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Uniquely strongly clean triangular matrix rings

Cui Jian Chen Jianlong

(Department of Mathematics, Southeast University, Nanjing 211189, China)

Abstract: An element a of a ring R is called uniquely strongly clean if it is the sum of an idempotent and a unit that commute, and in addition, this expression is unique. R is called uniquely strongly clean if every element of R is uniquely strongly clean. The uniquely strong cleanness of the triangular matrix ring is studied. Let R be a local ring. It is shown that any $n \times n$ upper triangular matrix ring over R is uniquely strongly clean if and only if R is uniquely bleached and $R/J(R) \cong Z_2$.

Key words: uniquely strongly clean ring; uniquely bleached local ring; triangular matrix ring

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Throughout, all rings are associative with unity. For a ring R, $T_n(R)$ denotes the ring of all $n \times n$ upper triangular matrices over R. Given a matrix $\boldsymbol{A}$, A_{ij} denotes the (i, j)-th entry of $\boldsymbol{A}$, and $\boldsymbol{I}_n$ denotes the $n \times n$ identity matrix. The symbols $U(R)$ and $J(R)$ stand for the group of units and the Jacobson radical of R, respectively.

An element of a ring is called strongly clean if it can be written as the sum of an idempotent and a unit which commute. A ring is strongly clean[1] if each of its elements is strongly clean. Some results on strongly clean rings can be referred to in Refs. [2 – 9]. In Ref. [10], Chen et al. generalized the concept of strongly clean rings and introduced the notion of uniquely strongly clean rings. An element of a ring is called uniquely strongly clean (or USC for short) if it has a uniquely strongly clean expression, and a ring is said to be uniquely strongly clean (or USC for short) if every element of the ring is uniquely strongly clean. It was shown in Ref. [10] that a ring R is uniquely clean (i. e., a ring in which every element is uniquely the sum of an idempotent and a unit[11]) iff R is an Abelian (that is, all its idempotents are central) USC ring, and the ring of all $n \times n$ matrices over any given ring R is not uniquely strongly clean whenever $n > 1$. The USC triangular matrix ring was investigated in Ref. [10], and it was proved that for a commutative ring R, R is uniquely clean if and only if $T_n(R)$ is USC for any $n > 1$.

In this paper, we continue the study of the uniquely strong cleanness of any $n \times n$ triangular matrix ring over a local ring. Let R be a local ring. We show that R is uniquely bleached with $R/J(R) \cong Z_2$ if and only if $T_n(R)$ is USC for any $n \geqslant 1$ if and only if $T_n(R)$ is USC for some $n \geqslant 2$.

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Biographies: Cui Jian (1984—), male, graduate; Chen Jianlong (corresponding author), male, doctor, professor, jlchen@seu.edu.cn.

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We start with the following elementary lemma which will be used freely in sequel.

Lemma 1 Let R be a ring, and let E, A, $B \in T_n(R)$.

1) If $E^2 = E$, then $(E_{ii})^2 = E_{ii}$ for $i = 1, 2, \dots, n$.

2) $A \in U(T_n(R))$ if and only if $A_{ii} \in U(R)$ for $i = 1, 2, \dots, n$.

3) $B \in J(T_n(R))$ if and only if $B_{ii} \in J(R)$ for $i = 1, 2, \dots, n$.

Lemma 2 Let R be a ring and $a \in R$. Then a is USC if and only if so is $1 - a$.

Proof It is easy to see that $a = e + u$ is a uniquely strongly clean expression in R if and only if so is $1 - a = (1 - e) + (-u)$.

Lemma 3 A local ring R is USC iff $R/J(R) \cong Z_2$.

Proof In view of Example 4 in Ref. [10], an Abelian ring is USC if and only if it is uniquely clean. By Theorem 15 in Ref. [11], a local ring R is uniquely clean iff $R/J(R) \cong Z_2$. Thus the result follows.

Let $a \in R$. l_a: $R \to R$ and r_a: $R \to R$ denote, respectively, the Abelian group endomorphisms given by $l_a(s) = as$ and $r_a(s) = sa$ for all $s \in R$.

Definition 1[2] A local ring R is called uniquely bleached if for any $j \in J(R)$ and any $u \in U(R)$; the Abelian group endomorphisms $l_u - r_j$ and $l_j - r_u$ of R are isomorphic.

According to Example 13 in Ref. [2], division rings, commutative local rings, local rings with nil Jacobson radicals, and local rings for which some power of each element of their Jacobson radicals is central are all uniquely bleached.

Lemma 4 Let R be a local ring such that $R/J(R) \cong Z_2$ and $u \in U(R)$, $j \in J(R)$. The following are equivalent.

1) $l_u - r_j$ is an isomorphism.

2) For any $r \in R$, the matrix $\begin{bmatrix} u & r \\ 0 & j \end{bmatrix} \in T_2(R)$ is USC.

Proof Denote $\bar{R} = R/J(R)$. For any given $r \in R$, write $\boldsymbol{A} = \begin{bmatrix} u & r \\ 0 & j \end{bmatrix}$. Since $\bar{R} \cong Z_2$, $\bar{u} = \bar{1}$. If $\boldsymbol{X}^2 = \boldsymbol{X} \in T_2(R)$ and $\boldsymbol{A} - \boldsymbol{X} \in T_2(R)$, then $\boldsymbol{X} = \begin{bmatrix} 0 & x_{12} \\ 0 & 1 \end{bmatrix}$ for $x_{12} \in R$.

1) $\Rightarrow$ 2). By hypothesis, there exists a unique $e_{12} \in R$ such that $ue_{12} - e_{12}j = -r$. Let $\boldsymbol{E} = \begin{bmatrix} 0 & e_{12} \\ 0 & 1 \end{bmatrix}$. Then $\boldsymbol{E}^2 = \boldsymbol{E}$ and $\boldsymbol{U} = \boldsymbol{A} - \boldsymbol{E} = \begin{bmatrix} u & r - e_{12} \\ 0 & j - 1 \end{bmatrix} \in U(T_2(R))$ by Lemma 1. Since $ue_{12} + r = e_{12}j$, it follows that $\boldsymbol{AE} = \boldsymbol{EA}$. So $\boldsymbol{A} = \boldsymbol{E} + \boldsymbol{U}$ is a strongly clean expression. Note that $e_{12} \in R$ is uniquely determined by r. Thus, $\boldsymbol{E} \in T_2(R)$ is unique such that $\boldsymbol{A} - \boldsymbol{E} \in U(T_2(R))$ and $\boldsymbol{AE} = \boldsymbol{EA}$. This proves that $\boldsymbol{A}$ is USC in $T_2(R)$.

2) $\Rightarrow$ 1). Let $\boldsymbol{E} = \begin{bmatrix} 0 & e_{12} \\ 0 & 1 \end{bmatrix} \in T_2(R)$ be such that $\boldsymbol{A} - \boldsymbol{E} \in U(T_2(R))$ and $\boldsymbol{AE} = \boldsymbol{EA}$. It follows that $ue_{12} - e_{12}j =$

$-r$. Since $\boldsymbol{A}$ is USC, $\boldsymbol{E}$ is uniquely determined by $\boldsymbol{A}$. So e_{12} is uniquely determined by r, which implies the isomorphism of $l_u - r_j$.

Similar to Lemma 4, we have the following result.

Lemma 5 Let R be a local ring such that $R/J(R) \cong Z_2$ and $u \in U(R)$, $j \in J(R)$. The following are equivalent.

1) $l_j - r_u$ is an isomorphism.

2) For any $r \in R$, the matrix $\begin{bmatrix} j & r \\ 0 & u \end{bmatrix} \in T_2(R)$ is USC.

Corollary 1 Let R be a local ring. The following are equivalent.

1) R is uniquely bleached and $R/J(R) \cong Z_2$.

2) $T_2(R)$ is USC.

Proof 1) $\Rightarrow$ 2). Let $\boldsymbol{A} = (a_{ij}) \in T_2(R)$. If $a_{ii} \in U(R)$ (resp., $a_{ii} \in J(R)$) for $i = 1, 2$, then let $\boldsymbol{E} = \boldsymbol{0}$ (resp., $\boldsymbol{E} = \boldsymbol{I}_2$). Since $R/J(R) \cong Z_2$, $1 - u \in J(R)$ for all $u \in U(R)$. Thus, $\boldsymbol{A} = \boldsymbol{E} + (\boldsymbol{A} - \boldsymbol{E})$ is a uniquely strongly clean expression in $T_2(R)$. If $a_{11} \in U(R)$ and $a_{22} \in J(R)$ (resp., $a_{11} \in J(R)$ and $a_{22} \in U(R)$), then $l_{a_{11}} - r_{a_{22}}$ is isomorphic by the hypothesis. In view of Lemma 4 (resp., Lemma 5), A is USC.

2) $\Rightarrow$ 1). Write $\boldsymbol{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $\boldsymbol{R} \cong \boldsymbol{E}T_2(R)\boldsymbol{E}$ is USC by Example 5 in Ref. [10]. In view of Lemma 3, $R/J(R) \cong Z_2$. For any given $u \in U(R)$, $j \in J(R)$, matrices in the form of $\begin{bmatrix} u & r \\ 0 & j \end{bmatrix}$ and $\begin{bmatrix} j & r \\ 0 & u \end{bmatrix}$ are USC in $T_2(R)$ for all $r \in R$. Thus, $l_u - r_j$ and $l_j - r_u$ are isomorphisms by Lemma 4 and Lemma 5. Hence R is uniquely bleached.

We write R^n (resp., R_n) for the set of all $1 \times n$ (resp., $n \times 1$) matrices over a ring R.

Theorem 1 Let R be a local ring. Then the following are equivalent.

1) R is uniquely bleached and $R/J(R) \cong Z_2$.

2) $T_n(R)$ is USC for any $n \geqslant 1$.

3) $T_n(R)$ is USC for some $n \geqslant 2$.

Proof 1) $\Rightarrow$ 2). We only need to prove the following claim.

Claim For every $\boldsymbol{A} \in T_n(R)$, there exists a unique $\boldsymbol{E} \in T_n(R)$ satisfying the property $P_n(\boldsymbol{A})$:

$$\boldsymbol{E}^2 = \boldsymbol{E}, \quad \boldsymbol{A} - \boldsymbol{E} \in U(T_n(R)), \ \boldsymbol{AE} = \boldsymbol{EA}$$

Notice that $R/J(R) \cong Z_2$. Any idempotent $\boldsymbol{E}$ satisfying the property $P_n(\boldsymbol{A})$ must be of the form $E_{ii} = 0$ whenever $A_{ii} \in U(R)$ and $E_{ii} = 1$ whenever $A_{ii} \in J(R)$.

If $\boldsymbol{A}$ is a unit or $\boldsymbol{A} \in J(T_n(R))$, then we take $\boldsymbol{E} = \boldsymbol{0}$ and $\boldsymbol{E} = \boldsymbol{I}_n$, respectively. Hence, we only need to consider $\boldsymbol{A} \notin J(T_n(R))$ and $\boldsymbol{A}$ is not invertible. The claim will be proved by induction on n.

By Lemma 3 and Corollary 1, we can assume that $n \geqslant 3$. Let

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & \boldsymbol{\alpha}_1 & a_{1n} \\ \boldsymbol{0} & \boldsymbol{A}_2 & \bar{\boldsymbol{\alpha}}_1 \\ 0 & \boldsymbol{0} & a_{nn} \end{bmatrix} \in T_n(R)$$

where $\boldsymbol{A}_2 \in T_{n-2}(R)$, $\boldsymbol{\alpha}_1 = \{a_{12}, \ldots, a_{1,n-1}\} \in R^{n-2}$ and $\bar{\boldsymbol{\alpha}}_1 = \{a_{2n}, \ldots, a_{n-1,n}\}^{\mathrm{T}} \in R_{n-2}$.

Write

$$\boldsymbol{A}_1 = \begin{bmatrix} a_{11} & \boldsymbol{\alpha}_1 \\ \boldsymbol{0} & \boldsymbol{A}_2 \end{bmatrix} \in T_{n-1}(R), \quad \boldsymbol{A}_1' = \begin{bmatrix} \boldsymbol{A}_2 & \bar{\boldsymbol{\alpha}}_1 \\ \boldsymbol{0} & a_{nn} \end{bmatrix} \in T_{n-1}(R)$$

By induction hypothesis, there exists $\boldsymbol{E}_1 = \begin{bmatrix} e_{11} & \boldsymbol{e}_1 \\ \boldsymbol{0} & \boldsymbol{E}_2 \end{bmatrix} \in T_{n-1}(R)$ satisfying the property $P_{n-1}(\boldsymbol{A}_1)$ where $\boldsymbol{E}_2 \in T_{n-2}(R)$ and $\boldsymbol{e}_1 \in R^{n-2}$, so we have $\boldsymbol{E}_2^2 = \boldsymbol{E}_2$, $\boldsymbol{A}_2\boldsymbol{E}_2 = \boldsymbol{E}_2\boldsymbol{A}_2$ and $\boldsymbol{A}_2 - \boldsymbol{E}_2 \in U(T_{n-2}(R))$. Similarly, there exists $\boldsymbol{E}_1' = \begin{bmatrix} \boldsymbol{E}_2' & \bar{\boldsymbol{e}}_1 \\ \boldsymbol{0} & e_{nn} \end{bmatrix} \in T_{n-1}(R)$ satisfying the property $P_{n-1}(\boldsymbol{A}_1')$ where $\boldsymbol{E}_2'^2 \in T_{n-2}(R)$ and $\bar{\boldsymbol{e}}_1 \in R_{n-2}$, and it follows that $\boldsymbol{E}_2'^2 = \boldsymbol{E}_2'$, $\boldsymbol{A}_2\boldsymbol{E}_2'^2 = \boldsymbol{E}_2'\boldsymbol{A}_2$ and $\boldsymbol{A}_2 - \boldsymbol{E}_2' \in U(T_{n-2}(R))$. Nevertheless, the induction hypothesis implies that there exists a unique $\boldsymbol{E}_0 \in T_{n-2}(R)$ such that $\boldsymbol{E}_0^2 = \boldsymbol{E}_0$, $\boldsymbol{A}_2 - \boldsymbol{E}_0 \in U(T_{n-2}(R))$ and $\boldsymbol{A}_2\boldsymbol{E}_0 = \boldsymbol{E}_0\boldsymbol{A}_2$. Thus, $\boldsymbol{E}_2 = \boldsymbol{E}_2' = \boldsymbol{E}_0$.

Now, write $\boldsymbol{E}_1 = \begin{bmatrix} e_{11} & \boldsymbol{e}_1 & e_{1n} \\ \boldsymbol{0} & \boldsymbol{E}_2 & \bar{\boldsymbol{e}}_1 \\ 0 & \boldsymbol{0} & e_{nn} \end{bmatrix} \in T_n(R)$. The remaining proof is to show that $\boldsymbol{A}$ has a uniquely strongly clean expression in $T_n(R)$. Indeed, it suffices to prove that there exists a unique element $e_{12} \in R$ such that E satisfies the property $P_n(\boldsymbol{A})$.

From $\boldsymbol{E}_1\boldsymbol{A}_1 = \boldsymbol{A}_1\boldsymbol{E}_1$, one obtains

$$e_{11}\boldsymbol{\alpha}_1 + \boldsymbol{e}_1\boldsymbol{A}_2 = a_{11}\boldsymbol{e}_1 + \boldsymbol{\alpha}_1\boldsymbol{E}_2 \tag{1}$$

And it follows from $\boldsymbol{E}_1'\boldsymbol{A}_1' = \boldsymbol{A}_1'\boldsymbol{E}_1'$ that

$$\boldsymbol{E}_2\bar{\boldsymbol{\alpha}}_1 + \bar{\boldsymbol{e}}_1 a_{nn} = \boldsymbol{A}_2\bar{\boldsymbol{e}}_1 + \bar{\boldsymbol{\alpha}}_1 e_{nn} \tag{2}$$

Multiplying Eq. (1) on the right by $\bar{\boldsymbol{e}}_1$ yields

$$e_{11}\boldsymbol{\alpha}_1\bar{\boldsymbol{e}}_1 + \boldsymbol{e}_1\boldsymbol{A}_2\bar{\boldsymbol{e}}_1 = a_{11}\boldsymbol{e}_1\bar{\boldsymbol{e}}_1 + \boldsymbol{\alpha}_1\boldsymbol{E}_2\bar{\boldsymbol{e}}_1 \tag{3}$$

and multiplying Eq. (2) on the left by $\boldsymbol{e}_1$, we have

$$\boldsymbol{e}_1\boldsymbol{E}_2\bar{\boldsymbol{\alpha}}_1 + \boldsymbol{e}_1\bar{\boldsymbol{e}}_1 a_{nn} = \boldsymbol{e}_1\boldsymbol{A}_2\bar{\boldsymbol{e}}_1 + \boldsymbol{e}_1\bar{\boldsymbol{\alpha}}_1 e_{nn} \tag{4}$$

By Lemma 2 and the condition $R/J(R) \cong Z_2$, we prove the result in two cases.

Case 1 a_{11}, $a_{nn} \in U(R)$. Then $e_{11} = e_{nn} = 0$. Let $e_{1n} = \boldsymbol{e}_1\bar{\boldsymbol{e}}_1$. Then $\boldsymbol{E} = \begin{bmatrix} 0 & \boldsymbol{e}_1 & \boldsymbol{e}_1\bar{\boldsymbol{e}}_1 \\ \boldsymbol{0} & \boldsymbol{E}_2 & \bar{\boldsymbol{e}}_1 \\ 0 & \boldsymbol{0} & 0 \end{bmatrix}$ is an idempotent of $T_n(R)$. Clearly, $\boldsymbol{A} - \boldsymbol{E} \in U(T_n(R))$. Since $\boldsymbol{E}_1^2 = \boldsymbol{E}_1$ and $(\boldsymbol{E}_1')^2 = \boldsymbol{E}_1'$, we obtain $\boldsymbol{e}_1 = \boldsymbol{e}_1\boldsymbol{E}_2$ and $\bar{\boldsymbol{e}}_1 = \boldsymbol{E}_2\bar{\boldsymbol{e}}_1$. As $e_{11} = e_{nn} = 0$, Eq. (3) becomes

$$\boldsymbol{e}_1\boldsymbol{A}_2\bar{\boldsymbol{e}}_1 = a_{11}\boldsymbol{e}_1\bar{\boldsymbol{e}}_1 + \boldsymbol{\alpha}_1\boldsymbol{E}_2\bar{\boldsymbol{e}}_1 = a_{11}\boldsymbol{e}_1\bar{\boldsymbol{e}}_1 + \boldsymbol{\alpha}_1\bar{\boldsymbol{e}}_1$$

and by Eq. (4) one has

$$\boldsymbol{e}_1\bar{\boldsymbol{\alpha}}_1 + \boldsymbol{e}_1\bar{\boldsymbol{e}}_1 a_{nn} = \boldsymbol{e}_1\boldsymbol{E}_2\bar{\boldsymbol{\alpha}}_1 + \boldsymbol{e}_1\bar{\boldsymbol{e}}_1 a_{nn} = \boldsymbol{e}_1\boldsymbol{A}_2\bar{\boldsymbol{e}}_1$$

Thus, $e_{11}\boldsymbol{\alpha}_1\bar{\boldsymbol{e}}_1 + \boldsymbol{\alpha}_1\bar{\boldsymbol{e}}_1 = \boldsymbol{e}_1\bar{\boldsymbol{\alpha}}_1 + \boldsymbol{e}_1\bar{\boldsymbol{e}}_1 a_{nn}$, implying $\boldsymbol{AE} = \boldsymbol{EA}$. The uniqueness of E is obvious.

Case 2 $a_{11} \in U(R)$, $a_{nn} \in J(R)$. Then $e_{11} = 0$, $e_{nn} = 1$ and $\boldsymbol{E} = \begin{bmatrix} 0 & \boldsymbol{e}_1 & e_{1n} \\ \boldsymbol{0} & \boldsymbol{E}_2 & \bar{\boldsymbol{e}}_1 \\ 0 & \boldsymbol{0} & 1 \end{bmatrix}$. Since $\boldsymbol{E}_1^2 = \boldsymbol{E}_1$ and $(\boldsymbol{E}_1')^2 = \boldsymbol{E}_1'$, it follows that $\boldsymbol{e}_1 = \boldsymbol{e}_1\boldsymbol{E}_2$ and $\boldsymbol{E}_2\bar{\boldsymbol{e}}_1 = \boldsymbol{0}$. Combining Eq. (3) with (4), we obtain

$$a_{11}\boldsymbol{e}_1\bar{\boldsymbol{e}}_1 = \boldsymbol{e}_1\boldsymbol{A}_2\bar{\boldsymbol{e}}_1 = \boldsymbol{e}_1\bar{\boldsymbol{e}}_1 a_{nn}$$

Thus, $a_{11}\boldsymbol{e}_1\bar{\boldsymbol{e}}_1 - \boldsymbol{e}_1\bar{\boldsymbol{e}}_1 a_{nn} = 0$. Since $l_{a_{11}} - r_{a_{nn}}$ is injective by hypothesis, $\boldsymbol{e}_1\bar{\boldsymbol{e}}_1 = 0$. It is clear that $\boldsymbol{E}^2 = \boldsymbol{E}$ and $\boldsymbol{A} - \boldsymbol{E} \in U(T_n(R))$. Now, to show that A is USC, we only need to prove that such idempotents satisfying $\boldsymbol{AE} = \boldsymbol{EA}$ are unique. Since $l_{a_{11}} - r_{a_{nn}}$ is isomorphic, there exists a unique $e_{1n} \in R$ such that

$$a_{11}e_{1n} - e_{1n}a_{nn} = \boldsymbol{e}_1\bar{\boldsymbol{\alpha}}_1 - \boldsymbol{\alpha}_1\bar{\boldsymbol{e}}_1 - a_{1n}$$

That is, $a_{11}e_{1n} + \boldsymbol{\alpha}_1\bar{\boldsymbol{e}}_1 + a_{1n} = \boldsymbol{e}_1\bar{\boldsymbol{\alpha}}_1 + e_{1n}a_{nn}$. It is equivalent to $\boldsymbol{AE} = \boldsymbol{EA}$, as desired.

2) $\Rightarrow$ 3) is obvious.

3) $\Rightarrow$ 1). Let $\boldsymbol{E} = \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix} \in T_n(R)$. Then $T_2(R) \cong \boldsymbol{E}T_n(R)\boldsymbol{E}$ is USC by Example 5 in Ref. [10]. By Corollary 1, we are done.

Corollary 2 If R is a commutative local ring and $R/J(R) \cong Z_2$, then $T_n(R)$ is USC for any $n \geqslant 1$.

Proof By Example 13 in Ref. [2], commutative local rings are uniquely bleached. So the result follows from Theorem 1.

Remark Note that the condition $R/J(R) \cong Z_2$ in Corollary 2 is necessary. Let $R = Z_{(3)}$ be the localization of Z at 3, and let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in T_2(R)$. Since $\boldsymbol{A} = \boldsymbol{0} + \boldsymbol{A}$ and $\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \boldsymbol{I}_2$ are two strongly clean expressions in $T_2(R)$, $T_2(R)$ is not USC.

In Ref. [5], Chen et al. introduced the notion of a skew triangular matrix ring over a given ring. For a ring R and an endomorphism α of R with $\alpha(1) = 1$, let $T_n(R, \alpha) = \{(a_{ij})_{n\times n}: a_{ij} \in R \text{ and } a_{ij} = 0 \text{ if } i > j\}$. For (a_{ij}), $(b_{ij}) \in T_n(R, \alpha)$, define $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$ and $(a_{ij}) * (b_{ij}) = (c_{ij})$, where $c_{ij} = 0$ for $i > j$ and $c_{ij} = \sum_{k=i}^{j} a_{ik}\alpha^{k-i}(b_{kj})$, $i \leqslant j$. Then $T_n(R, \alpha)$ is called a skew triangular matrix ring over R.

The proof of Theorem 1 can be slightly modified to show the following theorem.

Theorem 2 Let R be a local ring and α be an endomorphism with $\alpha(J(R)) = J(R)$. Then the following are equivalent.

1) R is uniquely bleached and $R/J(R) \cong Z_2$.

2) $T_n(R, \alpha)$ is USC for any $n \geqslant 1$.

3) $T_n(R, \alpha)$ is USC for some $n \geqslant 2$.

References

[1] Nicholson W K. Strongly clean rings and Fitting's lemma [J]. *Comm Algebra*, 1999, **27**(8): 3583 − 3592.

[2] Borooah G, Diesl A J, Dorsey T J. Strongly clean triangular matrix rings over local rings [J]. *J Algebra*, 2007, **312**(2): 773 − 797.

[3] Borooah G, Diesl A J, Dorsey T J. Strongly clean matrix rings over commutative local rings [J]. *J Pure Appl Algebra*, 2008, **212**(1): 281 − 296.

[4] Chen H Y, Chen M S. On clean ideals [J]. *Internat J Math Sci*, 2003, **62**: 3949 − 3956.

[5] Chen J L, Yang X D, Zhou Y Q. On strongly clean matrix and triangular matrix rings [J]. *Comm Algebra*, 2006, **34**(10): 3659 − 3674.

[6] Chen J L, Yang X D, Zhou Y Q. When is the 2 × 2 matrix ring over a commutative local ring strongly clean? [J]. *J Algebra*, 2006, **301**(1): 280 − 293.

[7] Li Y L. Strongly clean matrix rings over local rings [J]. *J Algebra*, 2007, **312**(1): 397 − 404.

[8] Wang Z, Chen J L. On two open problems about strongly clean rings [J]. *Bull Austral Math Soc*, 2004, **70**(2): 279 − 282.

[9] Yang X D, Zhou Y Q. Strong cleanness of the 2 × 2 matrix ring over a general local ring [J]. *J Algebra*, 2008, **320**(6): 2280 − 2290.

[10] Chen J L, Wang Z, Zhou Y Q. Rings in which elements are uniquely the sum of an idempotent and a unit that commute [J]. *J Pure Appl Algebra*, 2009, **213**(2): 215 − 223.

[11] Nicholson W K, Zhou Y Q. Rings in which elements are uniquely the sum of an idempotent and a unit [J]. *Glasgow Math J*, 2004, **46**(2): 227 − 236.

唯一强 clean 三角矩阵环

崔　建　　陈建龙

(东南大学数学系，南京 211189)

摘要：称一个环 R 中的元素 a 是唯一强 clean 的，如果 a 可以唯一地表示成幂等元和可逆元的和且二者可交换. 称环 R 是唯一强 clean 的，如果 R 中每一个元素都是唯一强 clean 元. 研究了 $n \times n$ 阶三角矩阵环的唯一强 clean 性. 设 R 为局部环，证明了环 R 上的任意 $n \times n$ 阶上三角矩阵环是唯一强 clean 的当且仅当 R 是唯一 bleached 的且 $R/J(R) \cong Z_2$.

关键词：唯一强 clean 环；唯一 bleached 局部环；三角矩阵环

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