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Monotonicity of the tail dependence for multivariate *t*-copula

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Abstract: This paper considers the upper orthant and extremal tail dependence indices for multivariate *t*-copula. Where, the multivariate *t*-copula is defined under a correlation structure. The explicit representations of the tail dependence parameters are deduced since the copula of continuous variables is invariant under strictly increasing transformation about the random variables, which are more simple than those obtained in previous research. Then, the local monotonicity of these indices about the correlation coefficient is discussed, and it is concluded that the upper extremal dependence index increases with the correlation coefficient, but the monotonicity of the upper orthant tail dependence index is complex. Some simulations are performed by the Monte Carlo method to verify the obtained results, which are found to be satisfactory. Meanwhile, it is concluded that the obtained conclusions can be extended to any distribution family in which the generating random variable has a regularly varying distribution.

Key words: multivariate *t*-copula; copula; inverse gamma distribution; monotonicity; regularly varying function; correlation coefficient

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During the last decade, dependencies between financial asset returns have increased due to globalization effects and relaxed market regulation. Tail dependence is described via the so-called tail dependence indices introduced by Joe[1], and it is suitable for a proper understanding of dependencies in financial markets[2–5].

Many researchers use various multivariate distributions with heavy tails to describe the extremal or tail dependence[5–6]. And in particular, the multivariate *t*-distribution and its copula[7] are frequently used in the context of modelling multivariate financial return data[8]. We say an n-dimensional random vector $\boldsymbol{X}$ follows a multivariate *t*-distribution with parameters $\boldsymbol{\mu}_{n\times 1}$, $\boldsymbol{\Sigma}_{n\times n} > 0$ and $\nu > 0$, denoted by $\boldsymbol{X} \sim t_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$, if it has the stochastic representation[9]

$$\boldsymbol{X} = \boldsymbol{\mu} + \sqrt{R}\boldsymbol{Z} \tag{1}$$

where $\boldsymbol{Z} \sim \boldsymbol{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$ is independent of $R \sim \mathrm{IG}(\nu/2, \nu/2)$; $\boldsymbol{\mu}$ is the mean vector of $\boldsymbol{X}$.

Using the copula method, Li[10] derived explicit expressions of the upper and lower orthant tail dependence for the Marshall-Olkin distribution and the multivariate Pareto distribution. Chan et al.[11] derived the formula of tail dependence indices of multivariate *t*-distribution under Eq. (1), and discussed the monotonicity of these indices with respect to σ_{ij}, the element of the covariance matrix $\boldsymbol{\Sigma}$.

Here, suppose $\boldsymbol{\Sigma} = (\sigma_{ij})$, $\boldsymbol{\Delta} = \mathrm{diag}(\sigma_{11}, \sigma_{22}, \dots, \sigma_{nn})$, then $\boldsymbol{\Sigma} = \boldsymbol{\Delta}^{1/2}\boldsymbol{\rho}\boldsymbol{\Delta}^{1/2}$. Therefore, $\boldsymbol{\rho}$ is the correlation matrix of $\boldsymbol{Z}$. Correspondingly, Eq. (1) becomes

$$\boldsymbol{X} = \boldsymbol{\mu} + \sqrt{R}\boldsymbol{\Delta}^{1/2}\boldsymbol{Z} \tag{2}$$

which is denoted by $\boldsymbol{X} \sim t_n(\boldsymbol{\mu}, \boldsymbol{\rho}, \nu)$. In the following, we will discuss the tail dependence indices of $t_n(\boldsymbol{\mu}, \boldsymbol{\rho}, \nu)$ and the properties of the dependence indices with respect to the correlation coefficient.

1 Tail Dependence and Regular Variation

According to Ref. [2], tail dependence plays an important role in extremal value theory, finance and insurance models. The tail dependence of a bivariate distribution has been discussed extensively in the statistics literature[1,6]. The general case for multivariate distributions is mostly related to their bivariate marginal distributions[5,10–13].

Definition 1 Let $\boldsymbol{X} = \{X_1, X_2, \dots, X_n\}^{\mathrm{T}}$ be a random vector with continuous marginal $F_1, F_2, \dots, F_n$ and copula C.

1) $\boldsymbol{X}$ is said to be upper-orthant tail dependent if for some nonempty set $J \subset \{1, 2, \dots, n\}$, the following limit exists and is positive.

$$\tau_J = \lim_{u\to 1^-} \Pr\{F_j(X_j) > u,\ \forall\, j \notin J \mid F_i(X_i) > u,\ \forall\, i \in J\} \tag{3}$$

If for all nonempty sets $J \subset \{1, 2, \dots, n\}$, $\tau_J = 0$, then we say $\boldsymbol{X}$ is upper-orthant tail independent.

2) $\boldsymbol{X}$ is said to be upper extremal dependent if the following limit exists and is positive.

$$\gamma = \lim_{u\to 1^-} \Pr\{F_j(X_j) > u,\ \forall\, j \in \{1, 2, \dots, n\} \mid F_i(X_i) > u,\ \exists\, i \in \{1, 2, \dots, n\}\} \tag{4}$$

If $\gamma = 0$, then we say $\boldsymbol{X}$ is upper extremal independent.

Remark 1 The tail dependence describes the conditional probability of joint exceedance over a large threshold when some components already exceed that threshold.

Remark 2 Similarly, one may define the lower tail dependence coefficient.

Remark 3 For any nonempty subset $J \subset \{1, 2, \dots, n\}$ and any random vector, we can obtain $\tau_J \geqslant \gamma$ immediately from the definition. This means that the lower bound of the upper-orthant tail dependence is the extremal dependence index.

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Lemma 1[11] Let R be the inverse gamma distribution $R \sim \mathrm{IG}(\nu/2, \nu/2)$. Then its survival function $\overline{G}(r) = L(r)/r^{\nu/2}$, and it is regularly varying with index $\nu/2$, where $L(r)$ is a slowly varying function.

2 Tail Dependence for Multivariate t-Copula

In this section, we investigate the tail dependence indices for $t_n(\boldsymbol{\mu}, \boldsymbol{\rho}, \nu)$ under Eq. (2). Since the upper tail dependence only depends on the tail behavior of the random variables, we only need to focus on $t_n(\mathbf{0}, \boldsymbol{\rho}, \nu)$. From Eq. (2), $\boldsymbol{X} = \sqrt{R}\boldsymbol{\Delta}^{1/2}\boldsymbol{Y}$, so $\boldsymbol{\Delta}^{-1/2}\boldsymbol{X} = \sqrt{R}\boldsymbol{Y}$. Note that $\boldsymbol{\Delta}^{-1/2}\boldsymbol{X} = \{\sigma_{11}^{-1/2}X_1, \sigma_{22}^{-1/2}X_2, \dots, \sigma_{nn}^{-1/2}X_n\}^{\mathrm{T}}$ is a separable, strictly increasing transform of $\{X_1, X_2, \dots, X_n\}^{\mathrm{T}}$, so these two random vectors have exactly the same copula and thus the same tail dependence indices[7]. On the other hand, for any $1 \leqslant i \leqslant n$, $\sigma_{ii}^{-1/2}X_i$ has the same distribution, say H, so we have

$$\begin{aligned}\tau_J &= \lim_{u\to 1^-}\Pr\{F_j(X_j) > u, \forall j \notin J \mid F_i(X_i) > u, \forall i \in J\} = \\ &\lim_{u\to 1^-}\Pr\{H(\sigma_{jj}^{-1/2}X_j) > u, \forall j \notin J \mid H(\sigma_{ii}^{-1/2}X_i) > u, \forall i \in J\} = \\ &\lim_{t\to\infty}\Pr\{\sigma_{jj}^{-1/2}X_j > t, \forall j \notin J \mid \sigma_{ii}^{-1/2}X_i > t, \forall i \in J\} = \\ &\lim_{t\to\infty}\Pr\{\sqrt{R}Y_j > t, \forall j \notin J \mid \sqrt{R}Y_i > t, \forall i \in J\} = \\ &\lim_{t\to\infty}\frac{\Pr\{R > (t/\overline{Y}_j)^2, \forall j \in \{1, 2, \dots, n\}\}}{\Pr\{R > (t/\overline{Y}_i)^2, \forall i \in J\}}\end{aligned} \tag{5}$$

where $\overline{Y}_i = Y_i \vee 0$, and $x \vee y$ is the maximum of x and y. Similarly, from Eq. (4) we obtain

$$\gamma = \lim_{t\to\infty}\frac{\Pr\{R > (t/\overline{Y}_j)^2, \forall j \in \{1, 2, \dots, n\}\}}{\Pr\{R > (t/\overline{Y}_i)^2, \exists i \in \{1, 2, \dots, n\}\}} \tag{6}$$

Let $\phi(\boldsymbol{y}, \boldsymbol{\rho})$ and $\Phi(\boldsymbol{y}, \boldsymbol{\rho})$ be the density and the cumulative distribution of the multivariate normal distribution $N_n(\mathbf{0}, \boldsymbol{\rho})$, respectively. Then we can obtain the expressions of the tail dependence indices in the following theorem.

Theorem 1 Let $\boldsymbol{X}$ be a multivariate t-distribution $t_n(\mathbf{0}, \boldsymbol{\rho}, \nu)$ as that in Eq. (2), then

1) For any nonempty set $J \subset \{1, 2, \dots, n\}$, the upper-orthant tail dependence indices are given by

$$\tau_J = \frac{E(\bigwedge_{j=1}^{n}\overline{Y}_j)^{\nu}}{E(\bigwedge_{i\in J}\overline{Y}_i)^{\nu}} \tag{7}$$

2) The upper extremal dependence index is given by

$$\gamma = \frac{E(\bigwedge_{j=1}^{n}\overline{Y}_j)^{\nu}}{E(\bigvee_{i=1}^{n}\overline{Y}_i)^{\nu}} \tag{8}$$

Proof For $R \sim \mathrm{IG}(\nu/2, \nu/2)$, according to Lemma 1,

$$\begin{aligned}\Pr\{R > t^2/(\bigwedge_{j=1}^{n}\overline{Y}_j)^2\} &= \overline{G}(t^2/(\bigwedge_{j=1}^{n}\overline{Y}_j)^2) = \\ &L(t^2/(\bigwedge_{j=1}^{n}\overline{Y}_j)^2)(\bigwedge_{j=1}^{n}\overline{Y}_j)^{\nu}/t^{\nu}\end{aligned}$$

where $L(\cdot)$ is a bounded function over $(0, \infty)$. Furthermore, following the dominated convergence theorem, from Eqs. (3) and (5), we have

$$\begin{aligned}\tau_J &= \lim_{t\to\infty}\frac{\Pr\{R > \bigvee_{j=1}^{n}t^2/\overline{Y}_j^2\}}{\Pr\{R > \bigvee_{i\in J}t^2/\overline{Y}_i^2\}} = \lim_{t\to\infty}\frac{\Pr\{R > t^2/\bigwedge_{j=1}^{n}\overline{Y}_j^2\}}{\Pr\{R > t^2/\bigwedge_{i\in J}\overline{Y}_i^2\}} = \\ &\lim_{t\to\infty}\frac{\int_{y_1>0,\dots,y_n>0}\Pr\{R > t^2/\bigwedge_{j=1}^{n}y_j^2\}\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{y}, \boldsymbol{\rho})}{\int_{y_1>0,\dots,y_n>0}\Pr\{R > t^2/\bigwedge_{i\in J}y_i^2\}\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{y}, \boldsymbol{\rho})} = \\ &\lim_{t\to\infty}\frac{\int_{y_1>0,\dots,y_n>0}L(t^2/\bigwedge_{j=1}^{n}y_j^2)(\bigwedge_{j=1}^{n}y_j)^{\nu}\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{y}, \boldsymbol{\rho})}{\int_{y_1>0,\dots,y_n>0}L(t^2/\bigwedge_{i\in J}y_i^2)(\bigwedge_{i\in J}y_i)^{\nu}\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{y}, \boldsymbol{\rho})} = \\ &\lim_{t\to\infty}\frac{\int_{y_1>0,\dots,y_n>0}(\bigwedge_{j=1}^{n}y_j)^{\nu}\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{y}, \boldsymbol{\rho})}{\int_{y_1>0,\dots,y_n>0}(\bigwedge_{i\in J}y_i)^{\nu}\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{y}, \boldsymbol{\rho})} = \frac{E(\bigwedge_{j=1}^{n}\overline{Y}_j)^{\nu}}{E(\bigwedge_{i\in J}\overline{Y}_i)^{\nu}}\end{aligned}$$

Similarly, from Eqs. (4) and (6), we have Eq. (8).

From the expressions of the tail dependence indices of the multivariate t-copula given in Eqs. (7) and (8), we can see that they are more concise than those obtained by Chan et al[11].

3 Monotonicity of the Multivariate t-Copula's Tail Dependence Indices

In this section, we consider the monotonicity of the tail dependence indices about the multivariate t-copula. We discuss the local monotonicity proposition of τ_J and γ about the correlation coefficient. First, we introduce two lemmas[14].

Lemma 2 Let $\{Y_1, Y_2, \dots, Y_n\}^{\mathrm{T}}$ and $\{\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_n\}^{\mathrm{T}}$ be normally distributed with distributions $N_n(\mathbf{0}, \boldsymbol{\rho})$ and $N_n(\mathbf{0}, \tilde{\boldsymbol{\rho}})$, respectively. $\boldsymbol{\rho} > \tilde{\boldsymbol{\rho}}$ entry-wise, then for any $t_1, t_2, \dots, t_n \in R$,

$$\begin{aligned}\Pr\{Y_1 \geqslant t_1, \dots, Y_n \geqslant t_n\} &\geqslant \Pr\{\tilde{Y}_1 \geqslant t_1, \dots, \tilde{Y}_n \geqslant t_n\} \\ \Pr\{Y_1 \leqslant t_1, \dots, Y_n \leqslant t_n\} &\geqslant \Pr\{\tilde{Y}_1 \leqslant t_1, \dots, \tilde{Y}_n \leqslant t_n\}\end{aligned}$$

For nonempty set $J \subset \{1, 2, \dots, n\}$, let $\boldsymbol{Y}_J = (Y_j, j \in J)$, then $\boldsymbol{Y}_J \sim N_{|J|}(\mathbf{0}, \boldsymbol{\rho}_J)$ with the density function $\phi_J(\boldsymbol{y}_J, \boldsymbol{\rho}_J)$. $\boldsymbol{K} = \boldsymbol{\rho}^{-1} = (k_{ij})$, and $\boldsymbol{K}_J = \boldsymbol{\rho}_J^{-1} = (k_{ij}^J)$. Then the following lemma (Plackett's Lemma)[14] is determined.

Lemma 3 For any $i \neq j \in J$,

$$\frac{\partial\phi(\boldsymbol{y}, \boldsymbol{\rho})}{\partial\rho_{ij}} = \frac{\partial^2\phi(\boldsymbol{y}, \boldsymbol{\rho})}{\partial y_i\partial y_j} = -k_{ij}\phi(\boldsymbol{y}, \boldsymbol{\rho})$$

and

$$\frac{\partial\phi_J(\boldsymbol{y}_J, \boldsymbol{\rho}_J)}{\partial\rho_{ij}} = \frac{\partial^2\phi_J(\boldsymbol{y}_J, \boldsymbol{\rho}_J)}{\partial y_i\partial y_j} = -k_{ij}^J\phi_J(\boldsymbol{y}_J, \boldsymbol{\rho}_J)$$

Theorem 2 Let $\boldsymbol{X} \sim t_n(\mathbf{0}, \boldsymbol{\rho}, \nu)$ with the stochastic representation as Eq. (2). Then the upper extremal dependence index γ increases monotonically in ρ_{ij}.

Proof For conciseness, let

$$A_{\nu} = E(\bigwedge_{j=1}^{n}\overline{Y}_j)^{\nu},\ B_{\nu} = E(\bigwedge_{i\in J}\overline{Y}_i)^{\nu},\ C_{\nu} = E(\bigvee_{j=1}^{n}\overline{Y}_j)^{\nu}$$

Then

$$\tau_J = \frac{A_{\nu}}{B_{\nu}},\quad \gamma = \frac{A_{\nu}}{C_{\nu}} \tag{9}$$

and

$$A_\nu = \int_0^\infty \Pr\{(\bigwedge_{j=1}^n \bar{Y}_j)^\nu > t\}\mathrm{d}t = \int_0^\infty \Pr\{Y_1 > t^{1/\nu}, \ldots, Y_n > t^{1/\nu}\}\mathrm{d}t$$

$$C_\nu = \int_0^\infty (1 - \Pr\{Y_1 \leqslant t^{1/\nu}, \ldots, Y_n \leqslant t^{1/\nu}\})\mathrm{d}t$$

According to Lemma 2, A_ν increases monotonically in ρ_{ij}, and C_ν decreases monotonically in ρ_{ij}. These lead to $\gamma = A_\nu / C_\nu$ increasing monotonically in ρ_{ij}.

Theorem 3 Under the same assumptions as in Theorem 2, we have

1) If $i \notin J$ or $j \notin J$, then τ_J increases monotonically in ρ_{ij}.

2) If $i, j \in J$ and $i \neq j$, then τ_J decreases monotonically in ρ_{ij} if and only if $k_{ij} > k_{ij}^J$.

Proof 1) For $i \notin J$ or $j \notin J$, similar to Theorem 2, we have $B_\nu = \int_0^\infty \Pr\{\bigcap_{i \in J}(Y_i > t^{1/\nu})\}\mathrm{d}t$, which means that B_ν is invariant when ρ_{ij} increases. On the other hand, we already know that A_ν increases monotonically in ρ_{ij}. So τ_J increases monotonically in ρ_{ij}.

2) For $i, j \in J$ and $i \neq j$, from Eq. (9), we have

$$\frac{\partial \tau_J}{\partial \rho_{ij}} = \frac{1}{B_\nu^2}\left(B_\nu \frac{\partial A_\nu}{\partial \rho_{ij}} - A_\nu \frac{\partial B_\nu}{\partial \rho_{ij}}\right)$$

and

$$A_\nu = \int_{y_1 > 0, \ldots, y_n > 0} (\bigwedge_{j=1}^n y_j)^\nu \mathrm{d}\boldsymbol{\Phi}(\boldsymbol{y}, \boldsymbol{\rho})$$

Using Lemma 3,

$$\frac{\partial A_\nu}{\partial \rho_{ij}} = \int_{y_1 > 0, \ldots, y_n > 0} (\bigwedge_{j=1}^n y_j)^\nu \frac{\partial \phi(\boldsymbol{y}, \boldsymbol{\rho})}{\partial \rho_{ij}} \mathrm{d}\boldsymbol{y} = \int_{y_1 > 0, \ldots, y_n > 0} (\bigwedge_{j=1}^n y_j)^\nu (-k_{ij}\phi(\boldsymbol{y}, \boldsymbol{\rho}))\mathrm{d}\boldsymbol{y} = -k_{ij}A_\nu$$

Similarly, $\frac{\partial B_\nu}{\partial \rho_{ij}} = -k_{ij}^J B_\nu$. So, we obtain

$$\frac{\partial \tau_J}{\partial \rho_{ij}} = \frac{1}{B_\nu^2}(B_\nu(-k_{ij}A_\nu) - A_\nu(-k_{ij}^J B_\nu)) = \frac{(-k_{ij} + k_{ij}^J)A_\nu B_\nu}{B_\nu^2}$$

Furthermore, we obtain

$$\frac{\partial \tau_J}{\partial \rho_{ij}} < 0 \Leftrightarrow -k_{ij} + k_{ij}^J < 0 \Leftrightarrow k_{ij} > k_{ij}^J$$

The proof is completed.

These two theorems show the local monotonicity propositions of the tail dependence indices τ_J and γ about the correlation coefficient of the normal distribution.

4 Numerical Simulations

In Section 2, we obtained formulae of the tail dependence indices for the multivariate t-copula under Eq. (2). Then in Section 3 we discussed the monotonicity of the two indices about the correlation coefficient. Now, we illustrate these monotonicities by some Monte Carlo simulation examples. Let $y^{(1)}, y^{(2)}, \ldots, y^{(m)}$ be a sample generated from the multivariate normal distribution $N_n(\mathbf{0}, \boldsymbol{\rho})$. Then the tail dependence indices of $t_n(\mathbf{0}, \boldsymbol{\rho}, \nu)$ can be estimated by

$$\hat{\tau}_J = \frac{\sum_{k=1}^m (\bigwedge_{j=1}^n |y_j^{(k)}|)^\nu I\{y^{(k)} > 0 \text{ or } y^{(k)} < 0\}}{\sum_{k=1}^m (\bigwedge_{i \in J} |y_i^{(k)}|)^\nu I\{y^{(k)} > 0 \text{ or } y^{(k)} < 0\}} \tag{10}$$

$$\hat{\gamma} = \frac{\sum_{k=1}^m (\bigwedge_{j=1}^n |y_j^{(k)}|)^\nu I\{y^{(k)} > 0 \text{ or } y^{(k)} < 0\}}{\sum_{k=1}^m (\bigvee_{j=1}^n |y_j^{(k)}|)^\nu I\{y^{(k)} > 0 \text{ or } y^{(k)} < 0\}} \tag{11}$$

Here we select a 3-dimensional t-distribution to estimate τ_J and γ. For each correlation matrix, we generate 8×10^4 pseudorandom vectors from the corresponding normal distribution, and then use Eqs. (10) and (11) to estimate the tail dependence indices for different ν. We do the following two simulations.

1) The estimation of τ_J and γ

We suppose the correlation matrix $\boldsymbol{\rho} = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.8 \\ 0.5 & 0.8 & 1 \end{bmatrix}$, and select $J = \{2\}$ and $J = \{1, 2\}$, respectively. The corresponding tail dependence indices are denoted by τ_2 and τ_{12}. The simulated values of τ_2, τ_{12} and γ about different ν are plotted in Fig. 1, which show that τ_2 and τ_{12} are bounded by γ. And we can see that τ_J and γ decrease and approach 0 quickly as ν increases to infinity, which are the tail dependence indices for multivariate normal-copula.

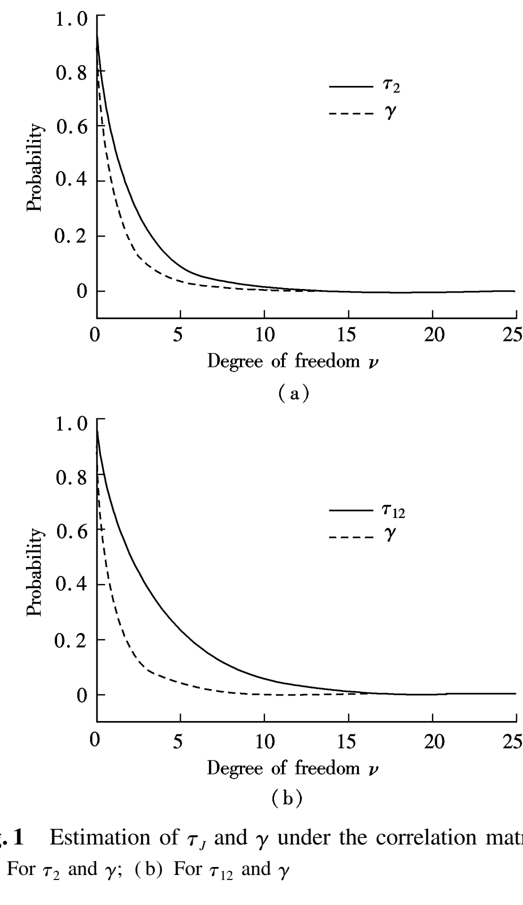

Fig. 1 Estimation of τ_J and γ under the correlation matrix $\boldsymbol{\rho}$. (a) For τ_2 and γ; (b) For τ_{12} and γ

2) The monotonicity of the tail dependence indices

We set the correlation matrices,

$$\boldsymbol{\rho}_1 = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.8 \\ 0.5 & 0.8 & 1 \end{bmatrix}, \quad \boldsymbol{\rho}_2 = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.8 \\ 0.5 & 0.8 & 1 \end{bmatrix}$$

Here ρ_{12} increases from 0 in $\boldsymbol{\rho}_1$ to 0.5 in $\boldsymbol{\rho}_2$, and others are the same.

We first select $J = \{1, 2\}$ and let $i = 1, j = 2$. Obviously, $i, j \in J$ but $i \neq j$. Further computing the inverse matrices $\boldsymbol{K}$ and $\boldsymbol{K}_J$, we obtain $k_{12} = 3.6364$, $k_{12}^J = 0$, so $k_{12} > k_{12}^J$. From 2) of Theorem 3, τ_{12} should decrease locally in ρ_{12}. This is verified by Fig. 2(a).

Then we select $i = 2$, $j = 3$, so $j \notin J$, τ_{23} should decrease locally in ρ_{12} according to 1) of Theorem 3. The simulated values are demonstrated in Fig. 2(b), which are consistent with Theorem 3.

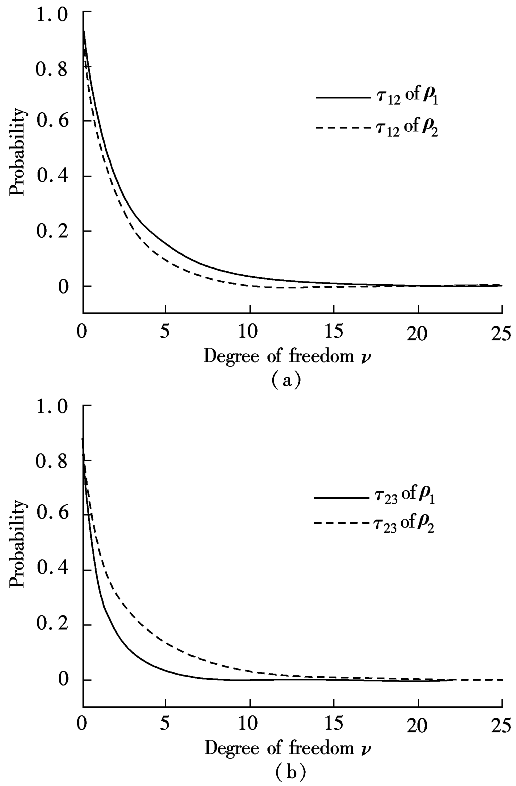

Fig. 2 Monotonicity of the tail dependence indices for $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$. (a) About τ_{12}; (b) About τ_{23}

5 Conclusion

In this paper, we consider the properties of the tail dependence indices introduced by Joe[1] for the multivariate t-copula, where the multivariate t-distribution is represented as the relative structure as Eq. (2). It is known that the copula of continuous variables is invariant under a strictly increasing transformation of the random variables[7]. Following this proposition, we first derive the formulae of tail dependence indices of the multivariate t-copula based on these representations. Then we present the monotonic properties of the indices about the correlation coefficient. The results we obtain here are similar to the Proposition 3.3 in Ref. [11], but our results have a much more clear-cut statistical meaning. In the proof of Theorem 1, we only use the regularly varying quality of the distribution function of the random variable R. Furthermore, the monotonic properties about the correlation coefficient presented in Theorem 1 and Theorem 2 have no relationship with the distribution of random variable R. Therefore, the results obtained here can be extended to any distribution family in which the random variable R has a regularly varying distribution.

All the conclusions are verified by the simulation examples. According to the representation of the tail dependence indices in Eqs. (7) and (8), there perhaps exists some relationship between the indices and the regular variation index of the survival function with an inverse gamma distribution. And according to the simulations, we infer that the tail dependence indices decrease with the regular variation index. We will discuss the correctness of this conjecture in the following work in detail.

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多元 t-copula 尾相依的单调性

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摘要: 考虑了多元 t-copula 的上尾象限相依指数和上尾极值相依指数,该 t-copula 是在相依结构下定义的. 由于多元连续型随机变量的 copula 函数关于严格单调递增变换具有不变性质,由此推导了多元 t-copula 的尾相依指数的精确表达式,得到的结果明显比以往文献给出的结论更加简洁. 然后,讨论了这 2 个相依指数关于相关系数的局部单调性质:上尾极值相依指数关于相关系数是严格单调递增的,但上尾象限相依指数的单调性比较复杂. 通过蒙特卡罗模拟数据验证了结果的正确性. 同时,发现所有结论可以推广到生成随机变量是正则变化的分布类中.

关键词: 多元 t-copula;copula;逆伽玛分布;单调性;正则变化函数;相关系数
中图分类号: O212. 4