftarrow S(h_3)_2) \otimes h_{22} l_2 = \\
&a_1(h_1 \rightarrow b_1 \leftarrow S(h_6)) \otimes h_3 l_1 \otimes a_2(h_2 \rightarrow b_2 \leftarrow S(h_5)) \otimes h_4 l_2 = \\
&a_1(h_1 \rightarrow b_1 \leftarrow S(h_4)) \otimes h_3 l_1 \otimes a_2(h_2 \rightarrow b_2 \leftarrow S(h_6)) \otimes h_5 l_2 = \\
&a_1(h_1 \rightarrow b_1 \leftarrow S(h_3)) \otimes h_2 l_1 \otimes a_2(h_4 \rightarrow b_2 \leftarrow S(h_6)) \otimes h_5 l_2 = \\
&(a_1 \otimes h_1)\Theta(b_1 \otimes l_1) \otimes (a_2 \otimes h_2)\Theta(b_2 \otimes l_2) = \\
&\Delta(a \otimes h)\Delta(b \otimes l)
\end{aligned}$$

The first equality and the second equality above use the definitions of multiplication and comultiplication of $A\Theta H$, respectively. The third equality uses the algebra morphisms of Δ_H and Δ_A. The fourth equality uses the properties of the left H-quasimodule algebra and the right H-quasimodule algebra. The fifth equality uses the anticomultiplication of the antipode. The sixth equality uses the condition (14) two times. The seventh equality uses the condition (13) two times. The eighth equality uses the definition of multiplication of $A\Theta H$.

It remains to check the Hopf quasigroup conditions (1) and (2). For all $a, b \in A, h, l \in H$, we compute

$$\begin{aligned}
&S((a \otimes h)_1)\Theta((a \otimes h)_2\Theta(b \otimes l)) = \\
&S((a \otimes h)_1)\Theta(a_2(h_2 \rightarrow b \leftarrow S(h_4)) \otimes h_3 l) = \\
&(S(h_3) \rightarrow S(a_1) \leftarrow S^2(h_1) \otimes S(h_2))\Theta(a_2(h_2 \rightarrow b \leftarrow S(h_4)) \otimes h_3 l) = \\
&(S(h_5) \rightarrow (S(a_1) \leftarrow S^2(h_1)))(S(h_4) \rightarrow ((a_2(h_6 \rightarrow b \leftarrow S(h_8))) \leftarrow \\
&S^2(h_2))) \otimes S(h_3)(h_7 l)S(h_4) \rightarrow ((S(a_1) \leftarrow \\
&S^2(h_1))((a_2(h_5 \rightarrow b \leftarrow S(h_7))) \leftarrow S^2(h_2)) \otimes S(h_3)(h_6 l)) = \\
&S(h_3) \rightarrow ((S(a_1)(a_2(h_4 \rightarrow b \leftarrow S(h_6)))) \leftarrow S^2(h_1)) \otimes S(h_2)(h_5 l) = \\
&\varepsilon(a)S(h_3) \rightarrow ((h_4 \rightarrow b \leftarrow S(h_6)) \leftarrow S^2(h_1)) \otimes S(h_2)(h_5 l) = \\
&\varepsilon(a)(S(h_3) \rightarrow (h_4 \rightarrow b \leftarrow S(h_6))) \leftarrow S^2(h_1) \otimes S(h_2)(h_5 l) = \\
&\varepsilon(a)(b \leftarrow S(h_4)) \leftarrow S^2(h_1) \otimes S(h_2)(h_3 l) = \\
&\varepsilon(a)\varepsilon(h)(b \leftarrow 1) \otimes l = \varepsilon(a)\varepsilon(h)b \otimes l
\end{aligned}$$

The first equality, the second equality and the third equality above use the definitions of multiplication, comultiplication and the antipode of $A\Theta H$. The fourth equality and the fifth equality use the properties of left quasimodule algebra and right quasimodule algebra. The sixth and the eighth use the properties of quasimodule. The seventh equality uses the property of biquasimodule. This proves that $\mu(\mathrm{id}\otimes\mu)(S\otimes \mathrm{id}\otimes\mathrm{id})(\Delta\otimes\mathrm{id}) = \varepsilon\otimes\mathrm{id}$. Next,

$$\begin{aligned}
&(a \otimes h)_1\Theta(S((a \otimes h)_2)\Theta(b \otimes l)) = \\
&(a_1 \otimes h_1)\Theta(((S(h_4) \rightarrow S(a_2) \leftarrow S^2(h_2)) \otimes S(h_3))\Theta(b \otimes l)) = \\
&(a_1 \otimes h_1)\Theta((S(h_6) \rightarrow S(a_2) \leftarrow S^2(h_2))(S(h_5) \rightarrow b \leftarrow S^2(h_3)) \otimes h_4 l) = \\
&a_1(h_1 \rightarrow (S(h_8) \rightarrow S(a_2) \leftarrow S^2(h_4))(S(h_7) \rightarrow b \leftarrow S^2(h_5)) \leftarrow \\
&S(h_3)) \otimes h_2(h_6 l) = a_1(h_1 \rightarrow ((S(h_7) \rightarrow S(a_2))(S(h_6) \rightarrow b) \leftarrow \\
&S^2(h_4)) \leftarrow S(h_3)) \otimes h_2(h_5 l) = \\
&a_1(h_1 \rightarrow ((S(h_5) \rightarrow S(a_2))(S(h_4) \rightarrow b)) \otimes h_2(h_3 l)) = \\
&a_1(h_1 \rightarrow (S(h_2) \rightarrow (S(a_2)b))) \otimes l = \\
&\varepsilon(h)a_1(S(a_2)b) \otimes l = \varepsilon(h)\varepsilon(a)b \otimes l
\end{aligned}$$

This proves that $\varepsilon \otimes \mathrm{id} = \mu(\mathrm{id} \otimes \mu)(\mathrm{id} \otimes S \otimes \mathrm{id})(\Delta \otimes \mathrm{id})$.

Now, we will prove that $\mu(\mu\otimes\mathrm{id})(\mathrm{id}\otimes S\otimes\mathrm{id})(\mathrm{id}\otimes\Delta) = \mathrm{id}\otimes\varepsilon$. For all $a, b \in A, h, l \in H$, we have

$$\begin{aligned}
&((b \otimes l)\Theta(a \otimes h)_1)\Theta S((a \otimes h)_2) = \\
&((b \otimes l)\Theta(a_1 \otimes h_1))\Theta(S(h_4) \rightarrow S(a_2) \leftarrow S(h_2) \otimes S(h_3)) = \\
&(b(l_1 \rightarrow a_1 \leftarrow S(l_3)) \otimes l_2 h_1)\Theta(S(h_4) \rightarrow S(a_2) \leftarrow S(h_2) \otimes S(h_3)) = \\
&b(l_1 \rightarrow a_1 \leftarrow S(l_5))((l_2 h_1) \rightarrow (S(h_6) \rightarrow S(a_2) \leftarrow S(h_4)) \leftarrow S(l_4 h_3) \otimes \\
&(l_3 h_2)S(h_5)) = b(l_1 \rightarrow a_1 \leftarrow S(l_5))((l_2 h_1) \rightarrow \\
&((S(h_6) \rightarrow S(a_2)) \leftarrow S(h_4)(S(h_3)S(l_4)))) \otimes (l_3 h_2)S(h_5) = \\
&b(l_1 \rightarrow a_1 \leftarrow S(l_5))((l_2 h_1) \rightarrow ((S(h_2) \rightarrow S(a_2)) \leftarrow S(l_4))) \otimes l_3 = \\
&\varepsilon(h)b(l_1 \rightarrow a_1 \leftarrow S(l_5))(l_2 \rightarrow S(a_2) \leftarrow S(l_4)) \otimes l_3 = \\
&b(l_1 \rightarrow (a_1 S(a_2)) \leftarrow S(l_3)) \otimes l_2 = \\
&b(l \rightarrow 1 \leftarrow S(l_3)) \otimes l_2\varepsilon(h)\varepsilon(a) = b \otimes l\varepsilon(h)\varepsilon(a)
\end{aligned}$$

The first equality, the second equality and the third equality use the definitions of multiplication, comultiplication and the antipode of $A\Theta H$. The fourth equality uses the antimultipication of the antipode. The fifth equality uses (10) and the property of Hopf quasigroup. The sixth equality uses (9) and the property of Hopf quasigroup. The seventh equality uses the properties of quasimodule algebra. This proves that $\mu(\mu\otimes\mathrm{id})(\mathrm{id}\otimes S\otimes\mathrm{id})(\mathrm{id}\otimes\Delta)=\mathrm{id}\otimes\varepsilon$. Similarly,

$$
\begin{aligned}
&((b\otimes l)\Theta S((a\otimes h)_1))\Theta(a\otimes h)_2=\\
&((b\otimes l)\Theta(S(h_3)\rightarrow S(a_1)\leftarrow S^2(h_1)\otimes S(h_2)))\Theta(a_2\otimes h_4)=\\
&b(l_1\rightarrow(S(h_3)\rightarrow S(a_1)\leftarrow S^2(h_1))\leftarrow S(l_3)\otimes l_2S(h_2))\Theta(a_2\otimes h_4)=\\
&b(l_1\rightarrow(S(h_5)\rightarrow S(a_1)\leftarrow S^2(h_1))\leftarrow S(l_5))(l_2S(h_4)\rightarrow\\
&a_2\leftarrow S^2(h_2)S(l_4))\otimes(l_3S(h_3))h_6=\\
&b(l_1S(h_5)\rightarrow(S(a_1)\leftarrow S^2(h_1)S(l_5)))(l_2S(h_4)\rightarrow\\
&(a_2\leftarrow S^2(h_2)S(l_4)))\otimes(l_3S(h_3))h_6=\\
&b(l_1S(h_4)\rightarrow(S(a_1)\leftarrow S^2(h_1)S(l_4))(a_2\leftarrow S^2(h_2)S(l_3)))\otimes(l_2S(h_3))h_5=\\
&b(l_1S(h_3)\rightarrow(S(a_1)a_2\leftarrow S^2(h_1)S(l_3)))\otimes(l_2S(h_2))h_4=\\
&\varepsilon(a)b(l_1S(h_2)\rightarrow 1)\otimes(l_2S(h_1))h_3=\\
&b\otimes(lS(h_1))h_2=b\otimes l\varepsilon(h)\varepsilon(a)
\end{aligned}
$$

This proves that $\mathrm{id}\otimes\varepsilon=\mu(\mu\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes S)(\mathrm{id}\otimes\Delta)=\mathrm{id}\otimes\varepsilon$.

(necessity) Since

$$\Delta((1\otimes h)\Theta(a\otimes 1))=\Delta(1\otimes h)\Delta(a\otimes 1)$$

we have

$$
\begin{aligned}
&(h_1\rightarrow a\leftarrow S(h_4))_1\otimes h_2\otimes(h_1\rightarrow a\leftarrow S(h_4))_2\otimes h_3=\\
&\quad(h_1\rightarrow a\leftarrow S(h_3))\otimes h_2\otimes(h_4\rightarrow a_2\leftarrow S(h_6))\otimes h_5
\end{aligned}\tag{15}
$$

Applying $\varepsilon_A\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}$ to (15) and using the property of quasimodule coalgebra, we obtain

$$h_2\otimes(h_1\rightarrow a\leftarrow S(h_4))\otimes h_3=h_1\otimes(h_2\rightarrow a\leftarrow S(h_4))\otimes h_3\tag{16}$$

Applying $(\mathrm{id}\otimes\leftarrow)(\mathrm{id}\otimes\mathrm{id}\otimes s^2)$ to (16) and using the property of quasimodule, we obtain (13). Applying $\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon\otimes\mathrm{id}$ to (15) and using the property of quasimodule coalgebra and (13), we have

$$h_2\rightarrow a\leftarrow S(h_4)\otimes h_1\otimes h_3=h_2\rightarrow a\leftarrow S(h_3)\otimes h_1\otimes h_4\tag{17}$$

i.e.,

$$h_1\otimes h_2\rightarrow a\leftarrow S(h_4)\otimes h_3=h_1\otimes h_2\rightarrow a\leftarrow S(h_3)\otimes h_4\tag{18}$$

Applying $(\rightarrow\otimes\mathrm{id})(s\otimes\mathrm{id}\otimes\mathrm{id})$ to (18) and using the property of left quasimodule, we have (14). This completes the proof.

If the right quasiaction is trivial, then (14) holds. By Theorem 1, we have the following corollary which is the main theorem in Ref. [8].

Corollary 1 Let H be a Hopf quasigroup, A is a left H-quasimodule Hopf quasigroup. Then a smash product Hopf quasigroup $A\times H$ built on $A\times H$ with the tensor product coproduct, counit and unit, and the product and the antipode given (11) and (12) is a Hopf quasigroup if and only if conditions (13) and (9) hold.

Remark 1 For a smash product, the condition (9) can be obtained by the proof of Theorem 4.3 in Ref. [8].

If the right quasiaction is trivial. It is easy to see that the module structure defined in Ref. [3] is a special case of the quasimodule in this note and the condition (9) holds by the definition of the module structure given[3]. If H is a cocommutative Hopf quasigroup, it is easy to find that the condition (13) holds. Thus, we have the following corollary which is an important result in Ref. [1].

Corollary 2 Let H be a cocommutative Hopf quasigroup and A a left H-quasimodule Hopf quasigroup. Then a smash product Hopf quasigroup $A\times H$ built on $A\otimes H$ with the tensor product coproduct, counit and unit, and the product and the antipode given in (11) and (12) is a Hopf quasigroup.

If A and H are both associative, A and H are usual Hopf algebra. Then Theorem 1 becomes the main result in Ref. [9].

Corollary 3 Let H be a Hopf algebra, A a left H-module bialgebra and a right H-module bialgebra. The twisted smash product algebra $A\star H$ with the tensor coproduct, where the product and the antipode given in (11) and (12) is a Hopf algebra if and only if conditions (9) and (10) hold.

By Corollary 3, we have the following corollary (see Theorem 2.3 in Ref. [10]).

Corollary 4 Let H be a cocommutative Hopf algebra and A be a Hopf algebra which is an H-module bialgebra. Then the tensor product coalgebra structure on $A\#H$ equipped with the smash product structure makes $A\#H$ into a Hopf algebra with antipode defined by $S(a\#h)=S(h_2)S(a)\#S(h_1)$.

Remark 2 By duality, we can also give a necessary and sufficient condition making the twisted smash coproduct and tensor product into a Hopf coquasigroup which is introduced in Ref. [1].

References

[1] Klim J, Majid S. Hopf quasigroups and the algebraic 7-sphere[J]. *J Algebra*, 2010, **323**(11): 3067 − 3110.

[2] Albuquerque H, Majid S. Quasialgebra structure of the octonions[J]. *J Algebra*, 1999, **220**(1): 188 − 224.

[3] Woronowicz S L. Differential calculus on compact matrix pseudogroup (quantum groups) [J]. *Comm Math Phys*, 1989, **122**(1): 125 − 170.

[4] Perez-Izquierdo J M, Shestakov I P. An envelope for Malcev algebras[J]. *J Algebra*, 2004, **272**(1): 379 − 393.

[5] Montgomery S. *Hopf algebras and their actions on rings* [M]. Providence, RI, USA: American Mathematical Society, 1993.

[6] Sweedler M. *Hopf algebra* [M]. New York: Benjamin, 1969.

[7] Brzezinski T. Hopf modules and the fundamental theorem for Hopf (co) quasigroups[J]. *International Electronic Journal of Algebra*, 2010, **8**: 114 − 128.

[8] Brzezinski T, Jiao Z M. Actions of Hopf quasigroups[EB/OL]. (2010-05-14) [2010-08-01]. http://arxiv.org/abs/1005.2496.

[9] Wang S H, Li J Q. On twisted smash products for bimodule algebras and the Drinfeld double[J]. *Comm Algebra*, 1998, **26**(8): 2435 − 2444.

[10] Molnar R K. Semi-direct products of Hopf algebras[J]. *J Algebra*, 1977, **47**(1): 29 − 51.

Hopf 拟群上扭曲冲积

方小利　　王栓宏

（东南大学数学系，南京 211189）

摘要：为了研究平行球面 s^7 的代数结构，引进了 Hopf 拟群上的拟模和双拟模代数的概念，由于这些概念的公理中模缺少结合性的条件，通过增加对极的条件来弥补结合性的条件. 并通过双拟模代数构造了扭曲冲积的概念，事实上这种扭曲冲积是 Hopf 代数上扭曲冲积的推广，并且证明了扭曲冲积与张量余积成为 Hopf 拟群的充要条件为当且仅当下列条件 $(h_1 \rightarrow a) \otimes h_2 = (h_2 \rightarrow a) \otimes h_1$，$(a \leftarrow S(h_1)) \otimes h_2 = (a \leftarrow S(h_2)) \otimes h_1$ 成立. 所得到的结果推广并改进了 Hopf 代数上扭曲冲积一些相应的结果.

关键词：Hopf 拟群；拟模；扭曲冲积

中图分类号：O153