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Constructing generalized Drinfel'd quantum double

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Abstract: Let H be a Hopf algebra and B an algebra with two linear maps σ, τ: $H\otimes H\rightarrow B$. The necessary and sufficient conditions for the twisted crossed product $B\#_{\sigma}^{\tau}H$ equipped with the tensor product coalgebra structure to be a bialgebra are proved. Then, $B\#_{\sigma}^{\tau}H$ is a coquasitriangular Hopf algebra under certain conditions. This coquasitriangular Hopf algerbra generalizes some known cross products. Finally, as an application, an explicit example is given.

Key words: twisted crossed product; coquasitriangular Hopf algebra; generalized Drinfel'd quantum double

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In 1993, Majid [1] showed that if (H,β) is a coquasitriangular Hopf algebra, then the bicrossproduct $H\bowtie_{\beta}H$ admits a coquasitriangular Hopf algebra structure given by

$$\beta(a\otimes h, b\otimes g) = \sum \sigma(a, b_1g_1)\sigma^{-1}(b_2g_2, h)$$

for all $a, b, h, g\in H$. This new construction $H\bowtie_{\beta}H$ is called Drinfel'd quantum double. An analogue for the twisted smash products introduced in Ref. [2] was studied by Wang in Ref. [3].

The classical Hopf crossed products were introduced by Blattner et al. [4–5]. In 1999, Kim et al. [6] found the sufficient and necessary conditions for the classical Hopf crossed product with the Hopf crossed coproduct coalgebra structure to be a Hopf algebra, generalizing the results given by Wang et al. [7], which mainly generalizes Radford's biproduct theorem in Ref. [8]. Motivated by the above referenced papers, in this paper we mainly construct a new algebra $B\#_{\sigma}^{\tau}H$ for a left H-weak module algebra B by a Hopf algebra H and study the conditions when $B\#_{\sigma}^{\tau}H$ is a braided Hopf algebra. Our result generalizes the main result in Ref. [3].

1 Definitions and Terminologies

Throughout this paper, K will denote a fixed field. All the algebras, coalgebras, (co)modules, $\otimes$ and Hom are over K. For the basic definitions and facts about coalgebras, Hopf algebras and comodules, we refer to Sweedler's book[9]. In particular, the comultiplication of a coalgebra C is denoted by $\Delta(c) = \sum c_1\otimes c_2$ for all $c\in C$, and the structure map of a left C-comodule V is denoted by $\rho(v) = \sum v_{(-1)}\otimes v_0$ for all $v\in V$.

Let A be an algebra with unit 1_A and C a coalgebra with counit ε_C. Then the convolution algebra $C*A=\mathrm{Hom}(C, A)$ (as vector spaces) has the multiplication (with the unit $1_A\varepsilon_C$) given by $(f*g)(c) = \sum f(c_1)g(c_2)$ for all $f, g\in \mathrm{Hom}(C, A)$ and $c\in C$. Moreover, we say that a linear map $f\in \mathrm{Hom}(C, A)$ is a convolution invertible linear map if there exists a $g\in\mathrm{Hom}(C, A)$ such that $f*g=g*f=1_A\varepsilon_C$. In this case, we say that f is invertible and we call g the convolution inverse of f, denoted by f^{-1}.

Let (B, μ_B, Δ_B) be a bialgebra. Then we denote the twisted product (resp. coproduct) of B by $\mu^{\mathrm{op}}(a\otimes b) = ba$ $\left(\text{resp. } \Delta^{\mathrm{cop}}(b) = \sum b_2\otimes b_1\right)$ for all $a, b\in B$, and denote the resulting bialgebra by B^{op} (resp. B^{cop}). Let S denote the antipode of a Hopf algebra H and, if S is bijective, $\overline{S}$ will denote its composition inverse.

We first recall from Ref. [10] that a coquasitriangular Hopf algebra is a pair (B,σ), where B is a Hopf algebra and σ is a convolution invertible linear map from $B\otimes B$ to K satisfying the following three conditions for all $x, y, z\in B$.

$$\left.\begin{aligned}\sigma(xy, z) &= \sum\sigma(x, z_1)\sigma(y, z_2)\\ \sigma(x, yz) &= \sum\sigma(x_1, z)\sigma(x_2, y)\\ \sum\sigma(x_1, y_1)x_2y_2 &= \sum y_1x_1\sigma(x_2, y_2)\end{aligned}\right\}\qquad(1)$$

Let (B,σ) be coquasitriangular and let σ^{-1} be the convolution inverse of σ. By Ref. [10], we have

$$\left.\begin{aligned}\sigma^{-1}(xy, z) &= \sum\sigma^{-1}(y, z_1)\sigma^{-1}(x, z_2)\\ \sigma^{-1}(x, yz) &= \sum\sigma^{-1}(x_1, y)\sigma^{-1}(x_2, z)\\ \sum\sigma^{-1}(x_1, y_1)y_2x_2 &= \sum x_1y_1\sigma^{-1}(x_2, y_2)\end{aligned}\right\}\qquad(2)$$

for all $x, y, z\in B$. It is a consequence of the above that $\sigma^{-1}(x, y)=\sigma(S(x), y)=\sigma(x, \overline{S}(y))$ and $\sigma(1, x)=\varepsilon(x)=\sigma(x, 1)$ for all $x, y\in B$. Moreover, if (B,σ) is coquasitriangular, then $(B^{\mathrm{op}}, \sigma^{-1})$ and $(B^{\mathrm{cop}}, \sigma^{-1})$ are also coquasitriangular.

Definition 1 Let B, H be Hopf algebras. A linear

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map $u: B\otimes H\to K$ is called a skew pairing if the following hold for all $a, b\in B, h, l\in H$.

$$\left.\begin{aligned} u(ab,h) &= \sum u(a,h_1)u(b,h_2)\\ u(a,hl) &= \sum u(a_1,l)u(a_2,h)\\ u(a,1) &= \varepsilon(a), \quad u(1,h) = \varepsilon(h)\end{aligned}\right\}\tag{3}$$

Any skew pairing u is invertible with the convolution inverse given by $u^{-1}(a,h)=u(S(a),h)$ for all $a\in B, h\in H$.

Given a skew pairing u on a pair of Hopf algebras (B, H), one may construct a new Hopf algebra, the quantum double $B\bowtie_u H$ or just $B\bowtie H$. As a coalgebra, the double is isomorphic to $B\otimes H$. The multiplication is given by the rule

$$(1\otimes h)(a\otimes 1) = \sum u(a_1,h_1)(a_2\otimes h_2)u^{-1}(a_3,h_3) \tag{4}$$

Let B and H be Hopf algebras. Let $p: B\otimes B\to K$, $\beta: H\otimes H\to K$, $\sigma: H\otimes H\to B$, $\tau: H\otimes H\to B$ and $u: B\otimes H\to K$ be some convolution invertible linear maps and let $v: H\otimes B\to K$ be a convolution invertible linear map. Then we have the following definitions.

Definition 2 (H,β) is called a quasi-braided Hopf algebra associated to parameters (σ, u, v) if the following conditions hold,

$$\left.\begin{aligned} &\sum \beta(h,l_1)\beta(g,l_2) =\\ &\qquad\sum u(\sigma(h_1,g_1)\tau(h_3,g_3),l_1)\beta(h_2g_2,l_2)\\ &\sum \beta(h_1,l)\beta(h_2,g) =\\ &\qquad\sum \tau(h_1,g_2l_2)v(h_2,\sigma(g_1,l_1)\tau(g_3,l_3))\\ &\sum \beta(h_1,g_1)(\sigma(h_2,g_2)\tau(h_4,g_4)\otimes h_3g_3) =\\ &\qquad\sum (\sigma(g_1,h_1)\tau(g_3,h_3)\otimes g_2h_2)\beta(h_4,g_4)\end{aligned}\right\}\tag{5}$$

for all $h, g, l\in H$.

Definition 3 The triple (B, H, u) is called a skew braided pairing associated to parameters (σ, τ, p) if the following conditions hold,

$$\left.\begin{aligned} &u(ab,z) = \sum u(a,z_1)u(b,z_2)\\ &\sum u(a_1,g_2l_2)p(a_2,\sigma(g_1,l_1)\tau(g_3,l_3)) = \sum u(a_1,l)u(a_2,g)\\ &\sum u(a_1,g_1)(a_2\otimes g_2) = \sum (g_1a_1\otimes g_2)u(a_2,g_3)\end{aligned}\right\}\tag{6}$$

for all $g, l, z\in H, a, b\in B$.

Definition 4 We call (H, B, v) a skew opposite braided pairing associated to parameters (σ, τ, p) if the following conditions hold,

$$\left.\begin{aligned} &v(h,bc) = \sum v(h_1,c)v(h_2,b)\\ &\sum p(\sigma(h_1,g_1)\tau(h_3,g_3),c_1)v(h_2g_2,c_2) = \sum v(h,c_1)v(g,c_2)\\ &\sum v(h_1,b_1)(h_2 b_2\otimes h_3) = \sum (b_1\otimes h_1)v(h_2,b_2)\end{aligned}\right\}\tag{7}$$

for all $h, g\in H, b, c\in B$.

Remark 1 Let $v=u^{-1}T$, where T is the flip map. Then u satisfies (6) if and only if v satisfies (7).

2 A Coquasitriangular Structure on $B\#_\sigma^\tau H$

In this section, we will study the necessary and sufficient conditions for $B\#_\sigma^\tau H$ to be a coquasitriangular Hopf algebra, generalizing the main results in Ref. [3].

Let H act weakly on B [4]. This means that Hopf algebra H measures algebra B, i. e, there is a linear map $H\otimes B\to B$, given by $h\otimes a\mapsto ha$, such that $h1=\varepsilon(h)1$ and $h(ab) = \sum (h_1a)(h_2b)$, for all $h\in H, a, b\in B$. Let $\sigma: H\otimes H\to B$ and $\tau: H\otimes H\to B$ be two linear maps. If we define $F: H\otimes H\to B\otimes H$ and $X: H\otimes B\to B\otimes H$ by $F(h\otimes k) = \sum \sigma(h_1,k_1)\tau(h_3,k_3)\otimes h_2k_2$ and $X(h\otimes a) = \sum h_1a\otimes h_2$, for $h, k\in H, a\in B$, as in Ref. [11] respectively, then we obtain a multiplication

$$(a\otimes h)(b\otimes k) = \sum a(h_1b)\sigma(h_2,k_1)\tau(h_4,k_3)\otimes h_3k_2$$

on $B\otimes H$ for all $a, b\in B$, $h, k\in H$ and denote the resulting (possibly non-associative) algebra by $B\#_\sigma^\tau H$.

If $B\#_\sigma^\tau H$ is an associative algebra with $1\#1$ as an identity element, then we call $B\#_\sigma^\tau H$ a twisted crossed product.

In what follows, we always assume that τ has a convolution inverse which is denoted by τ^{-1} and τ is a right 2-cocycle, i. e., it satisfies the following condition:

$$\sum \tau(h_1g_1,z)\tau(h_2,g_2) = \sum \tau(h,g_1z_1)\tau(g_2,z_2)$$

By Ref. [11], the following proposition is obvious.

Proposition 1 Let $B\#_\sigma^\tau H$ be defined as above. Assume that σ has a convolution inverse. If $\tau(H\otimes H)\subseteq Z(B)$, i. e, the center of B, then $B\#_\sigma^\tau H$ is a twisted crossed product if and only if σ and τ satisfy the following conditions:

$$\left.\begin{aligned} &\sum \sigma(1,k_1)\tau(1,k_3)\otimes k_2 = 1\otimes k\\ &\sum \sigma(h_1,1)\tau(h_3,1)\otimes h_2 = 1\otimes h\\ &\sum h_1[\sigma(k_1,g_1)\tau(k_5,g_5)]\sigma(h_2,k_2g_2)\tau(h_4,k_4g_4)\otimes h_3k_3g_3 =\\ &\qquad\sum \sigma(h_1,k_1)\tau(h_5,k_5)\sigma(h_2k_2,g_1)\tau(h_4k_4,g_3)\otimes h_3k_3g_2\\ &\sum (h_1(k_1b))\sigma(h_2,k_2)\tau(h_4,k_4)\otimes h_3k_3 =\\ &\qquad\sum \sigma(h_1,k_1)\tau(h_4,k_4)(h_2k_2b)\otimes h_3k_3\end{aligned}\right\}\tag{8}$$

for all $b\in B, h, k, g\in H$.

Example 1 1) Consider the case when τ is trivial, that is, $\tau(h,k)=\varepsilon(h)\varepsilon(k)$, for all $h, k\in H$. Then Proposition 1 simply says that $B\#_\sigma^\tau H = B\#_\sigma H$ is the usual crossed product (see Refs. [4, 6, 11]).

2) If τ and σ are all trivial, then we obtain the usual smash product $B\#H = B\#_\sigma^\tau H$ (see Ref. [9]).

Example 2 Let $B=K$ be the field in Proposition 1.

Then we have the following special cases:

1) A bitwisted algebra ${}_\sigma H_\tau$ with the multiplication given by

$$h * k = \sum \sigma(h_1, k_1) h_2 k_2 \tau(h_3, k_3)$$

for all $h, k \in H$;

2) When $\tau = \sigma^{-1}$ in 1), we have the twisted algebra $K_\sigma[\mathrm{H}]$ (see Ref. [12]);

3) When τ is trivial in 1), we obtain the left twisted algebra H^σ(see Ref. [13]);

4) When σ is trivial in 1), we obtain the right twisted algebra H_τ(see Ref. [13]).

The proof of the following proposition is straightforward.

Proposition 2 Let $B\#_\sigma^\tau H$ be a twisted crossed product. Then for all $a, b \in B$, $h, k \in H$, we have

1) $(1\otimes h)(1\otimes k) = \sum \sigma(h_1, k_1)\tau(h_3, k_3)\otimes h_2 k_2$;

2) $(a\otimes 1)(b\otimes 1) = (ab\otimes 1)$;

3) $(a\otimes 1)(1\otimes k) = a\otimes k$;

4) $(1\otimes h)(b\otimes 1) = \sum (h_1 b)\otimes h_2$;

5) B is a subalgebra of $B\#_\sigma^\tau H$ with the inclusion algebra map $i: B\to B\#_\sigma^\tau H$ ($b \mapsto b\otimes 1$).

Now by Ref. [14], we can obtain the following theorem.

Theorem 1 Let B be a bialgebra and $B\#_\sigma^\tau H$ a twisted crossed product. Assume that σ is a convolution invertible map. If $\tau(H\otimes H)\subseteq Z(B)$, then the twisted crossed product $B\#_\sigma^\tau H$ equipped with the tensor product coalgebra structure is a bialgebra if and only if the following conditions hold:

$$\left.\begin{array}{l}\Delta(hb) = \sum h_1 b_1\otimes h_2 b_2, \quad \varepsilon(hb) = \varepsilon(h)\varepsilon(b)\\ \sum \varepsilon(\sigma(h_1, k_1)\tau(h_2, k_2)) = \varepsilon(h)\varepsilon(k)\end{array}\right\} \quad (9)$$

$$\sum \Delta(\sigma(h_1, k_1)\tau(h_3, k_3))\otimes\Delta(h_2 k_2) = \sum \sigma(h_1, k_1)\tau(h_3, k_3)\otimes\sigma(h_4, k_4)\tau(h_6, k_6)\otimes h_2 k_2\otimes h_5 k_5$$

for all $b \in B$, $h, k, g \in H$.

As a corollary of this theorem, we have

Corollary 1[1] Let $\sigma: H\otimes H\to K$ be a convolution invertible map satisfying the left 2-cocycle condition. Then the twisted algebra $K_\sigma[H]$ is a bialgebra with the multiplication

$$h * k = \sum \sigma(h_1, k_1) h_2 k_2 \sigma^{-1}(h_3, k_3)$$

for all $h, k \in H$. Furthermore, $K_\sigma[H]$ is a Hopf algebra with the antipode given by

$$S^\sigma(h) = \sum \sigma(h_1, S(h_2))S(h_3)\sigma^{-1}(S(h_4), h_5)$$

where S is the antipode of H.

Proof Let $B = K$ be the field and $\tau = \sigma^{-1}$ in Theorem 1. Then, by Example 2, the twisted algebra $K_\sigma[H]$ has the above multiplication. Finally, it is obvious that the conditions (9) in Theorem 1 hold. The proof of the statement is straightforward.

Next, we will investigate relationships between coquasitriangular Hopf algebra structures on B and $B\#_\sigma^\tau H$. Let $(B\#_\sigma^\tau H, \tilde{\sigma})$ be a braided Hopf algebra. Then set

$$\begin{aligned} p(a, b) &= \tilde{\sigma}(a\otimes 1, b\otimes 1)\\ \beta(h, l) &= \tilde{\sigma}(1\otimes h, 1\otimes l)\\ u(a, h) &= \tilde{\sigma}(a\otimes 1, 1\otimes h)\\ v(h, a) &= \tilde{\sigma}(1\otimes h, a\otimes 1)\end{aligned}$$

for all $a, b \in B$ and $h, l \in H$.

Proposition 3 With the notations as above. Let $(B\#_\sigma^\tau H, \tilde{\sigma})$ be a coquasitriangular Hopf algebra. Then we have

$$\tilde{\sigma}(a\otimes h, b\otimes g) = \sum u(a_1, g_1)p(a_2, b_1)\beta(h_1, g_2)v(h_2, b_2) \quad (10)$$

for any $a, b \in B$ and $h, g \in H$.

Proposition 4 With the notations as above. Let $(B\#_\sigma^\tau H, \tilde{\sigma})$ be a coquasitriangular Hopf algebra. Then

1) (B, p) is a coquasitriangular Hopf algebra;

2) (H, β) is a coquasitriangular Hopf algebra associated to parameters (σ, u, v);

3) (B, H, u) is a skew braided pairing associated to parameters (σ, τ, p);

4) (H, B, v) is a skew opposite braided pairing associated to parameters (σ, τ, p).

Theorem 2 Let (B, p) be a braided Hopf algebra, (H, β) be a quasi-braided Hopf algebra, (H, B, V) a skew opposite braided pairing, and (B, H, u) a (σ, τ, p)-skew braided pairing. Then $(B\#_\sigma^\tau H, \tilde{\sigma})$ with $\tilde{\sigma}$ given by (10) is a coquasitriangular Hopf algebra if and only if the following conditions hold:

$$\left.\begin{array}{l}\sum v(h, c_1)p(b, c_2) = \sum v(h_2, c_2)p(h_1 b, c_1)\\ \sum \beta(h, l_1)u(b, l_2) = \sum \beta(h_2, l_2)p(h_1 b, l_1)\\ \sum \beta(h_1, g_2)v(h_2, g_1 c) = \sum v(h_1, c)\beta(h_2, g)\\ \sum u(a_1, g_2)p(a_2, g_1 c) = \sum p(a_1, c)u(a_2, g)\end{array}\right\} \quad (11)$$

for all $a, b, c \in B$ and $h, g, l \in H$.

Proof Consider the condition (1) for $(B\#_\sigma^\tau H, \tilde{\sigma})$. First, we have

$$\begin{aligned}&\sum u(a_1(h_1 b)_1(\sigma(h_2, g_1)\tau(h_4, g_3))_1, l_1)p(a_2(h_1 b)_2 \cdot\\ &(\sigma(h_2, g_1)\tau(h_4, g_3))_2, c_1)\beta(h_3 g_2, l_2)v(h_4 g_3, c_2) \overset{(9)}{=\!=}\\ &\sum u(a_1(h_1 b_1)(\sigma(h_3, g_1)\tau(h_6, g_4))_1, l_1)p(a_2(h_2 b_2) \cdot\\ &(\sigma(h_3, g_1)\tau(h_6, g_4))_2, c_1)\beta(h_4 g_2, l_2)v(h_5 g_3, c_2) \overset{(9)}{=\!=}\\ &\sum u(a_1(h_1 b_1)\sigma(h_3, g_1)\tau(h_5, g_3), l_1)p(a_2(h_2 b_2) \cdot\\ &\sigma(h_6, g_4)\tau(h_8, g_6), c_1)\beta(h_4 g_2, l_2)v(h_7 g_5, c_2) \overset{(6,1)}{=\!=}\\ &\sum u(a_1, l_1)u(h_1 b_1, l_2)u(\sigma(h_3, g_1)\tau(h_5, g_3), l_3) \cdot\end{aligned}$$

$\beta(h_4g_2, l_4)p(a_2, c_1)p(h_2b_2, c_2)p(\sigma(h_6, g_4)\tau(h_8, g_6), c_3)\cdot$
$v(h_7g_5, c_4) \overset{(7,1)}{=\!=}$
$\sum u(a_1, l_1)u(h_1b_1, l_2)\beta(h_3, l_3)\beta(g_1, l_4)p(a_2, c_1)\cdot$
$p(h_2b_2, c_2)v(h_4, c_3)v(g_2, c_4)$

Letting $l=1$ and $c=1$ in the two sides of the above equation, and using (9), we have (11). Conversely, it follows from (11) that the above equation holds.

Similarly, consider the condition (1) for $(B\#^{\tau}_{\sigma}H, \tilde{\sigma})$ if and only if we have (11). Finally, by a similar method to the one in Ref. [3], we can prove the condition (1) on $(B\#^{\tau}_{\sigma}H, \tilde{\sigma})$.

3 Applications

This section is devoted to the special cases of section 2 and we derive a generalized Drinfel'd double.

Case 1 If u and v are trivial, then we have that $hb = \varepsilon(h)b$, i.e., it is trivial. Hence, we have that (B, p) is a braided Hopf algebra and (H, H, β) is a skew pairing Hopf algebra, and the following conditions hold:

$$\sum \beta(h_1, g_1)(\sigma(h_2, g_2)\tau(h_4, g_4)\otimes h_3g_3) = \sum (\sigma(g_1, h_1)\tau(g_3, h_3)\otimes g_2h_2)\beta(h_4, g_4)$$
$$\sum p(a, \sigma(g_1, l_1)\tau(g_2, l_2)) = \varepsilon(a)\varepsilon(g)\varepsilon(l)$$
$$\sum p(\sigma(h_1, g_1)\tau(h_2, g_2), c) = \varepsilon(c)\varepsilon(g)\varepsilon(\mathrm{h})$$

In particular, if we assume that $\tau = \sigma^{-1}$ then (6) and (7) automatically hold and (5) becomes

$$\sum \beta(h_1, g_1)(\sigma(h_2, g_2)\sigma^{-1}(h_4, g_4)\otimes h_3g_3) = \sum (\sigma(g_1, h_1)\sigma^{-1}(g_3, h_3)\otimes g_2h_2)\beta(h_4, g_4)$$

In this case, we have that $(B\#^{\sigma^{-1}}_{\sigma} H, \tilde{\sigma})$ is a coquasitriangular Hopf algebra, where

$$\tilde{\sigma}(a\otimes h, b\otimes g) = p(a, b)\beta(h, g)$$

Case 2 If $\sigma: H\otimes H\to K$ and $\tau = \sigma^{-1}$, then (B, p) and (H^{σ}, β) are coquasitriangular Hopf algebras. (B, H, u) and (H, B, v) are skew pairing Hopf algebras, and the following conditions hold:

$$\sum u(a_1, g_1)(a_2\otimes g_2) = \sum (g_1a_1\otimes g_2)u(a_2, g_3)$$
$$\sum v(h_1, b_1)(h_2b_2\otimes h_3) = \sum (b_1\otimes h_1)v(h_2, b_2)$$

Notice that if we choose $v = u^{-1}$ then it follows from (6) that

$$\sum g_1a\otimes g_2 = \sum u(a_1, g_1)(a_2\otimes g_2)u^{-1}(a_3, g_3)$$

So we have

$$(a\#h)(b\#g) = \sum a(h_1b)\#h_2{}^{\sigma}g = \sum au(b_1, h_1)b_2\#h_2^{\sigma}gu^{-1}(b_3, h_3)$$

showing that $B\#H^{\sigma} = B \bowtie_u H^{\sigma}$, a generalized Drinfel'd double.

In this case $(B\bowtie_u H^{\sigma}, \tilde{\sigma})$ is a coquasitriangular Hopf algebra, here

$$\tilde{\sigma}(a\otimes h, b\otimes g) = p(a, b)\beta(h, g)$$

Case 3 Let $B = K$. Then we have the following proposition.

Proposition 5 If (H^{σ}, β) is a coquasitriangular Hopf algebra, then (H, α) is a coquasitriangular Hopf algebra with

$$\alpha(h, g) = \sum \sigma^{-1}(g_1, h_1)\beta(h_2, g_2)\sigma(h_3, g_3)$$

for all $h, g\in H$.

Proof By the condition (1) for (H^{σ}, β), we have

$$\sum \sigma(h_1, g_1)\sigma^{-1}(h_3, g_3)\beta(h_2g_2, l) = \sum \beta(h, l_1)\beta(g, l_2)$$

Thus,

$$\beta(hg, l) = \sum \sigma(h_1, g_1)\sigma^{-1}(h_2, g_2)\beta(h_3g_3, l)\sigma^{-1}(h_4, g_4)\sigma(h_5, g_5) = \sum \sigma(h_1, g_1)\beta(h_2, l_1)\beta(g_2, l_2)\sigma(h_3, g_3)$$

and the definition of $\alpha(h, g)$ implies that

$$\sum \sigma(l_1, h_1g_1)\alpha(h_2g_2, l_2)\sigma^{-1}(h_3g_3, l_4) = \sum \sigma^{-1}(h_1, g_1)\sigma(l_1, h_2)\alpha(h_3, l_2)\sigma^{-1}(h_4, l_3)\cdot \sigma(l_4, g_2)\alpha(g_3, l_5)\sigma^{-1}(g_4, l_6)\sigma(h_5, g_5)$$

Now, one can do a calculation for all $h, g, l\in H$ as follows:

$$\begin{aligned}\alpha(hg, l) = &\sum \sigma(l_2, h_3)\alpha(h_4, l_3)\sigma^{-1}(h_5, l_4)\sigma(l_5, g_3)\cdot \\ &\alpha(g_4, l_6)\sigma^{-1}(g_5, l_7)\sigma(h_6, g_6)\sigma(h_7g_7, l_8) = \\ &\sum \sigma^{-1}(l_1h_1, g_1)\underline{\sigma^{-1}(l_2, h_2)\sigma(l_3, h_3)}\cdot \\ &\alpha(h_4, l_4)\sigma^{-1}(h_5, l_5)\sigma(l_6, g_2)\alpha(g_3, l_7)\sigma^{-1}(g_4, l_8) = \\ &\sum \sigma^{-1}(l_1h_1, g_1)\alpha(h_2, l_2)\sigma^{-1}(h_3, l_3)\cdot \\ &\sigma(l_4, g_2)\alpha(g_3, l_5)\sigma(h_4, g_4l_6) = \\ &\sum \alpha(h_1, l_1)\underline{\sigma^{-1}(h_2l_2, g_1)\sigma^{-1}(h_3, l_3)}\cdot \\ &\sigma(l_4, g_2)\alpha(g_4, l_6)\sigma(h_4, l_5g_3)\ (\text{by (1) for } \beta) = \\ &\sum \alpha(h, l_1)\alpha(g, l_2)\end{aligned}$$

From the above discussions, it is not hard to obtain the following result.

Theorem 3 Let $B\#^{\sigma^{-1}}_{\sigma} H$ be a twisted crossed product. Let $\sigma(H\times H)\subset K$. Let (B, p) and (H, α) be coquasitriangular Hopf algebras. Then $(B\#^{\sigma^{-1}}_{\sigma} H, \tilde{\sigma})$ is a coquasitriangular Hopf algebra with

$$\tilde{\sigma}(a\otimes h, b\otimes g) = \sum u(a_1, g_1)p(a_2, b_1)\sigma(g_2, h_1)\cdot \alpha(h_2, g_3)\sigma^{-1}(h_3, g_4)v(h_4, b_2)$$

As corollaries of Theorem 3, we have the following corollary.

Corollary 2 Let (H, α) be a coquasitriangular Hopf algebra and σ a normal 2-cocycle. Then $(H^{\sigma}, \tilde{\sigma})$ is a coquasitriangular Hopf algebra, where $\tilde{\sigma}$ is given by

$$\tilde{\sigma}(h, g) = \sum \sigma(g_1, h_1)\alpha(h_2, g_2)\sigma^{-1}(h_3, g_3)$$

Corollary 3 Let $B \bowtie_u H$ be the quantum double. Suppose that (B, p) and (H, α) are coquasitriangular Hopf algebras. Then $(B \bowtie_u H, \tilde{\sigma})$ is a coquasitriangular Hopf algebra, where $\tilde{\sigma}$ is given by

$$\tilde{\sigma}(a \otimes h, b \otimes g) = u(a_1, g_1)p(a_2, b_1)\alpha(h_1, g_2)u(S_B(b_2), h_2)$$

for all $a, b \in B$ and $h, g \in H$.

Example 3 Let G be the cyclic group of order n generated by g. Fix a bicharacter $\varepsilon: G \times G \to k^*$ (that is, ε is a homomorphism in both entries). Assume that ε is antisymmetric (that is, $\varepsilon(g, h) = \varepsilon(h, g)^{-1}$ for all $g, h \in G$). We define a linear map $\alpha: k[G] \otimes k[G] \to k$ by $\alpha(g^i, g^j) = \omega^{ij}$ for all $g \in G$. Then $(k[G], \alpha)$ is a coquasitriangular Hopf algebra.

We note that the quantum double $D(G) = k[G] \bowtie_{\varepsilon} k[G]$ is in fact tensor product Hopf algebra of $k[G]$ with itself. In this case, by Corollary 3, the coquasitriangular Hopf algebra structure $\tilde{\sigma}$ on $D(G)$ is determined by

$$\tilde{\sigma}(g^i \otimes g^j, g^s \otimes g^t) = \omega^{is-tj}\varepsilon(g^i, g^t)\varepsilon(g^j, g^s)$$

for all $i, j, s, t \in \{1, 2, \ldots, n\}$.

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广义 Drinfel'd 量子偶的构造

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摘要：设 H 是 Hopf 代数，B 是代数，H 和 B 带有 2 个线性映射 $\sigma, \tau: H \otimes H \to B$. 设 B 是一个左 H-弱模代数，利用 σ 和 τ 可以定义 $B\#_{\sigma}^{\tau}H$ 上的一个乘法结构，给出了该乘法结构和张量余代数构成双代数的充要条件. 同时，讨论了双代数 $B\#_{\sigma}^{\tau}H$ 构成余拟三角 Hopf 代数的条件，所构造的余拟三角 Hopf 代数 $B\#_{\sigma}^{\tau}H$ 推广了现有的一些关于余拟三角 Hopf 代数的结果. 最后，给出了具体的应用实例.

关键词：扭结交叉积；余拟三角 Hopf 代数；广义 Drinfel'd 量子偶

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